Markov-modulated Diffusion Risk Models

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Abstract In this paper we consider Markov-modulated diffusion risk reserve processes. Using diffusion approximation we show the relation to classical Markov-modulated risk reserve processes. In particular we derive a representation for the adjustment coefficient and prove some comparison results. Among others we show that increasing the volatility of the diffusion increases the probability of ruin.

Key words: Ruin probability; Adjustment coefficient; Diffusion approximation; Stochastic orderings

1 Introduction

A key topic of risk theory still is the probability of ruin of a risk reserve process. This process is a simple mathematical model for the differences of assets and liabilities of an insurance company. Good references to such models are Asmussen (2000) and Rolski et al. (1999). In this paper we investigate Markov-modulated diffusion risk reserve processes. These models are given by equation (1) below. They extend Asmussen (1989) where a classical risk process in a Markovian environment has been investigated without diffusion. In Schmidli (1995) among others, the author considers such a Markov-modulated risk model where a diffusion is added. In contrast to this model, in our case it is possible that all data, including the premium rate and the volatility of the diffusion depends on the external Markov chain. The aim of this paper is to show that by continuity properties interesting data of the Markov-modulated diffusion risk model can be approximated by the respective data of a classical Markov-modulated risk model.

The paper is organized as follows: After introducing the model in Section 2 we derive the adjustment coefficient of such a risk reserve process in Section 3. Here a similar result has been obtained by Schmidli (1995), however our model is slightly different and we also give a different representation of the adjustment coefficient which is more convenient for our purpose. Whereas Schmidli (1995) uses a change of measure technique we essentially follow the paper by Björk and Grandell (1988). In Section 4 we derive a diffusion approximation for classical Markov-modulated risk reserve processes. The idea is that a Markov-modulated diffusion risk model can be seen as an approximation of a classical Markov-modulated risk model with small and frequent claims. Taking the appropriate limit we are able to carry over results which are already known to the diffusion case. In particular we show that the adjustment coefficients of a classical properly scaled sequence of risk models converge to the adjustment coefficient computed in Section 3. The diffusion approximation is also used in Section 5 to derive some comparison results for these models. We show that increasing the volatility of a diffusion risk model increases the probability of ruin and averaging the system parameters reduces the risk.

2 The Model

We consider a diffusion risk reserve process where the underlying data changes according to a continuous-time Markov chain with finite state space. More precisely, we denote by $J = \{J_t,\ t \geq 0\}$ an irreducible continuous-time Markov chain with finite state space $E = \{1, \dots, d\}$ and intensities q_{ij}. If not stated otherwise, the distribution of J_0 is arbitrary. J_t can be interpreted as the general economic conditions which are present at time t. J_t influences the premium rate, the arrival intensity of claims, the claim size distribution and the volatility of the diffusion process as follows: the premium income rate at time t is c_{J_t}, i.e. as long as $J_t = i$ we have a linear income stream at rate c_i. Claim arrivals are according to a Poisson-process with rate λ_{J_t}. Thus, $N = \{N_t,\ t \geq 0\}$ is a Markov-modulated Poisson-process. A claim U_k which occurs at time t has distribution Q_{J_t}, where Q_i is some distribution concentrated on $(0, \infty)$ for $i \in E$. As usual claim sizes $U_1, U_2, \dots$ are assumed to be conditionally independent given J and μ_i is the finite expectation of Q_i, for $i \in E$. The volatility of the diffusion at time t is given by σ_{J_t}. If $u \geq 0$ denotes the initial risk reserve, and $W = \{W_t,\ t \geq 0\}$ is a standard Brownian motion, the Markov-modulated diffusion risk reserve process $\{X_t,\ t \geq 0\}$ is given by

$$X_t = u + \int_0^t c_{J_s} ds - \sum_{k=1}^{N_t} U_k + \int_0^t \sigma_{J_s} dW_s. \tag{1}$$

In what follows we will only be interested in ruin probabilities for this model, i.e. we are only interested in the question whether the trajectories of X stay above 0 or not. By applying the time change $\hat{X}_t := X_{T(t)}$ with $T(t) := \int_0^t \frac{1}{c_{J_s}} ds$ the structure of the model does not change and we can w.l.o.g. assume that $c(\cdot) \equiv 1$. Thus, in our paper we suppose that

$$X_t = u + t - \sum_{k=1}^{N_t} U_k + \int_0^t \sigma_{J_s} dW_s. \tag{2}$$

The probability of ruin in infinite time is then for $u \geq 0$ defined by

$$\psi(u) = P\left(\inf_{t \geq 0} X_t < 0 \mid X_0 = u\right).$$

If $\tau := \inf\{t \geq 0 \mid X_t < 0\}$ is the time of ruin, then obviously $\psi(u) = P(\tau < \infty \mid X_0 = u)$. If we denote by $\pi = (\pi_i)_{i \in E}$ the stationary distribution of J (which exists and is unique since J is irreducible and has a finite state space) and define $\rho := 1 - \sum_{i \in E} \pi_i \lambda_i \mu_i$. ρ is the difference between the premium income in one time unit and the expected payout in one time unit. We obtain:

Lemma 1. *Suppose $\rho \leq 0$. Then for all $u \geq 0$ it holds that*

$$\psi(u) = 1.$$

The proof of this statement is omitted since it is standard. For the remaining sections we assume that $\rho > 0$, i.e. we have a positive safety loading.

3 The Adjustment Coefficient

In this section we impose some further conditions on our data. In order to obtain the adjustment coefficient we assume that the moment generating functions of the claim size distributions are finite near zero, i.e. for every $i \in E$ there exists a (possibly infinite) constant $r_\infty^{(i)} \in (0,\infty]$ such that for $r \geq 0$

$$h_i(r) := \int_0^\infty e^{rx}\, dQ_i(x) - 1 < \infty$$

for every $r < r_\infty^{(i)}$ with $h_i(r) \to \infty$ as $r \to r_\infty^{(i)}$. Thus, the tail of the distribution Q_i decreases at least exponentially fast. This case is sometimes called the small claim case in contrast to models with heavy claim size distributions. Our aim is to find a constant $R > 0$ such that for all $\varepsilon > 0$:

$$\lim_{u\to\infty} \psi(u) e^{(R-\varepsilon)u} = 0 \tag{3}$$

$$\lim_{u\to\infty} \psi(u) e^{(R+\varepsilon)u} = \infty. \tag{4}$$

R is then called the *adjustment coefficient*. There are different methods available for obtaining the adjustment coefficient (see e.g. Rolski et al. (1999)). We use the so-called martingale method. For the next result we denote by $\mathcal{F}^X = \{\mathcal{F}_t^X,\ t \geq 0\}$ the natural filtration of the risk reserve process and by $\mathcal{F}^J = \{\mathcal{F}_t^J,\ t \geq 0\}$ the natural filtration of the environment process. Finally we define $\mathcal{F} = \{\mathcal{F}_t,\ t \geq 0\}$ by $\mathcal{F}_t := \mathcal{F}_t^X \vee \mathcal{F}_\infty^J$. This means in particular that J_t is $\mathcal{F}_0$-measurable. Moreover, we define the time which the environment process J spends in some state $i \in E$ until time $t \geq 0$ by $\xi_i(t)$, i.e. $\xi_i(t) := \int_0^t \delta_{\{i\}}(J_s)\, ds$.

Lemma 2. *Let $u, r \geq 0$ be arbitrary but fixed. Then the process $M = \{M_t,\ t \geq 0\}$ defined by*

$$M_t := \frac{\exp(-rX_t)}{\exp\Big(\sum_{i\in E}\big[\lambda_i h_i(r) + \frac{1}{2}r^2\sigma_i^2 - r\big]\xi_i(t)\Big)}$$

is an $\mathcal{F}$-martingale.

Proof. By $E^{\mathcal{F}_\infty^J}$ we denote the conditional expectation given $\mathcal{F}_\infty^J$. $W^1, \ldots, W^d$ are d independent Brownian motions and $N^1, \ldots, N^d$ are independent Poisson processes with intensities $\lambda_1, \ldots, \lambda_d$ respectively. The random variables U_k^i have distribution Q_i and are all independent. It is straightforward to compute

$$E^{\mathcal{F}_\infty^J}\Big(\exp\big(-r\sum_{i\in E}\sigma_i W^i_{\xi_i(t)}\big)\Big) = \prod_{i\in E} E^{\mathcal{F}_\infty^J}\Big(\exp\big(-r\sigma_i W^i_{\xi_i(t)}\big)\Big) = \prod_{i\in E}\exp\Big(\frac{r^2\sigma_i^2}{2}\,\xi_i(t)\Big)$$

and

$$\begin{aligned}
E^{\mathcal{F}_\infty^J}\Big(\exp\big(r\sum_{i\in E}\sum_{k=1}^{N^i_{\xi_i(t)}} U_k^i\big)\Big) &= \prod_{i\in E} E^{\mathcal{F}_\infty^J}\Big(\exp\big(r\sum_{k=1}^{N^i_{\xi_i(t)}} U_k^i\big)\Big) \\
&= \prod_{i\in E}\sum_{m=0}^\infty E^{\mathcal{F}_\infty^J}\Big(e^{r\sum_{k=1}^m U_k^i}\Big) P^{\mathcal{F}_\infty^J}\big(N^i_{\xi_i(t)} = m\big) \\
&= \prod_{i\in E}\sum_{m=0}^\infty \big(1 + h_i(r)\big)^m e^{-\lambda_i\xi_i(t)}\,\frac{(\lambda_i\xi_i(t))^m}{m!} \\
&= \prod_{i\in E} e^{\lambda_i h_i(r)\xi_i(t)} = \exp\Big(\sum_{i\in E}\lambda_i\, h_i(r)\,\xi_i(t)\Big).
\end{aligned}$$

Using these results, we obtain

$$E\big[M_t\big|\mathcal{F}_s\big] = M_s \cdot E^{\mathcal{F}^J_\infty}\left[\frac{\exp\Big(-r\sum_{i\in E}\sigma_i(W^i_{\xi_i(t)} - W^i_{\xi_i(s)}) + r\sum_{i\in E}\sum_{k=N^i_{\xi_i(s)+1}}^{N^i_{\xi_i(t)}} U^i_k\Big)}{\exp\Big(\sum_{i\in E}\big(\lambda_i h_i(r) + \frac{1}{2}r^2\sigma_i^2\big)\big(\xi_i(t)-\xi_i(s)\big)\Big)}\middle|\mathcal{F}^X_s\right]$$

$$= M_s \cdot \frac{E^{\mathcal{F}^J_\infty}\Big(\exp\big(-r\sum_{i\in E}\sigma_i W^i_{\xi_i(t)-\xi_i(s)}\big)\Big)}{\exp\Big(\sum_{i\in E}\frac{r^2\sigma_i^2}{2}\big(\xi_i(t)-\xi_i(s)\big)\Big)}\;\frac{E^{\mathcal{F}^J_\infty}\Big(\exp\big(r\sum_{i\in E}\sum_{k=1}^{N^i_{\xi_i(t)-\xi_i(s)}} U^i_k\big)\Big)}{\exp\Big(\sum_{i\in E}\lambda_i h_i(r)\big(\xi_i(t)-\xi_i(s)\big)\Big)} = M_s$$

for $0 \le s \le t$. Since $EM_t < \infty$ for all $t \ge 0$ the statement follows. □

Exploiting the martingale property we immediately derive the following inequality:

Lemma 3. *Let $r > 0$ be fixed. Then*

$$\psi(u) \le e^{-ru}\, C(r)$$

for all $u \ge 0$ where

$$C(r) := E\left(\sup_{t\ge 0}\ \exp\left(\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2\sigma_i^2}{2} - r\right]\xi_i(t)\right)\right).$$

Proof. Define by $\tilde{M}_t := M_{t\wedge\tau}$ the process stopped at the time of ruin. $\tilde{M}$ is also an $\mathcal{F}$-martingale. For $r > 0$ and $u \ge 0$ it therefore follows that

$$e^{-ru} = \tilde{M}_0 = E\Big[\tilde{M}_t\Big|\mathcal{F}_0\Big] = E^{\mathcal{F}^J_\infty}\Big(\tilde{M}_t\Big) \ge E^{\mathcal{F}^J_\infty}\Big[M_\tau\Big|\tau \le t\Big]\ P^{\mathcal{F}^J_\infty}(\tau \le t)$$

$$= E^{\mathcal{F}^J_\infty}\left[\frac{\exp(-rX_\tau)}{\exp\Big(\sum_{i\in E}\Big[\lambda_i h_i(r) + \frac{r^2\sigma_i^2}{2} - r\Big]\xi_i(\tau)\Big)}\middle|\tau \le t\right]\cdot P^{\mathcal{F}^J_\infty}(\tau \le t)$$

$$\ge \frac{P^{\mathcal{F}^J_\infty}(\tau \le t)}{\sup_{0\le v\le t}\ \exp\Big(\sum_{i\in E}\Big[\lambda_i h_i(r) + \frac{r^2\sigma_i^2}{2} - r\Big]\xi_i(v)\Big)}$$

and hence

$$P^{\mathcal{F}^J_\infty}(\tau \le t) \le e^{-ru}\sup_{0\le v\le t}\ \exp\left(\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2\sigma_i^2}{2} - r\right]\xi_i(v)\right).$$

Letting $t \to \infty$ and taking the expectation on both sides we obtain

$$\psi(u) = P(\tau < \infty) \le e^{-ru}\, C(r).$$

□

To get a good bound we have to choose $r > 0$ as large as possible while $C(r) < \infty$. The way to find such a maximizing $r > 0$ is similar to what Björk and Grandell (1988) do for the ordinary Cox model. Let the time epoch of the n^{th} entry of the environment process to state $j \in E$ be denoted by $\tau^{(j)}_n$ where $\tau^{(j)}_0 \equiv 0$. We put $\tau^{(j)} := \tau^{(j)}_1$. For $j, k \in E$ we now have to consider the function θ_{kj} defined by

$$\theta_{kj}(r) := E_k\left(\exp\left(\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2\sigma_i^2}{2} - r\right]\xi_i(\tau^{(j)})\right)\right)$$

where $r \geq 0$ and E_k is the expectation, given $J_0 = k$. Using these functions we are able to state a necessary condition for $C(r)$ being finite.

Lemma 4. *Let $r > 0$ be fixed. Then $C(r) < \infty$ implies $\theta_{jj}(r) < 1$ for all $j \in E$.*

The proof is rather technical and can be found in the appendix. Let us now define

$$R = \sup \Big\{ r > 0 \mid \theta_{jj}(r) < 1 \ \forall j \in E \Big\}. \tag{5}$$

R will be the adjustment coefficient. We can now show

Lemma 5. *Suppose that R defined by (5) exists. For $0 < r < R$ we have $C(r) < \infty$.*

The proof can again be found in the Appendix. The so-called Lundberg inequality now follows directly form Lemma 3:

Theorem 6 (Lundberg-inequality)**.** *Suppose that R defined by (5) exists. For any $r < R$ we have*

$$\psi(u) \leq e^{-ru}\, C(r)$$

with $C(r) < \infty$ for all $u \geq 0$.

The Lundberg-inequality now immediately implies the convergence result (3).

Corollary 7. *Suppose that R defined by (5) exists. For any $\varepsilon > 0$ we obtain*

$$\lim_{u \to \infty} \psi(u) e^{(R-\varepsilon)u} = 0.$$

In order to obtain the second convergence result (4) we need a little bit stronger assumption.

Theorem 8. *Suppose that R defined by (5) exists and that there is a $\delta > 0$ such that $\theta_{jj}(R+\delta) < \infty$ for all $j \in E$. For any $\varepsilon > 0$ we obtain*

$$\lim_{u \to \infty} \psi(u) e^{(R+\varepsilon)u} = \infty.$$

Proof. If we denote by

$$\psi_j(u) = P\left(\inf_{t \geq 0} X_t < 0 \mid X_0 = u, J_0 = j \right)$$

then $\psi(u) = \sum_{j \in E} P(J_0 = j)\psi_j(u)$ and it suffices to show that $\lim_{u \to \infty} \psi_j(u) e^{(R+\varepsilon)u} = \infty$ for some $j \in E$ with $P(J_0 = j) > 0$.
Since θ_{jj} is convex and therefore continuous on the interior of its domain it follows from our assumption and the definition of R in (5) that $\theta_{jj}(R) = 1$ for this $j \in E$ (it can indeed be shown that if $\theta_{jj}(R) = 1$ for one $j \in E$, then the equation is satisfied for all $j \in E$).
Define $Y_t := X_t - u$. It is not difficult to see that

$$\theta_{jj}(r) = E_j\left(e^{-rY_{\tau(j)}} \right).$$

Moreover, $\left(Y_{\tau_n^{(j)}}\right)_{n\in\mathbb{N}}$ is a random walk under the assumption that $J_0 = j$. The ruin probability of this random walk is defined by

$$\psi_j^{rw}(u) = P\Big(\inf_{n\in\mathbb{N}} Y_{\tau_n^{(j)}} < -u \mid J_0 = j\Big).$$

It is obvious that $\psi_j^{rw}(u) \leq \psi_j(u)$ for all $u \geq 0$. Note that the distribution of $Y_{\tau^{(j)}}$, i.e. the distribution of the generic random variable for the steps, is clearly non-lattice. Thus, it follows from Theorem 6.5.7 and the associated remark in Rolski et al. (1999), p. 258, that

$$\lim_{u\to\infty} \frac{\psi_j^{rw}(u)}{e^{-Ru}} = \tilde{C}$$

for some constant $\tilde{C} > 0$. We therefore get

$$\lim_{u\to\infty} \frac{\psi_j(u)}{e^{-(R+\varepsilon)u}} \geq \lim_{u\to\infty} \frac{\psi_j^{rw}(u)}{e^{-(R+\varepsilon)u}} = \infty$$

for all $\varepsilon > 0$.

□

4 Diffusion Approximation

In principle the structure of the diffusion risk model differs from the classical risk model where trajectories are linear with jumps. However, it is well-known that a diffusion arises as a limit from properly scaled classical risk processes. This means, the diffusion can be approximately interpreted as a risk process with very small and frequent claims. Our hope is to carry over results form the classical model to the diffusion model by taking limits. This idea has also been exploited by Sarkar and Sen (2005). In order to work this idea out we have to establish a limit result for the Markov-modulated model. Since we have not found such a statement in the literature we give a proof below. But first let us recall the diffusion approximation for the classical risk model (this can be found e.g. in Grandell (1977), Grandell (1978)). Suppose $N^0 = \{N_t^0,\ t \geq 0\}$ is a Poisson process with intensity 1. Let $\tilde{U}_1, \tilde{U}_2, \ldots$ be a sequence of independent and identically distributed random variables with finite expectation $E\tilde{U} = \tilde{\mu}$ and finite variance $Var(\tilde{U}) = \tilde{s}^2$. Now define for $n \in \mathbb{N}$ the (martingale) processes

$$M_t^{(n)} := \frac{1}{\sqrt{n}}\left(\tilde{\mu}nt - \sum_{k=1}^{N_{tn}^0} \tilde{U}_k\right).$$

Then with $n \to \infty$ we obtain

$$M^{(n)} \Rightarrow \sigma W$$

where $\Rightarrow$ denotes weak convergence with respect to the Skorohod topology and $\sigma^2 = \left(\tilde{\mu}^2 + \tilde{s}^2\right)$. W is as in Section 2 a standard Brownian motion. Moreover, the ruin probabilities of the $M^{(n)}$ processes converge to the ruin probability of σW. For the Markov-modulated model we suppose that when the claim $\tilde{U}_k$ occurs at time t it has distribution $\tilde{Q}_{J_t}$. All distributions $\tilde{Q}_1, \ldots, \tilde{Q}_d$ have the same finite expectation $\tilde{\mu}$ and finite variances $\tilde{s}_i^2 = \int (x-\tilde{\mu})^2 Q_i(dx)$, $i = 1, \ldots, d$. We choose $\tilde{\mu}, \tilde{s}_i^2, \ldots, \tilde{s}_d^2$ such that $\sigma_i^2 = \tilde{\mu}^2 + \tilde{s}_i^2$ where σ_i are given as in Section 2. We set

$$X_t^{(n)} := u + t - \sum_{k=1}^{N_t} U_k + \frac{1}{\sqrt{n}}\left(\tilde{\mu}nt - \sum_{k=1}^{N_{tn}^0} \tilde{U}_k\right). \tag{6}$$

We can think of (6) as a classical Markov-modulated Cramér-Lundberg model as follows:

$$X_t^{(n)} \stackrel{d}{=} u + t(1+\tilde{\mu}\sqrt{n}) - \sum_{k=1}^{\hat{N}_t} \hat{U}_k$$

where $\hat{N}$ is a Markov-modulated Poisson-process with intensities $\lambda_1+n, \ldots, \lambda_d+n$. If $J_t = i$ and the claim $\hat{U}_k$ appears at time t, then it has the distribution $\hat{Q}_i^{(n)}(x) = \frac{\lambda_i}{\lambda_i+n} Q_i(x) + \frac{n}{\lambda_i+n}\tilde{Q}_i(\sqrt{n}x)$. The ruin probability for the approximating sequence is given by

$$\psi^{(n)}(u) = P\left(\inf_{t\geq 0} X_t^{(n)} < 0 \mid X_0^{(n)} = u\right).$$

In what follows, the process X is the diffusion risk reserve process given in equation (2).

Theorem 9. *Suppose the processes $X^{(n)}$, $n \in I\!N$ and X are given. Then*

a) $X^{(n)} \Rightarrow X$.

b) $\lim_{n\to\infty} \psi^{(n)}(u) = \psi(u)$ for all $u \geq 0$.

Proof. a) It holds that

$$X_t^{(n)} = u + t - \sum_{k=1}^{N_t} U_k + \frac{1}{\sqrt{n}}\left(\tilde{\mu}nt - \sum_{i\in E}\sum_{k=1}^{N^0_{\xi_i(t)n}} \tilde{U}_k^i\right).$$

where the random variables $\tilde{U}_k^i$ are independent and identically distributed and have distribution $\tilde{Q}_i$. It follows from the d-dimensional Donsker FCLT (see e.g. Whitt (2002) Theorem 4.3.5) that

$$\left(\left\{\frac{1}{\sqrt{n}}\Big(\tilde{\mu}nt - \sum_{k=1}^{N^0_{tn}} \tilde{U}_k^i\Big),\ t\geq 0\right\}, i\in E\right) \Rightarrow \left(\left\{\sigma_i W_t^i,\ t\geq 0\right\}, i\in E\right)$$

with $\sigma_i^2 = \tilde{\mu}^2 + \tilde{s}_i^2$ and $W^1, \ldots, W^d$ are independent standard Brownian motions. Indeed, in order to apply the Donsker FCLT to the compound Poisson process, a number of arguments are necessary but they are standard, so we skip them here. For details of this procedure see e.g. Bäuerle (2004). Applying the time transformation $t \mapsto \xi_i(t)$ we obtain

$$\left(\left\{\frac{1}{\sqrt{n}}\Big(\tilde{\mu}n\xi_i(t) - \sum_{k=1}^{N^0_{\xi_i(t)n}} \tilde{U}_k^i\Big), t\geq 0\right\}, i\in E\right) \Rightarrow \left(\left\{\sigma_i W^i_{\xi_i(t)},\ t\geq 0\right\}, i\in E\right)$$

Adding up these processes it follows from the continuous mapping theorem that

$$\left\{\frac{1}{\sqrt{n}}\Big(\tilde{\mu}nt - \sum_{i\in E}\sum_{k=1}^{N^0_{\xi_i(t)n}} \tilde{U}_k^i\Big),\ t\geq 0\right\} \Rightarrow \left\{\sum_{i\in E}\sigma_i W^i_{\xi_i(t)},\ t\geq 0\right\} \stackrel{d}{=} \left\{\int_0^t \sigma_{J_s} dW_s,\ t\geq 0\right\}$$

and the statement follows.

b) Note that

$$\begin{aligned}\psi^{(n)}(u) &= P\left(\inf_{t\geq 0} X_t^{(n)} + u < 0 \mid X_0^{(n)} = 0\right)\\ &= P\left(\sup_{t\geq 0} -X_t^{(n)} - u > 0 \mid X_0^{(n)} = 0\right).\end{aligned}$$

Thus, it suffices to show that $\sup_{t\geq 0} -X_t^{(n)} \Rightarrow \sup_{t\geq 0} -X_t$. This follows from part a) if we can show that

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\left(\sup_{t\geq k} -X_t^{(n)} > 0 \mid X_0^{(n)} = 0\right) = 0$$

(see e.g. Grandell (1977), Grandell (1978) or Billingsley (1999)). This statement will now be shown in the remaining part of this proof. We start by observing the following: From the Ergodic Theorem for Markov chains we know that

$$\lim_{t\to\infty} \frac{1}{t}\xi_i(t) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \delta_{\{i\}}(J_s)ds = \pi_i \quad \text{a.s.}$$

and consequently

$$\lim_{t\to\infty} \frac{1}{t}\Big(t - \sum_{i\in E} \lambda_i\mu_i\xi_i(t)\Big) = \rho \quad \text{a.s.}$$

Thus, we obtain in particular for

$$A_k := \Big\{t - \sum_{i\in E} \lambda_i\mu_i\xi_i(t) \geq \frac{\rho}{2}t, \ \ \forall t \geq k\Big\}$$

that $\lim_{k\to\infty} P(A_k) = 1$. Next let us define

$$Y_t^{(n)} := \sum_{k=1}^{N_t} U_k - \int_0^t \lambda_{J_s}\mu_{J_s}ds + \frac{1}{\sqrt{n}}\Big(\sum_{k=1}^{N_{tn}^0} \tilde{U}_k - \tilde{\mu}nt\Big).$$

Then

$$\begin{aligned}P\left(\sup_{t\geq k} -X_t^{(n)} > 0 \mid X_0^{(n)} = 0\right) &= P\left(\sup_{t\geq k} Y_t^{(n)} - (t - \sum_{i\in E}\lambda_i\mu_i\xi_i(t)) > 0\right)\\ &\leq P\Big(\big\{\sup_{t\geq k} Y_t^{(n)} - (t - \sum_{i\in E}\lambda_i\mu_i\xi_i(t)) > 0\big\} \cap A_k\Big) + 1 - P(A_k)\\ &\leq P\Big(\big\{\sup_{t\geq k} Y_t^{(n)} - \frac{\rho}{2}t > 0\big\} \cap A_k\Big) + 1 - P(A_k)\\ &= E\Big(I_{A_k} P^{\mathcal{F}_\infty^J}\big(\sup_{t\geq k} Y_t^{(n)} - \frac{\rho}{2}t > 0\big)\Big) + 1 - P(A_k)\end{aligned}$$

As done in Grandell (1978) we intend to use the Hájek-Rényi inequality to bound this probability. We use the version given in Frank (1966) Theorem 2. Thus, for any $h \in (0,1)$ we obtain:

$$\begin{aligned}P^{\mathcal{F}_\infty^J}\Big(\sup_{j\geq\lfloor\frac{k}{h}\rfloor+1} Y_{jh}^{(n)} - \frac{\rho}{2}jh > 0\Big) &= P^{\mathcal{F}_\infty^J}\Big(\sup_{j\geq\lfloor\frac{k}{h}\rfloor+1} \frac{Y_{jh}^{(n)}}{\frac{\rho}{2}jh} > 1\Big)\\ &\leq P^{\mathcal{F}_\infty^J}\Big(\sup_{j\geq\lfloor\frac{k}{h}\rfloor+1} \left|\frac{Y_{jh}^{(n)}}{\frac{\rho}{2}jh}\right| > 1\Big)\end{aligned}$$

In order to apply the Hájek-Rényi inequality, note that the sequence $X_j := Y^{(n)}_{jh} - Y^{(n)}_{(j-1)h}$ satisfies

$$E^{\mathcal{F}^J_\infty}(X_j \mid X_{j-1},\ldots,X_1) = 0$$

and that a simple but lengthy calculation gives

$$\begin{aligned}
\sigma_j^2 &:= E^{\mathcal{F}^J_\infty}(X_j^2) = E^{\mathcal{F}^J_\infty}\Big(\sum_{k=N_{(j-1)h}+1}^{N_{jh}} U_k - \int_{(j-1)h}^{jh} \lambda_{J_s}\mu_{J_s} ds\Big)^2 \\
&\quad + E^{\mathcal{F}^J_\infty}\Big(\sum_{k=N^0_{(j-1)hn}+1}^{N^0_{jhn}} \frac{\tilde{U}_k}{\sqrt{n}} - \sqrt{n}\tilde{\mu}h\Big)^2 \\
&= \sum_{i\in E}\lambda_i E\big((U^i)^2\big)\Big(\xi_i(jh) - \xi_i((j-1)h)\Big) + \sum_{i\in E} E\big((\tilde{U}^i)^2\big)\Big(\xi_i(jh) - \xi_i((j-1)h)\Big) \\
&\leq h\Big(\max_{i\in E}\lambda_i E\big((U^i)^2\big) + \max_{i\in E} E\big((\tilde{U}^i)^2\big)\Big) =: hC.
\end{aligned}$$

Applying the Hájek-Rényi inequality we obtain

$$\begin{aligned}
& P^{\mathcal{F}^J_\infty}\Big(\sup_{j\geq\lfloor\frac{k}{h}\rfloor+1}\left|\frac{Y^{(n)}_{jh}}{\frac{\rho}{2}jh}\right| > 1\Big) \leq \Big(\Big\lfloor\frac{k}{h}\Big\rfloor h\frac{\rho}{2}\Big)^{-2}\sum_{j=1}^{\lfloor\frac{k}{h}\rfloor} Ch + \sum_{j=\lfloor\frac{k}{h}\rfloor+1}^{\infty}\big(jh\frac{\rho}{2}\big)^{-2}Ch \\
&= \frac{4C}{\rho^2}\left(\frac{1}{\lfloor\frac{k}{h}\rfloor h} + \frac{1}{h}\sum_{j=\lfloor\frac{k}{h}\rfloor+1}^{\infty}\frac{1}{j^2}\right) \leq \frac{4C}{\rho^2}\frac{2}{\lfloor\frac{k}{h}\rfloor h} \leq \frac{8C}{(k-1)\rho^2}.
\end{aligned}$$

Since this bound is independent of $h \in (0,1)$ and $n \in I\!N$ it follows that

$$P^{\mathcal{F}^J_\infty}\Big(\sup_{t\geq k+1} Y^{(n)}_t - \frac{\rho}{2}t > 0\Big) \leq \frac{8C}{(k-1)\rho^2}.$$

Hence plugging all things together, it follows that

$$\begin{aligned}
& \lim_{k\to\infty}\limsup_{n\to\infty} P\Big(\sup_{t\geq k} -X^{(n)}_t > 0 \mid X^{(n)}_0 = 0\Big) \\
\leq\; & \lim_{k\to\infty}\limsup_{n\to\infty} E\Big(I_{A_k}P^{\mathcal{F}^J_\infty}\big(\sup_{t\geq k} -X^{(n)}_t > 0 \mid X^{(n)}_0 = 0\big)\Big) \\
\leq\; & \lim_{k\to\infty}\frac{8C}{(k-2)\rho^2}E(I_{A_k}) = 0
\end{aligned}$$

which finally implies the statement.

□

From Theorem 9 it should follow that the sequence of adjustment coefficients $(R^{(n)})_{n\in I\!N}$ belonging to the risk processes $(X^{(n)})_{n\in I\!N}$ as defined in equation (6) converges to the adjustment coefficient R given in equation (5). This can be seen directly by showing that $\theta^{(n)}_{jj}(r) \to \theta_{jj}(r)$ for $n \to \infty$. It follows from Björk and Grandell (1988) that for the classical Markov-modulated risk model

$$\theta^{(n)}_{jj} := E_j\left(\exp\left(\sum_{i\in E}\Big[(\lambda_i+n)\hat{h}^{(n)}_i(r) - r(1+\tilde{\mu}\sqrt{n})\Big]\,\xi_i(\tau^{(j)})\right)\right)$$

where

$$\hat{h}_i^{(n)}(r) = \frac{\lambda_i}{\lambda_i+n}E(e^{rU^i}) + \frac{n}{\lambda_i+n}E(e^{r\frac{\tilde{U}^i}{\sqrt{n}}}) - 1$$
$$= \frac{\lambda_i}{\lambda_i+n}h_i(r) + \frac{n}{\lambda_i+n}\left(E(e^{r\frac{\tilde{U}^i}{\sqrt{n}}}) - 1\right).$$

Moreover, using a Taylor series expansion we get

$$(\lambda_i+n)\hat{h}_i^{(n)}(r) - r(1+\tilde{\mu}\sqrt{n}) = \lambda_i h_i(r) + n\left(E(e^{r\frac{\tilde{U}^i}{\sqrt{n}}}) - 1\right) - r(1+\tilde{\mu}\sqrt{n})$$
$$= \lambda_i h_i(r) + \sqrt{n}r\tilde{\mu} + \frac{r^2}{2}E\Big((\tilde{U}^i)^2\Big) + O\Big(\frac{1}{\sqrt{n}}\Big) - r - r\tilde{\mu}\sqrt{n}$$
$$\rightarrow \lambda_i h_i(r) + \frac{r^2}{2}\sigma_i^2 - r$$

for $n \to \infty$ which yields $R^{(n)} \to R$. Hence, instead of computing the adjustment coefficient in the diffusion risk model it is possible to approximately compute it in the classical model sufficiently close to the limit.

5 Comparison Results

We use our findings from the previous sections to obtain some comparison results between diffusion risk models. Before we start we need a further notion form stochastic orderings (for a survey of stochastic orderings we refer the reader to Müller and Stoyan (2002)).

Definition 10. For given random variables X and Y we define the order relation $X \leq_{cx} Y$ if $Ef(X) \leq Ef(Y)$ for all convex functions $f : I\!R \to I\!R$ for which the expectations exist.

Note that $X \leq_{cx} Y$ implies in particular that the expectations of X and Y are the same. For actuarial applications it is important to keep in mind that $X \leq_{cx} Y$ is equivalent to $E(X) = E(Y)$ and $E(\max\{X-t,0\}) \leq E(\max\{Y-t,0\})$ an ordering of the stopp-loss premiums for all $t \in I\!R$.

5.1 Increasing the Volatility of the Diffusion

In this subsection we look at the special case $d = 1$, i.e. we have no Markov-modulation. Suppose we have two diffusion risk processes X and X' given as in equation (2) where the process X' has higher volatility. More precisely we suppose that all data for the processes are the same except however for the diffusion volatility. I.e.

$$X_t = u + t - \sum_{k=1}^{N_t} U_k + \sigma W_t.$$

$$X'_t = u + t - \sum_{k=1}^{N_t} U_k + \sigma' W_t.$$

We suppose that $\sigma \leq \sigma'$. Thus, the expectations of X_t and X'_t coincide but the process X' has a higher variability. By $\psi'(u)$ we denote the ruin probability of the X' process. We can now prove that an increase in the volatility of the diffusion implies a higher risk in terms of an increase in the probability of ruin.

Theorem 11. *Suppose that two diffusion risk reserve processes X and X' are given as defined with $d = 1$. Then for all $u \geq 0$ we get*

$$\psi(u) \leq \psi'(u).$$

Proof. Let $d = 1$. Besides $X^{(n)}$ defined in equation (6) we need

$$X'^{(n)}_t := u + t - \sum_{k=1}^{N_t} U_k + \frac{1}{\sqrt{n}} \left(\tilde{\mu} n t - \sum_{k=1}^{N^0_{tn}} \tilde{U}'_k \right)$$

where $\tilde{U}'_1, \tilde{U}'_2, \ldots$ are independent and identically distributed with $Var(\tilde{U}') = (\tilde{s}')^2, (\sigma')^2 = \tilde{\mu}^2 + (\tilde{s}')^2$ and $\tilde{U}' \geq_{cx} U$ (thus $\tilde{\mu}$ is both the expectation of $\tilde{U}$ and U). As mentioned at the beginning of Section 4 we can think of (6) as a classical Cramér-Lundberg model. In particular in the case $d = 1$ we obtain for $X^{(n)}$ and $X'^{(n)}$:

$$X^{(n)}_t = u + t(1 + \tilde{\mu}\sqrt{n}) - \sum_{k=1}^{\hat{N}_t} \frac{1}{\sqrt{n}} \hat{U}_k$$

$$X'^{(n)}_t = u + t(1 + \tilde{\mu}\sqrt{n}) - \sum_{k=1}^{\hat{N}_t} \frac{1}{\sqrt{n}} \hat{U}'_k$$

where $\hat{N}$ is a Poisson-process with intensity $\lambda + n$, $\hat{U}_1, \hat{U}_2, \ldots$ and $\hat{U}'_1, \hat{U}'_2, \ldots$ are independent and identically distributed with distribution $\frac{\lambda}{\lambda+n} Q(x) + \frac{n}{\lambda+n} \tilde{Q}(\sqrt{n}x)$ and $\frac{\lambda}{\lambda+n} Q(x) + \frac{n}{\lambda+n} \tilde{Q}'(\sqrt{n}x)$ respectively. Alternatively, if $I_1, I_2, \ldots$ is a sequence of independent and identically distributed random variables with

$$P(I = 1) = \frac{\lambda}{\lambda + n} = 1 - P(I = 0)$$

then

$$\hat{U}_k \overset{d}{=} I_k U_k + (1 - I_k) \frac{1}{\sqrt{n}} \tilde{U}_k$$

and analogously for $\hat{U}'_k$. From this representation it follows easily that $\hat{U} \leq_{cx} \hat{U}'$. By $\psi'^{(n)}$ we denote the ruin probability of the process $X'^{(n)}$. Thus, it follows for example from Theorem 6.3.5 a) in Müller and Stoyan (2002) that for all $u \geq 0$:

$$\psi^{(n)}(u) \leq \psi'^{(n)}(u).$$

Since

$$\{X^{(n)}_t,\ t \geq 0\} \Rightarrow \left\{ u + t - \sum_{k=1}^{N_t} U_k + \sigma W_t,\ t \geq 0 \right\}$$

$$\{X'^{(n)}_t,\ t \geq 0\} \Rightarrow \left\{ u + t - \sum_{k=1}^{N_t} U_k + \sigma' W_t,\ t \geq 0 \right\}$$

the statement follows from Theorem 9.b).

□

5.2 Comparison to average diffusion risk models

Now we compare the diffusion risk process to one where the parameters depending on the Markov chain are replaced by their average value. This is a classical question. However to the best of our knowledge this has not been investigated so far for the diffusion risk model. Now let the process X^* be defined by

$$X_t^* = u + t - \sum_{k=1}^{N_t^*} U_k^* + \sigma^* W_t$$

where $N^* = \{N_t^*,\ t \geq 0\}$ is a Poisson process with intensity $\lambda^* = \sum_{i=1}^d \pi_i \lambda_i$, the claim sizes $U_1^*, U_2^*, \dots$ are independent and identically distributed with distribution $Q^* = \sum_{i=1}^d \frac{\pi_i \lambda_i}{\lambda^*} Q_i$ and the diffusion volatility is $(\sigma^*)^2 = \sum_{i=1}^d \pi_i \sigma_i^2$. The ruin probability of the star-process is given by

$$\psi^*(u) = P\left(\inf_{t \geq 0} X_t^* < 0 \mid X_0^* = u\right)$$

and we denote by R^* the corresponding adjustment coefficient whenever it exists. Note that in X^* the claim intensity, the claim size distribution and the volatility of the diffusion are replaced by their average values.

Theorem 12. *Suppose that X and X^* are given and the adjustment coefficients R and R^* exist. Then we get*

$$R^* \geq R.$$

Remark 13. In the special case $\sigma \equiv 0$, i.e. no diffusion, Theorem 12 reduces to Theorem 3 in Asmussen and O'Cinneide (2002). Our proof shows that the result in Asmussen and O'Cinneide (2002) can be derived in a simpler way by a direct comparison of the adjustment coefficients.

Proof. Note that we get R^* from the definition of R in (5) in the special case where the data is the same for all states $j \in E$, i.e.

$$\theta^*(r) := E_j\left(\exp\left(\left[\lambda^* h^*(r) + \frac{r^2(\sigma^*)^2}{2} - r\right]\tau^{(j)}\right)\right).$$

$\theta^*(r) < 1$ if and only if $\lambda^* h^*(r) + \frac{r^2(\sigma^*)^2}{2} - r < 0$. Thus,

$$R^* = \sup\{r > 0 \mid \lambda^* h^*(r) + \frac{r^2(\sigma^*)^2}{2} - r < 0\}.$$

Next note that

$$\begin{aligned} h^*(r) &= \int_0^\infty e^{rx} dQ^*(x) - 1 = \left(\sum_{i \in E} \frac{\pi_i \lambda_i}{\lambda^*} \int_0^\infty e^{rx} dQ_i(x)\right) - 1 \\ &= \sum_{i \in E} \frac{\pi_i \lambda_i}{\lambda^*} h_i(r). \end{aligned}$$

From Jensen's inequality it follows that

$$\begin{aligned}\theta_{jj}(r) &\geq \exp\left(E_j\left(\sum_{i\in E}\left[\lambda_i h_i(r)+\frac{r^2\sigma_i^2}{2}-r\right]\xi_i(\tau^{(j)})\right)\right)\\ &= \exp\left(\sum_{i\in E}\left[\lambda_i h_i(r)+\frac{r^2\sigma_i^2}{2}-r\right]\pi_i E_j(\tau^{(j)})\right)\\ &= \exp\left(\left[\lambda^* h^*(r)+\frac{r^2(\sigma^*)^2}{2}-r\right]E_j(\tau^{(j)})\right)\end{aligned}$$

Thus, since $E_j(\tau^{(j)})>0$ we have that $r\geq R^*$ implies $\theta_{jj}(r)\geq 1$ and thus $R^*\geq R$.

□

Next we try to compare the ruin probabilities $\psi(u)$ and $\psi^*(u)$ itself. In order to simplify things we suppose that $Q_i\equiv Q$, i.e. the claim size distribution does not depend on the environment process J. We need the following further conditions:

1. $\tilde{\pi}=(\tilde{\pi}_i, i\in E)$ with

$$\tilde{\pi}_i=\frac{\sigma_i^2\pi_i}{\sum_{j\in E}\sigma_j^2\pi_j}$$

   is the initial distribution of J.

2. Suppose w.l.o.g. that $\lambda_1\leq\ldots\leq\lambda_d$. Then

$$\frac{\sigma_i^2}{\sigma_j^2}\geq\frac{\lambda_i}{\lambda_j}$$

   for $i,j\in E$ with $i\leq j$.

3. For all $j\leq k$ and $l\leq j$ or $l>k$ we have

$$\sum_{i\geq l}q_{ji}\ <\ \frac{\sigma_j^2}{\sigma_k^2}\cdot\sum_{i\geq l}q_{ki}.$$

Theorem 14. *Under assumptions (1)-(3) we obtain for the ruin probabilities of X and X^* respectively for all $u\geq 0$:*

$$\psi(u)\geq\psi^*(u).$$

Remark 15. In Asmussen et al. (1995) one can find such a comparison result in the classical case without diffusion, i.e. $\sigma_i\equiv 0$. In the case $\sigma_i\equiv\sigma$ conditions (1)-(3) reduce to the conditions given in Asmussen et al. (1995).

Proof. We approximate X by the sequence of processes

$$X_t^{(n)}:=u+t-\sum_{k=1}^{N_t}U_k+\frac{1}{\sqrt{n}}\left(\tilde{\mu}n\int_0^t\tilde{\lambda}_{J_s}ds-\sum_{k=1}^{\tilde{N}_t}\tilde{U}_k\right)$$

where $\tilde{N}$ is a Markov-modulated Poisson process with intensities $n\tilde{\lambda}_1,\ldots,n\tilde{\lambda}_d$ and $\tilde{U}_1,\tilde{U}_2,\ldots$ is a sequence of independent and identically distributed random variables with distribution $\tilde{Q}$. $\tilde{Q}$ has finite expectation $\tilde{\mu}$ and finite variance $\tilde{s}^2=\int(x-\tilde{\mu})^2\tilde{Q}(dx)$. The parameters are chosen such that $\sigma_i^2 = \tilde\lambda_i(\tilde\mu^2+\tilde s^2)$. In order to get a risk reserve process with premium rate 1 we apply the time change $T(t) = \int_0^t (1+\sqrt{n}\tilde\mu\tilde\lambda_{J_s})^{-1} ds$ as explained in Section 2. The probability of ruin of $X^{(n)}$ is the same as the probability of ruin of the process

$$Y_t^{(n)} := u + t - \sum_{k=1}^{\hat N_t} \hat U_k$$

where $\hat N$ is a Markov-modulated Poisson process with intensities $\frac{\lambda_1+n\tilde\lambda_1}{1+\sqrt{n}\tilde\mu\tilde\lambda_1}, \ldots, \frac{\lambda_d+n\tilde\lambda_d}{1+\sqrt{n}\tilde\mu\tilde\lambda_d}$ and if $\hat U_k$ occurs at time t and $J_t = i$, then the distribution of $\hat U_k$ is given by

$$P(\hat U_k \le x) = \frac{\lambda_i}{\lambda_i + n\tilde\lambda_i} Q(x) + \frac{n\tilde\lambda_i}{\lambda_i+n\tilde\lambda_i}\tilde Q(\sqrt{n}x).$$

Note that the time change also changes the environment process J. The new Markov chain $\hat J^{(n)}$ has intensities $\hat q_{ij}^{(n)}$ with

$$\hat q_{ij}^{(n)} = \frac{q_{ij}}{1+\sqrt{n}\tilde\mu\tilde\lambda_i}$$

and new stationary distribution $\tilde\pi^{(n)} = (\tilde\pi_i^{(n)}, i \in E)$ with

$$\tilde\pi_i^{(n)} = \frac{(1+\sqrt{n}\tilde\mu\tilde\lambda_i)\pi_i}{\sum_{j\in E}(1+\sqrt{n}\tilde\mu\tilde\lambda_j)\pi_j}.$$

Now we apply Theorem 1.1 in Asmussen et al. (1995). For n large enough their assumptions are satisfied: First we have to check that $\hat J^{(n)}$ is stochastically monotone, i.e. that for $0 \le s \le t$, $i, j, k \in E$ and $i \le j$

$$P(\hat J_t^{(n)} \le k \mid \hat J_s^{(n)} = j) \le P(\hat J_t^{(n)} \le k \mid \hat J_s^{(n)} = i).$$

However, this is equivalent to (see e.g. Müller and Stoyan (2002) Chapter 5)

$$\sum_{i\ge l} \hat q_{ji}^{(n)} \;\le\; \sum_{i \ge l} \hat q_{ki}^{(n)} \quad\Longleftrightarrow\quad \sum_{i\ge l} q_{ji} \;\le\; \frac{1+\sqrt{n}\tilde\mu\tilde\lambda_j}{1+\sqrt{n}\tilde\mu\tilde\lambda_k}\cdot \sum_{i\ge l} q_{ki}.$$

Due to assumption (3) and

$$\frac{1+\sqrt{n}\tilde\mu\tilde\lambda_j}{1+\sqrt{n}\tilde\mu\tilde\lambda_k} \quad\to\quad \frac{\tilde\lambda_j}{\tilde\lambda_k} = \frac{\sigma_j^2}{\sigma_k^2}$$

for $n \to \infty$ this inequality is true. Now suppose $\lambda_1 \le \ldots \le \lambda_d$. We then have to show that

$$\frac{\lambda_j}{\lambda_j+n\tilde\lambda_j}Q(x) + \frac{n\tilde\lambda_j}{\lambda_j+n\tilde\lambda_j}\tilde Q(\sqrt{n}x) \le \frac{\lambda_i}{\lambda_i+n\tilde\lambda_i}Q(x) + \frac{n\tilde\lambda_i}{\lambda_i+n\tilde\lambda_i}\tilde Q(\sqrt{n}x)$$

for $i \le j$ and all $x \ge 0$. Some simple algebra reveals that this is implied by (2). Finally $\hat J^{(n)}$ has to start with the stationary distribution $\tilde\pi^{(n)}$. An application of Theorem 1.1 in Asmussen et al. (1995) now gives that for all $u \ge 0$

$$\psi^{(n)}(u) \ge \psi^{(n),*}(u)$$

where $\psi^{(n),*}(u)$ is the probability of ruin of the process

$$Y_t^{(n),*} := u + t - \sum_{k=1}^{\hat{N}_t^*} \hat{U}_k^*$$

where $\hat{N}^*$ is a Poisson process with intensity

$$\sum_{i=1}^{d} \tilde{\pi}_i^{(n)} \frac{\lambda_i + n\tilde{\lambda}_i}{1 + \sqrt{n}\tilde{\mu}\tilde{\lambda}_i} = \frac{\lambda^* + n\tilde{\lambda}^*}{1 + \sqrt{n}\tilde{\mu}\tilde{\lambda}^*}$$

where $\lambda^* = \sum_{i\in E} \pi_i \lambda_i$ and $\tilde{\lambda}^* = \sum_{i\in E} \pi_i \tilde{\lambda}_i$. $\hat{U}_1^*, \hat{U}_2^*, \ldots$ are independent and identically distributed with distribution

$$P(\hat{U}_1^* \leq x) = \frac{\lambda^*}{\lambda^* + n\tilde{\lambda}^*} Q(x) + \frac{n\tilde{\lambda}^*}{\lambda^* + n\tilde{\lambda}^*} \tilde{Q}(\sqrt{n}x).$$

Applying once again a time change we see that $Y^{(n),*}$ has the same probability of ruin as $X^{(n),*}$ defined by

$$X_t^{(n),*} := u + t - \sum_{k=1}^{N_t^*} U_k + \frac{1}{\sqrt{n}} \left(\tilde{\mu} n \tilde{\lambda}^* t - \sum_{k=1}^{\tilde{N}_t^*} \tilde{U}_k \right)$$

where $\tilde{N}^*$ is a Poisson process with intensity $n\tilde{\lambda}^*$. For $n \to \infty$ we obtain with Theorem 9 (note that $\tilde{\pi}^{(n)} \to \tilde{\pi}$ for $n \to \infty$ and the result also holds whenever the initial distributions converge) that $\psi^{(n),*}(u) \to \psi^*(u)$ and $\psi^{(n)}(u) \to \psi(u)$ which implies the statement. □

6 Appendix

We will first prove Lemma 4.

Lemma 16. *Let $r > 0$ be fixed. $C(r) < \infty$ implies $\theta_{jj}(r) < 1$ for all $j \in E$.*

Proof. First note that it follows from the properties of a Markov chain that $\theta_{jj}(r) < 1$ implies $\theta_{kj}(r) < \infty$ for all $k \in E$. Now let $r > 0$ be fixed. For any given $\omega \in \Omega$ the function $\sum_{i\in E} \left[\lambda_i h_i(r) + \frac{r^2\sigma_i^2}{2} - r\right] \xi_i(t)$ is piecewise linear in t. Hence for $C(r)$, it suffices to examine this function at the jump times $\left(\tau_n^{(j)}\right)_{n\in\mathbb{N}}$, $j \in E$, of the environment process J. In order to ease notation we define for $n \in I\!N$ and $j \in E$

$$Z_n^{(j)} := \sum_{i\in E} \left[\lambda_i h_i(r) + \frac{r^2\sigma_i^2}{2} - r\right] \left(\xi_i(\tau_n^{(j)}) - \xi_i(\tau_{n-1}^{(j)})\right)$$

and

$$W_n^{(j)} := \exp\left(\sum_{i\in E} \left[\lambda_i h_i(r) + \frac{r^2\sigma_i^2}{2} - r\right] \xi_i(\tau_n^{(j)})\right) = \prod_{m=1}^{n} \exp\left(Z_m^{(j)}\right).$$

We obtain

$$\begin{aligned} C(r) = E\left(\sup_{t\geq 0} \exp\left(\sum_{i\in E} \left[\lambda_i h_i(r) + \frac{r^2\sigma_i^2}{2} - r\right] \xi_i(t)\right)\right) < \infty \\ \Leftrightarrow E\left(\max_{j\in E} \sup_{n\in I\!N} W_n^{(j)}\right) < \infty \\ \Leftrightarrow E\left(\sup_{n\in I\!N} W_n^{(j)}\right) < \infty \quad \forall j \in E \end{aligned}$$

where the last equivalence follows since the set E is finite. Without loss of generality we next assume that $J_0 = k$ and consider a fixed $j \in E$. Since the $Z_n^{(j)}$ are mutually independent for all $n \in I\!N$ and also identically distributed for $n \geq 2$ we get

$$E_k\left(W_n^{(j)}\right) = E_k\left(e^{Z_1^{(j)}}\right) \prod_{m=2}^{n} E_k\left(e^{Z_m^{(j)}}\right) = \theta_{kj}(r)\left(\theta_{jj}(r)\right)^{n-1}. \tag{7}$$

Thus, $\theta_{jj}(r) > 1$ for at least one $j \in E$ would imply $E_k(W_n^{(j)}) \to \infty$ as $n \to \infty$ and thus $C(r) = \infty$ which contradicts our assumption.

Now suppose that $\theta_{jj}(r) = 1$. Recall that the $Z_n^{(j)}$ are independent and identically distributed for $n \geq 2$. $(W_n^{(j)})_{n \in I\!N}$ is therefore a martingale with respect to its natural filtration. Jensen's inequality yields $\exp\big(E(Z_n^{(j)})\big) < E\big(e^{Z_n^{(j)}}\big) = \theta_{jj}(r) = 1$, i.e. $E\big(Z_n^{(j)}\big) < 0$, for $n \geq 2$. From (7) it follows that $\theta_{kj}(r) < \infty$ and therefore $Z_1^{(j)} < \infty$ a.s. We thus have $\lim_{n\to\infty} \sum_{k=1}^{n} Z_k^{(j)} = -\infty$ a.s. and consequently $\lim_{n\to\infty} W_n^{(j)} = 0$ a.s.

We have already shown that $C(r) < \infty$ implies $E(\sup_{n \in I\!N} W_n^{(j)}) < \infty$ which means that $(W_n^{(j)})_{n \in I\!N}$ is uniformly integrable. By standard martingale theory the existence of a random variable $W_\infty^{(j)}$ with $W_\infty^{(j)} = \lim_{n\to\infty} W_n^{(j)}$ a.s. and $E[W_\infty^{(j)} | W_n^{(j)}] = W_n^{(j)}$ for all $n \in I\!N$ follows. Knowing that $\lim_{n\to\infty} W_n^{(j)} = 0$ a.s. we conclude that $W_\infty^{(j)} = 0$ a.s. and accordingly $W_n^{(j)} = 0$ a.s. for all $n \in I\!N$ in contradiction to e.g. $W_2^{(j)} > 0$ a.s. Hence, we must have $\theta_{jj}(r) < 1$ for all $j \in E$. □

Next, Lemma 5 is shown in two parts. We define for $r \geq 0$, $\delta \geq 0$ and $j, k \in E$

$$\theta_{kj}(r,\delta) := E_k\left(\exp\left((1+\delta)\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2\sigma_i^2}{2} - r\right]\xi_i(\tau^{(j)})\right)\right) = E_k\big((W_1^{(j)})^{1+\delta}\big).$$

Note that $\theta_{kj}(r,0) = \theta_{kj}(r)$. Using these functions we can give a sufficient condition for $C(r) < \infty$.

Lemma 17. *Let $r > 0$ be fixed. The existence of a $\delta > 0$ such that $\theta_{jj}(r,\delta) < 1$ for all $j \in E$ is a sufficient condition for $C(r) < \infty$.*

Proof. Let $r > 0$ and suppose that there exists a $\delta > 0$ such that $\theta_{jj}(r,\delta) < 1$ holds for all $j \in E$. For this $\delta > 0$ and a given $j \in E$, it is not difficult to see that $\big((W_n^{(j)})^{1+\delta}\big)_{n \in I\!N}$ is a positive supermartingale with respect to its natural filtration. A supermartingale inequality yields

$$\alpha\, P\Big(\sup_{n \in I\!N} (W_n^{(j)})^{1+\delta} \geq \alpha\Big) \leq E\Big((W_1^{(j)})^{1+\delta}\Big) =: D$$

for $\alpha \geq 0$ as for example shown in Lemma 3.21 in Elliott (1982), p. 23. Due to the properties of a Markov chain one can show that $\theta_{jj}(r,\delta) < 1$ implies $\theta_{kj}(r,\delta) < \infty$ for all $k \in E$. Hence, $E\big((W_1^{(j)})^{1+\delta}\big) = D$ is finite under our assumptions. This implies

$$\begin{aligned} P\left(\sup_{n \in I\!N} W_n^{(j)} \geq t\right) &= P\left(\left(\sup_{n \in I\!N} W_n^{(j)}\right)^{1+\delta} \geq t^{1+\delta}\right) \\ &= P\left(\sup_{n \in I\!N} (W_n^{(j)})^{1+\delta} \geq t^{1+\delta}\right) \leq D\, t^{-(1+\delta)} \end{aligned}$$

for all $t > 0$ and therefore

$$E\left(\sup_{n\in\mathbb{N}} W_n^{(j)}\right) = \int_0^\infty P\left(\sup_{n\in\mathbb{N}} W_n^{(j)} > t\right) dt \le 1 + D\int_1^\infty t^{-(1+\delta)}\,dt < \infty\,.$$

Together with the fact that $C(r) < \infty$ if and only if $E\big(\sup_{n\in\mathbb{N}} W_n^{(j)}\big) < \infty$ for all $j \in E$ the result follows.

□

Lemma 18. *Suppose that R defined by (5) exists. For $0 < r < R$ we have $C(r) < \infty$.*

Proof. Let us assume that R exists and consider any $0 < r < R$. Next, choose some $\delta > 0$ sufficiently small such that $r' := (1+\delta)r < R$. Since θ_{jj} is convex with $\theta_{jj}(0) = 1$ it follows that $\theta_{jj}(r') < 1$, $j \in E$. We then get

$$\begin{aligned}
&\theta_{jj}(r') - \theta_{jj}(r,\delta)\\
&= E_j\left(\exp\left(\sum_{i\in E}\left[\lambda_i h_i(r') + \frac{r'^2\sigma_i^2}{2} - r'\right]\xi_i(\tau^{(j)})\right)\right.\\
&\qquad\qquad \left. - \exp\left((1+\delta)\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2\sigma_i^2}{2} - r\right]\xi_i(\tau^{(j)})\right)\right)\\
&= E_j\left(\exp\left(-(1+\delta)r\tau^{(j)}\right)\cdot\left[\exp\left(\sum_{i\in E}\left[\lambda_i h_i\big((1+\delta)r\big) + (1+\delta)^2\frac{r^2\sigma_i^2}{2}\right]\xi_i(\tau^{(j)})\right)\right.\right.\\
&\qquad\qquad \left.\left. - \exp\left(\sum_{i\in E}\left[(1+\delta)\lambda_i h_i(r) + (1+\delta)\frac{r^2\sigma_i^2}{2}\right]\xi_i(\tau^{(j)})\right)\right]\right)\\
&\ge 0
\end{aligned}$$

since $(1+\delta)^2 \ge (1+\delta)$ and $h_i\big((1+\delta)r\big) \ge (1+\delta)h_i(r)$ for each $i \in E$. The last inequality follows due to the fact that h_i ist convex with $h_i(0) = 0$ for all $i \in E$.
We therefore have $\theta_{jj}(r,\delta) \le \theta_{jj}(r') < 1$ for all $j \in E$. The statement follows now from Lemma 17. □

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