Applied Stochastic Models (SS 09)

Problem Set 4

Problem 1
Find the mean and variance of the Weibull distribution with hazard rate (or failure rate)

\[ r(t) = \frac{f(t)}{1 - F(t)} = \lambda \alpha (\lambda t)^{\alpha - 1}, \lambda > 0, \alpha > 0, t \geq 0. \]

Solution: The hazard rate function \( \lambda(t) = \frac{f(t)}{1 - F(t)}, t > 0, \) of some random lifetime \( T \) uniquely determines the distribution of \( T \), since necessarily

\[ F^T(t) = 1 - e^{-\int_0^t \lambda(u)du}, \quad f^T(t) = \lambda(t)e^{-\int_0^t \lambda(u)du}, \quad t > 0. \]

A random lifetime \( T \) with the above failure rate

\[ r(t) = \lambda \alpha (\lambda t)^{\alpha - 1}, \lambda > 0, \alpha > 0, t \geq 0, \]

thus has the density

\[ f^T(t) = r(t)e^{-\int_0^t r(u)du} = \lambda \alpha (\lambda t)^{\alpha - 1}e^{-\lambda t}, \lambda > 0, \alpha > 0, t \geq 0. \]

A direct calculation yields

\[
\mathbb{E}T = \int_0^\infty t\lambda \alpha t^{\alpha - 1}e^{-\lambda t^\alpha} dt = \alpha \int_0^\infty (\lambda t)^\alpha e^{-\lambda t} dt.
\]

Substituting \( u := (\lambda t)^\alpha \) yields

\[
\mathbb{E}T = \frac{1}{\lambda} \int_0^\infty u^{\frac{1}{\alpha}} e^{-u} du = \frac{1}{\lambda} \Gamma\left( \frac{1}{\alpha} + 1 \right).
\]

A similar calculation yields for the second moment

\[
\mathbb{E}T^2 = \int_0^\infty t^2\lambda \alpha (\lambda t)^{\alpha - 1}e^{-\lambda t^\alpha} dt,
\]

where the same substitution as above yields

\[
\mathbb{E}T^2 = \frac{1}{\lambda^2} \int_0^\infty u^{\frac{2}{\alpha}} e^{-u} du = \frac{1}{\lambda^2} \Gamma\left( \frac{2}{\alpha} + 1 \right).
\]
This means that the variance of $T$ is given by

$$Var(T) = ET^2 - (ET)^2 = \frac{1}{\lambda^2}(\Gamma(\frac{2}{\alpha} + 1) - \Gamma(\frac{1}{\alpha} + 1)^2).$$

**Problem 2**

Let $N(t)$ be a Poisson process, and let $Y_1, Y_2, Y_3, \ldots$ be independent and identically distributed random variables. Find the mean and variance of

$$\sum_{i=1}^{N(t)} Y_i.$$

**Solution:**

Conditioning by $N(t)$ yields for the expectation

$$E \left( \sum_{i=1}^{N(t)} Y_i \right) = E \left[ E[\sum_{i=1}^{N(t)} Y_i|N(t)] \right] = E[N(t)E[Y_1]] = E[N(t)]E[Y_1] = \left( \int_0^t \lambda(u)du \right) E[Y_1],$$

while the same procedure yields for the second moment

$$E \left( \sum_{i=1}^{N(t)} Y_i \right)^2 = E \left[ E[(\sum_{i=1}^{N(t)} Y_i)^2|N(t)] \right] = E \left[ E[\sum_{i=1}^{N(t)} Y_i^2 + \sum_{i \neq j} Y_i Y_j|N(t)] \right]$$

$$= E[N(t)]E[Y_1^2] + E[N(t)^2 - N(t)](E[Y_1]^2) = EN(t)VarY_1 + EN(t)^2(EY_1)^2.$$

Hence the variance is given by

$$Var(\sum_{i=1}^{N(t)} Y_i) = EN(t)VarY_1 + VarN(t)(EY_1)^2$$

$$= \left( \int_0^t \lambda(u)du \right) (VarY_1 + (EY_1)^2) = \left( \int_0^t \lambda(u)du \right) EY_1^2.$$

**Problem 3**

Consider a homogeneous Poisson process $N(t)$ with random (and with respect to time constant) intensity $\lambda$ which takes two values $\lambda_1, \lambda_2$ with equal probabilities. (This means that first, someone rolls the intensity according to a fair coinflip, and then generates the Poisson process with this intensity.) Find the probability generating function of $N(t)$.

**Solution:**
If $N(t)$ is a homogeneous Poisson process with deterministic rate $\lambda$, then its pgf is given by

$$
\phi(z) = E[z^{N(t)}] = \sum_{k \geq 0} P(N(t) = k)z^k = \sum_{k \geq 0} e^{-\lambda t} \frac{(\lambda t)^k}{k!} z^k = e^{-\lambda(1-z)}.
$$

If the intensity is given by a Bernoulli experiment, we may derive by conditioning that

$$
\phi(z) = E[z^{N(t)}] = E[E[z^{N(t)}|\lambda]] = \frac{1}{2} \left( e^{-\lambda t(1-z)} + e^{-\lambda z(1-z)} \right).
$$

Problem 4
Let the times between the events of a renewal process $N(t)$ be uniformly distributed on $(0, 1)$. Find the mean and variance of $N(t)$ for $0 < t < 1$.

Solution:
Let $0 < t < 1$ and consider the pgf of $N(t)$:

$$
G(z, t) := E \left[ z^{N(t)} \right], \quad z \in \mathbb{R}.
$$

Let $T_1, T_2, \ldots$ denote the times between the events of the renewal process, which are iid and $U(0, 1)$-distributed. Then with $S_n := T_1 + T_2 + \ldots + T_n$ we may write

$$
N(\omega, t) = \sum_{n \geq 1} \mathbb{1}\{S_n \leq t\}.
$$

We derive a differential equation for $G(z, t)$ as follows:

$$
G(z, t) = E \left[ z^{N(t)} \right] = \int z^{N(\omega, t)} P(d\omega) = \int z^{N(\omega, t)} \mathbb{1}\{T_1 > t\} P(d\omega) + \int z^{N(\omega, t)} \mathbb{1}\{T_1 < t\} P(d\omega)
$$

$$
= \int z^0 \mathbb{1}\{T_1 > t\} P(d\omega) + \int z^{\sum_{n \geq 1} \mathbb{1}\{S_n \leq t\}} \mathbb{1}\{T_1 < t\} P(d\omega)
$$

$$
= (1 - t) + \int z^{\sum_{n \geq 1} \mathbb{1}\{t_1 + \ldots + t_n \leq t\}} \mathbb{1}\{T_1 < t\} P^{(T_1, T_2, \ldots)}(d(t_1, t_2, \ldots))
$$

$$
= (1 - t) + \int \int z^{\sum_{n \geq 1} \mathbb{1}\{t_1 + \ldots + t_n \leq t\}} \mathbb{1}\{t_1 < t\} P^{T_1}(dt_1) P^{(T_2, T_3, \ldots)}(d(t_2, t_3, \ldots))
$$

$$
= (1 - t) + \int \int_0^t z^{\sum_{n \geq 2} \mathbb{1}\{t_2 + \ldots + t_n \leq t-t_1\}} P^{T_1}(dt_1) P^{(T_2, T_3, \ldots)}(d(t_2, t_3, \ldots))
$$

$$
= (1 - t) + \int \int_0^t z^{\sum_{n \geq 2} \mathbb{1}\{T_2 + \ldots + T_n \leq t-t_1\}} P^{(T_2, T_3, \ldots)}(dt_2) P^{T_1}(dt_1)
$$

$$
= (1 - t) + \int \int_0^t z^{\sum_{n \geq 2} \mathbb{1}\{T_2 + \ldots + T_n \leq t-t_1\}} P(dt_2) P^{T_1}(dt_1).
$$

Since $(T_2, T_3, \ldots) \overset{d}{=} (T_1, T_2, \ldots)$, this means that

$$
G(z, t) = (1 - t) + z \int_0^t G(z, t - t_1) dt_1 = (1 - t) + z \int_0^t G(z, u) du.
$$
Differentiating with respect to $t$ yields

$$\frac{d}{dt} G(z, t) = -1 + z G(z, t).$$

With respect to $t$ this is a first order linear differential equation with constant coefficients. The homogeneous solutions are obviously given by $\text{const} \cdot e^{zt}$, while a particular solution is the constant function $t \mapsto \frac{1}{z}$. Since necessarily $G(z, 0) = 1$, this yields

$$G(z, t) = \frac{1 - (1 - z)e^{zt}}{z}. \quad (1)$$

To find the mean and variance of $N(t)$, we need to investigate the first and second derivative of its pgf $G$ with respect to $z$. Equation (1) is equivalent to

$$zG(z, t) = 1 - (1 - z)e^{zt},$$

and differentiating this twice yields (\` denotes differentiation with respect to $z$)

$$zG'(z, t) + G(z, t) = (1 - (1 - z)t)e^{zt}, \quad (2)$$

and

$$zG''(z, t) + 2G'(z, t) = t(1 - (1 - z)t)e^{zt} + te^{zt}. \quad (3)$$

Evaluating (2) at $z = 1$ yields

$$E[N(t)] = e^t - 1, \quad 0 < t < 1,$$

while evaluating (3) at $z = 1$ yields

$$E[N(t)(N(t) - 1)] + 2E[N(t)] = 2te^t, \quad 0 < t < 1,$$

which means for the variance that

$$\text{Var}(N(t)) = E[N(t)(N(t) - 1)] + E[N(t)] - (E[N(t)])^2 = 2te^t - (e^t - 1) - (e^t - 1)^2, \quad 0 < t < 1.$$

For $t > 1$ the situation becomes pretty messy, since the differential equation for $G(z, t)$ takes a different form, and its solution seems to require a recursive consideration of the intervals $(k, k + 1), k \in \mathbb{N}$ where at each step the solution on $(k, k + 1)$ becomes more and more complicated. Another approach is to use the renewal equation and its Laplace transform, which yields a nice closed form of $\hat{m}(s)$. But then again, the inversion of the transform is difficult.