Applied Stochastic Models (SS 09)

Problem Set 5

Problem 1
The $r^{th}$ point $T_r$ of a Poisson process $N(t)$ of constant intensity $\lambda$ gives rise to an effect

$$X_r e^{-\alpha(t-T_r)}$$

at time $t \geq T_r$, where the $X_r$ are independent and identically distributed with finite variance. Find the characteristic function (or Laplace transform) of the total effect

$$S(t) = \sum_{r=1}^{N(t)} X_r e^{-\alpha(t-T_r)}$$

and its first two moments in terms of the first two moments of the $X_r$, and calculate $\text{Cov}(S(s), S(t))$. Show that

$$\rho(S(s), S(s+\nu)) \to e^{-\alpha\nu} \text{ as } s \to \infty.$$

Solution:
Conditional on $N(t)$ the vector $(T_1, ..., T_N(t))$ has the same distribution as the vector of the order statistics $(U_{(1)}, U_{(2)}, ..., U_{(N(t))})$ of $N(t)$ on $(0, t)$ uniformly distributed random variables $U_1, ..., U_{N(t)}$. Let $\pi$ denote the random permutation that satisfies

$$(U_{\pi(1)}, U_{\pi(2)}, ..., U_{\pi(N(t))}) = (U_{(1)}, U_{(2)}, ..., U_{(N(t))}).$$

Note that since the $X_i$ are iid, we have

$$(X_{\pi(1)}, ..., X_{\pi(N(t))}) \overset{d}{=} (X_1, ..., X_{N(t)})$$

for any fixed permutation $\tau$. This is also true for random permutations $\tau$ being independent from the $X_i$. In particular

$$(X_{\pi^{-1}(1)}, ..., X_{\pi^{-1}(N(t))}) \overset{d}{=} (X_1, ..., X_{N(t)}).$$

Using this, we derive

$$E[e^{-sS(t)}] = E[E[e^{-sS(t)}|N(t)] = E[E[\prod_{i=1}^{N(t)} e^{-sX_i e^{-\alpha(t-T_i)}}|N(t)]]$$

$$= E[E[\prod_{i=1}^{N(t)} e^{-sX_i e^{-\alpha(t-U_i)}}|N(t)] = E[E[\prod_{i=1}^{N(t)} e^{-sX_i e^{-\alpha(t-U_{\pi(i)}})|N(t)]]$$

$$= E[E[\prod_{i=1}^{N(t)} e^{-sX_i e^{-\alpha(t-U_{\pi(i)}}}|N(t)] \overset{(1)}{=} E[E[\prod_{i=1}^{N(t)} e^{-sX_i e^{-\alpha(t-U_{1})}}|N(t)]$$

$$= E[(E[e^{-sX_1 e^{-\alpha(t-U_{1})}}])]^{N(t)} = E[(E[e^{-sX_1 e^{-\alpha U_{1}}})]^{N(t)}].$$
The pgf of a homogenous Poisson process $N(t)$ with parameter $\lambda$ is given by

$$E[z^{N(t)}] = e^{\lambda(z-1)}.$$ 

Hence, we may proceed as follows:

$$E[e^{-sS(t)}] = e^{\lambda t(E[e^{-sX_1 e^{-at}}] - 1)} = e^{\lambda t \int_0^t E[e^{-sX_1 e^{-at}}] - 1} du)$$

$$= e^{\lambda t \hat{\phi}(se^{-at}) - 1} du,$$

where $\hat{\phi}$ denotes the Laplace transform of $X_1$. This means for the first two moments of $S(t)$:

$$E[S(t)] = -\frac{d}{ds} E[e^{-sS(t)}]_{s=0} = -\lambda \int_0^t -uX \alpha e^{-\alpha u} du = \frac{\lambda}{\alpha} (EX)(1 - e^{-\alpha t}),$$

$$E[S(t)^2] = \frac{d^2}{ds^2} E[e^{-sS(t)}]_{s=0} = \frac{\lambda}{2\alpha} E[X^2](1 - e^{-2\alpha t}) + \frac{\lambda^2}{\alpha^2} (E[X])^2(1 - e^{-\alpha t})^2.$$ 

For $s < t$ we can write $S(t)$ as follows:

$$S(t) = S(e^{-\alpha(t-s)} + \sum_{r=N(s)+1}^{N(t)} X_r e^{-\alpha(t-T_r)} = S(s)e^{-\alpha(t-s)} + \hat{S}(s,t).$$

Here $\hat{S}(s,t)$ is clearly independent from $S(s)$ and by linearity of the covariance this yields for $s < t$

$$\text{Cov}(S(s), S(t)) = \text{Var}(S(s))e^{-\alpha(t-s)} \rightarrow \frac{\lambda E[X^2]}{2\alpha}(1 - e^{-2\alpha s})e^{-\alpha(t-s)},$$

which means that

$$\text{Cov}(S(s), S(s+v)) \rightarrow \frac{\lambda E[X^2]}{2\alpha}e^{-\alpha v} \text{ as } s \rightarrow \infty.$$ 

In turn, this yields

$$\rho(S(s), S(s+v)) \rightarrow e^{-\alpha v} \text{ as } s \rightarrow \infty.$$ 

**Problem 2**

Let $N(t)$ be a non-homogeneous Poisson process with intensity function $\lambda(t)$. Show that the joint density function of the first two inter-event times is given by

$$\lambda(x)\lambda(x+y)e^{-\int_0^{x+y} \lambda(u) du}$$

and deduce that they are not in general independent.

*Hint*: Start with $P(T_1 \leq x, T_2 - T_1 > y)$.

**Solution**: Writing

$$\Lambda(x) := \int_0^x \lambda(u) du,$$

we have

$$P(T_1 \leq x, T_2 - T_1 > y) = \int_0^x \int_t^x 1 \{t_1 \leq x, t_2 - t_1 > y\} \mathbb{P}_{T_2 - T_1 | T_1 = t_1}(dt_2) \mathbb{P}(dt_1).$$
Here
\[ P_{T_2 - T_1 | T_1 = t_1} (T_2 - t_1 > y) = e^{-\int_{t_1}^{t_1+y} \lambda(u) \, du} = e^{-\Lambda(t_1+y) - \Lambda(t_1)} \]
and hence
\[ P(T_1 \leq x, T_2 - T_1 > y) = \int_0^x \lambda(t_1)e^{-\Lambda(t_1)}e^{-(\Lambda(t_1+y)-\Lambda(t_1))} \, dt_1 = \int_0^x \lambda(t_1)e^{-(\Lambda(t_1+y)} \, dt_1. \]
Hence the joint density of \( T_1 \) and \( T_2 - T_1 \) is given by
\[
\frac{d^2}{dx dy} (P(T_1 \leq x) - P(T_1 \leq x, T_2 - T_1 > y)) = \frac{d^2}{dx dy} (1 - P(T_1 \geq x) - P(T_1 \leq x, T_2 - T_1 > y)) = \frac{d^2}{dx dy} \left( 1 - e^{-\Lambda(x)} - \int_0^x \lambda(t_1)e^{-\Lambda(t_1+y)} \, dt_1 \right) = \lambda(x)\lambda(x+y)e^{-\Lambda(x+y)}.
\]
Since the density of \( T_1 \) is given by
\[
\frac{d}{dx} (1 - F(T_1 > x)) = \lambda(x)e^{-\Lambda(x)},
\]
independence of \( T_1 \) and \( T_2 - T_1 \) would necessarily imply
\[
F_{T_2 - T_1}(y) = \lambda(x + y)e^{\Lambda(x+y) - \Lambda(x)}
\]
for any \( x \), which cannot depend on \( x \). This in turn is true iff \( \lambda \equiv const. \) Hence in any other case \( T_1 \) and \( T_2 - T_1 \) are not independent.

**Problem 3**

Show that for a structure function \( \phi \)

(a) if \( \phi(0,0,...,0) = 0, \phi(1,1,...,1) = 1 \), then \( \min x_i \leq \phi(x) \leq \max x_i \),

(b) \( \phi(\max (x,y)) \geq \max (\phi(x),\phi(y)) \),

(c) \( \phi(\min (x,y)) \leq \min (\phi(x),\phi(y)) \).

**Solution:**

(a) By assumption \( \phi(0,0,...,0) = 0, \phi(1,1,...,1) = 1 \). There are the following three cases:

- Case 1: \( \min x_i = 0, \max x_i = 1 \). Then clearly
  \[ 0 = \min_i x_i \leq \phi(x) \leq \max_i x_i = 1. \]

- Case 2: \( \max x_i = 0 \). Then \( x = (0,...,0) \) and by assumption \( \phi(x) = 0 \). The inequality holds trivially.

- Case 3: \( \min x_i = 1 \). Then \( x = (1,...,1) \) and by assumption \( \phi(x) = 1 \). The inequality holds trivially.
(b) By monotonicity of a structure function, we have
\[ \phi(x) \leq \phi(\max(x, y)), \quad \phi(y) \leq \phi(\max(x, y)), \]
and hence
\[ \max\{\phi(x), \phi(y)\} \leq \phi(\max(x, y)). \]

(c) By monotonicity of a structure function, we have
\[ \phi(x) \geq \phi(\min(x, y)), \quad \phi(y) \geq \phi(\min(x, y)), \]
and hence
\[ \min\{\phi(x), \phi(y)\} \geq \phi(\min(x, y)). \]

**Problem 4**

For any structure function \( \phi \), we define the dual structure \( \phi^D \) by
\[
\phi^D(x) = 1 - \phi(1 - x), \quad \text{where } 1 := (1, 1, \ldots, 1).
\]

(a) Show that the dual of a parallel (series) system is a series (parallel) system.

(b) Show that the dual of the dual structure is the original structure.

(c) What is the dual of a \( k \)-out-of-\( n \) structure?

(d) Show that a minimal path (cut) set of the dual system is a minimal cut (path) set of the original structure.

**Solution:**

(a) If \( \phi \) corresponds to a series system, then it has the form
\[ \phi(x) = x_1x_2\ldots x_n. \]

The respective dual structure is given by
\[ \phi^D(x) = 1 - (1 - x_1)(1 - x_2)\ldots(1 - x_n), \]
which is the structure function of a parallel system. Part (b) implies that the dual of a parallel system is a series system again.

(b) We have
\[ \phi^{DD}(x) = 1 - \phi^D(1 - x) = 1 - (1 - \phi(1 - (1 - x))) = \phi(x). \]

(c) Let \( A_1, A_2, \ldots, A_l \) denote the \( \binom{n}{k} \) different minimal path sets of a \( k \)-out-of-\( n \)-structure (i.e. \( l = \binom{n}{k} \)). The structure function can be written as
\[
\phi(x) = \max_j \prod_{i \in A_j} x_i = \begin{cases} 1, & \sum x_i \geq k \\ 0, & \sum x_i < k \end{cases},
\]
and the dual is then given by

\[\phi^D(x) = 1 - \max_j \prod_{i \in A_j} (1 - x_i) = \begin{cases} 1, & \sum x_i \geq n - k + 1 \\ 0, & \sum x_i < n - k + 1 \end{cases},\]

which is a \((n - k + 1)\)-out-of-\(n\) structure.

(d) Let \(A\) be a minimal path set of \(\phi^D\) and let \(x\) be the respective minimal path vector of \(A\) in the dual structure. This means

\[\phi^D(x) = 1 \text{ and } \phi^D(y) = 0, \quad y < x.\]

According to part (b), dualizing is self-inverse, which means

\[\phi(x) = 1 - \phi^D(1 - x).\]

Then \(1 - x\) is a minimal cut vector of the original structure: First

\[\phi(1 - x) = 1 - \phi^D(x) = 1 - 1 = 0\]

and second for each \(y > 1 - x\), we have \(x > 1 - y\) and hence \(\phi^D(1 - y) = 0\) which means

\[\phi(y) = 1 - \phi^D(1 - y) = 1.\]

\(1 - x\) being a minimal cut vector of \(\phi\) just means that \(A\) is a minimal cut set of the original structure. The dual statement can be proved in the same way.