Problem 0
Let \( X_1, X_2 \sim U(\theta - 1/2, \theta + 1/2) \) be independent. Show that the probability density function (pdf) of \( X_1 - X_2 \) is independent of \( \theta \).

Problem 1
Let \( X \) and \( Y \) be random variables with distribution functions \( F \) and \( G \) respectively. We say that \( X \) is \textit{stochastically dominated} or \textit{dominated in distribution} by \( Y \) if

\[
F(x) \geq G(x), \quad x \in \mathbb{R}.
\]

Denote this relation by \( X \preceq Y \). Further, if \( F \) is a distribution function, its generalized inverse \( F^{-1} : (0, 1) \to \mathbb{R} \) is defined by

\[
F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}, \quad u \in (0, 1).
\]

(a) Prove that \( F^{-1}(u) \leq x \) if and only if \( u \leq F(x) \).

(b) Prove that if \( U \sim U[0, 1] \), then \( X \overset{d}{=} F^{-1}(U) \).

(c) Prove that \( X \preceq Y \) if and only if there exist two random variables \( \tilde{X} \) and \( \tilde{Y} \) such that \( \tilde{X} \overset{d}{=} X \) and \( \tilde{Y} \overset{d}{=} Y \) (i.e. two copies of \( X \) and \( Y \)) with the property \( \tilde{X} \preceq \tilde{Y} \).

Problem 2
Suppose \( X \) is distributed according to a Poisson distribution with random parameter \( \Lambda \) which is \( \Gamma(a, c) \)-distributed \((a > 0, c > 0)\). Assume in addition \( c \in \mathbb{N} \). Find the distribution of \( X \).

Problem 3
Let \( X, Y \sim U(0, 1) \) be independent. Write \( U := \min\{X, Y\} \) and \( V := \max\{X, Y\} \). Find \( E[U], E[V] \) and calculate \( \text{Cov}(U, V) \).

Problem 4
Let \( n \in \mathbb{N} \) and \( Y \sim \text{Bin}(n, X) \), where \( X \) is a random variable following a beta distribution on \((0, 1)\) (with parameters \( p, q > 0 \)). Find the distribution of \( Y \). What happens if \( X \) is uniform on \((0, 1)\)?