Problem Sheet 7

due date: 1.2.2012

Problem 1: Solve

$$-u''(x) = f(x) \text{ for } x \in \Omega = (0,1), \qquad u(0) = u(1) = 0$$

in Matlab using linear finite elements on an equidistant grid with $h = 0.1$ and the local shape functions $N_1(x) = 1 - x, N_2(x) = x$ as given in the lecture. For $f(x) = 1$ and $f(x) = x^2$ calculate the exact solutions explicitly and study the convergence of the L^2-norm of the error in your numerical solution in terms of h. One can approximate the L^2-norm of the error via the l^2-norm of the nodal values of the error scaled by $\sqrt{h}$ because $\|f\|^2_{L^2(\Omega)} \approx h \sum_{j=1}^N f^2(x_j)$, where $(x_1, x_2, \ldots, x_N)$ is the equispaced grid on $[0,1]$ with spacing h.

Note: You do NOT have to use the sparse format for the stiffness matrix K_h.

Problem 2: Consider $K \subset \mathbb{R}$ bounded. Prove for $v \in H^1_0(K)$ the following form of Poincaré's inequality

$$\|v\|_{L^2(K)} \leq \frac{\text{diam}(K)}{\pi} |v|_{1,K}.$$

Hint: Shift the function v to $x \in [0, L]$ with $L := \text{diam}(K)$ and extend it in an odd fashion to $[-L, L]$. Express the new function using the sine Fourier series. Use the Parseval's identity.

Problem 3: Consider $\Omega \subset \mathbb{R}$, its discretizaton $\mathcal{M}_h$, the linear Lagrange finite element test space X_h and the corresponding interpolation operator I_h. Let $K \in \mathcal{M}_h$ and $v \in H^2(K)$, and let $h_K := \text{diam}(K)$.

(a) Show

$$\|v - I_h(v)\|_{L^2(K)} \leq \tfrac{h_K^2}{2\sqrt{3}} |v|_{2,K}$$

with $|v|^2_{2,K} := \int_K |v''(x)|^2 dx$ as in the lecture.

Hint: Make use of the following representation of v via Taylor's formula: $v(x) = p_1(x) + \int_{x_0}^x (x - s) v''(s) ds$ with $p_1 \in \mathcal{P}_1$ (the space of linear polynomials).

(b) Using Rolle's theorem and Poincaré's inequality from Problem 2, show

$$|v - I_h(v)|_{1,K} \leq \tfrac{h_K}{\pi} |v|_{2,K},$$

where $|v|_{1,K} := \int_K |v'(x)|^2 dx$ as in the lecture.