OPTIMAL DETERMINISTIC INVESTMENT STRATEGIES FOR INSURERS

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Abstract. We consider an insurance company whose risk reserve is given by a Brownian motion with drift and which is able to invest the money into a Black-Scholes financial market. As optimization criteria we treat Mean-Variance problems, problems with other risk measures, exponential utility and probability of ruin. Following recent research we assume that investment strategies have to be deterministic. This leads to deterministic control problems which are quite easy to solve. Moreover, it turns out that there are some interesting links between the optimal investment strategies of these problems. Finally we also show that this approach works in the Lévy process framework.

Key words: Deterministic Control Problem; Mean-Variance; Risk Measure; Lévy Process

AMS subject classifications: 91G10, 91B30, 49L20, 60G51

1. Introduction

Inspired by [8] we first consider a Mean-Variance problem for an insurance company where the investment strategy has to be deterministic or in other words: pre-determined at time zero. Mathematically, the strategy has to be \( \mathbb{F}_0 \)-measurable. We assume that the risk reserve is given by a Brownian motion with drift and allow investments into a Black Scholes market with one bond and \( d \) risky assets. Investment strategies are determined by the amount of money which is invested in the assets. Such a model has been considered in [6] with one stock but different optimization criteria and in [18] with the emphasis towards time-consistency. Here we present first the solution of the classical Mean-Variance problem where we optimize over adapted wealth-dependent investment strategies. The solution procedure uses a standard HJB approach and follows along established lines, like in [10] or [19]. More interestingly in the second part we consider the same problem with deterministic investment strategies. The authors in [8] motivate this approach by remarking that such kind of investment strategies are easier to implement, communicate and compare to alternatives. However they consider a Mean-Variance problem with additional consumption and their investment strategies are given in terms of fraction of wealth invested in the single risky asset. We would like to add that our deterministic investment strategies are mathematically easier to obtain and that there are some interesting and surprising links to optimal investment strategies for other optimization criteria as we will explain below.

Mathematically, the Mean-Variance problem for the restricted class of strategies leads to a deterministic control problem directly without the problem of facing non-separability of the target function. In the classical adapted case it is necessary to link the Mean-Variance problem to an auxiliary linear-quadratic problem first (see e.g. [12, 19, 15]) denoted by \( QP(\eta) \) in section 3. This step is not necessary in the deterministic case. Moreover, we will also show that in this special model with deterministic strategies, the Mean-Variance optimal strategy is also optimal for an arbitrary Mean-Risk problem where the variance is replaced by an arbitrary law-invariant and positive homogeneous risk measure for the deviation of the terminal wealth from the mean. This is mainly due to the fact that the terminal wealth under a deterministic investment strategy has a normal distribution with mean and variance depending on the strategy. This observation can also be used to solve the control problem for other optimization criteria, like e.g. expected exponential utility or probability of ruin. Surprisingly it will turn out that the
classical optimal investment strategy for a company with exponential utility (within the class of adapted strategies) is deterministic and coincides with the optimal deterministic strategy for the Mean-Variance problem. Finally we also show that this approach works when the involved processes are Lévy processes. In order to explain our procedure we restrict the presentation to the most important case where the risk reserve process is given as in the Cramér-Lundberg model, i.e. the risk reserve process is a compound Poisson process. Since the jumps vanish under expectation we can proceed in almost the same way. In the classical setting with adapted strategies it is also possible to deal with Lévy processes, see e.g. [11, 9] for LQ- and Mean-Variance problems, [3] for the exponential utility or more general [16]. In [2], a reinsurance problem with a Lévy market has been considered and it turned out that the optimal reinsurance strategy is deterministic in the larger class of adapted strategies already.

In this paper we do not deal with questions of time-consistency of the optimal investment strategy. This seems to be a key point in recent research. We just point to the recent papers [1, 18, 13] where time-consistency is discussed. The deterministic investment strategies depend on time only and are consistent for the deterministic control problem.

The paper is organized as follows: In the next section we introduce the insurance model and the Mean-Variance problem along with some standing assumptions. Then we explain how to reduce the problem in general to a stochastic linear-quadratic problem. Next we solve the problem within the classical framework of adapted, i.e. wealth dependent investment strategies. In section 5 we consider the Mean-Variance problem with deterministic investment strategies. We show how the problem is turned into a deterministic control problem and solve it. In a special case we study the increase in variance for the smaller class of strategies. The next section is dedicated to more general Mean-Risk problems and other optimization criteria. We show that for deterministic investment strategies, the optimal one is insensitive to the choice of risk measure, as long as it is law-invariant and positive homogeneous. Finally in the last section we deal with the Lévy process framework. We assume that the risk reserve process follows a compound Poisson process like in the Cramér-Lundberg model.

2. The Model

We suppose that the risk reserve process \((Y_t)\) of the insurance company is given by the following stochastic differential equation
\[
dY_t = \alpha dt + \beta d\tilde{W}_t
\]
(2.1)
where \(\tilde{W} = (\tilde{W}_t)\) is a Brownian motion and \(\alpha, \beta\) are arbitrary real constants with \(\beta \geq 0\) and it is reasonable but not mathematically necessary to assume that \(\alpha \geq 0\). The initial capital is given by \(Y_0 = x_0 > 0\). The risk reserve can be invested into a financial market which is given by a riskless bond with price process \((S_0(t))\) where
\[
S_0(t) := e^{rt},
\]
and \(r \geq 0\) denotes the deterministic interest rate. Further there are \(d\) risky assets and the price process \((S_i(t))\) of asset \(i\) is given by the stochastic differential equation
\[
dS_i(t) = S_i(t)\left(b_i dt + \sum_{j=1}^{k} \sigma_{ij} dW_j(t)\right),
\]
with \(S_i(0) = 1\). The process \(W = (W_1^j, \ldots, W_k^j)\) is a \(k\)-dimensional Brownian motion which may be correlated with the driving Brownian motion of the risk reserve process. More precisely we assume that \((\tilde{W}, W_j) = \rho_j\) for \(j = 1, \ldots, k\) and \(\rho := (\rho_1, \ldots, \rho_k)\). In what follows we set \(b := (b_1, \ldots, b_d)^\top \in \mathbb{R}^d\) and \(\sigma = (\sigma_{ij}) \in \mathbb{R}^{d \times k}\). We assume that all processes are defined on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\), that \((\mathcal{F}_t)\) is the filtration generated by all Brownian motions and that there is a final time horizon \(T > 0\).

The insurance company is now allowed to invest the risk reserve into the financial market. A classical trading strategy \(\pi = (\pi_t)\) is an \((\mathcal{F}_t)\)-adapted stochastic process where \(\pi_t = \ldots\)
(\pi_1(t), \ldots, \pi_d(t)) \in \mathbb{R}^d and \pi_i(t) is the amount of money invested in stock \(i\) at time \(t\). Note that short-selling is allowed and that the bond investment \((\pi_0(t))\) is given by the self-financing condition. Adaptedness means that we assume that the decision maker is able to observe all Brownian motions and thus the risk reserve and the evolution of the financial market and is able to react to it. Given a trading strategy \(\pi\) and the notation \(\mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{R}^d\), the corresponding wealth process of the insurance company follows the stochastic differential equation
\[
\begin{align*}
\d X^\pi_t &= \left[rX^\pi_t + \alpha + (b - r\mathbf{1})^\top \pi_t\right]dt + \beta \d W_t + \pi_t^\top \sigma \d W_t, \\
X^\pi_0 &= x_0.
\end{align*}
\]
In what follows let us denote \(\Sigma := \sigma \sigma^\top\) which we assume to be positive definite. Since the quadratic variation of \((X^\pi_t)\) is given by
\[
d(X^\pi)_t = \left[\pi_t^\top \Sigma \pi_t + \beta^2 + 2\beta \pi_t^\top \sigma \rho\right]dt
\]
the process \((X^\pi_t)\) is in distribution equal to
\[
dX^\pi_t = \left[rX^\pi_t + \alpha + (b - r\mathbf{1})^\top \pi_t\right]dt + \sqrt{\pi_t^\top \Sigma \pi_t + \beta^2 + 2\beta \pi_t^\top \sigma \rho} \d \tilde{W}_t
\]
for a generic Brownian motion \(\tilde{W}\). The generator of the controlled Markov process \((X^\pi_t)\) for \(v \in C^{1,2}\) given by
\[
\mathcal{A}^\pi v(t, x) = v_t + v_x (rx + \alpha + (b - r\mathbf{1})^\top \pi) + \frac{1}{2} v_{xx} (\pi^\top \Sigma \pi + \beta^2 + 2\beta \pi^\top \sigma \rho).
\]
We call an investment strategy \(\pi\) admissible if all integrals in \((2.4)\) exist and \(\mathbb{E}_{x_0}[\{X^\pi_T\}^2] < \infty\). In what follows we are interested in the dynamic mean-variance problem of the form for \(\mu \in \mathbb{R}\)
\[
(MV) \quad \begin{cases} 
\operatorname{Var}_{x_0}[X^\pi_T] \to \min \\
\mathbb{E}_{x_0}[X^\pi_T] \geq \mu \\
\pi \text{ is an admissible investment strategy.}
\end{cases}
\]
In the next section we explain the standard way to transform this problem into a classical stochastic control problem which will then be solved in the subsequent sections. In order to obtain non-trivial problems we assume that
\[
\mu > x_0 e^{rT} + (e^{rT} - 1) \left(\frac{\alpha}{r} - \frac{\beta}{r} (b - r\mathbf{1})^\top \Sigma^{-1} \sigma \rho\right).
\]
We will discuss this condition later in Remark 5.3.

3. Transformation of MV to an Ordinary Stochastic Control Problem

Problem \((MV)\) can be solved via the well-known Lagrange multiplier technique. The discussion in this section follows [2], chapter 4.6. Let \(L_{x_0}(\pi, \lambda)\) be the Lagrange-function, i.e.
\[
L_{x_0}(\pi, \lambda) := \var{X^\pi_T} + 2\lambda (\mu - \mathbb{E}_{x_0}[X^\pi_T])
\]
for \(\pi\) is an admissible investment strategy and \(\lambda \geq 0\). As usual, \((\pi^*, \lambda^*)\) with \(\lambda^* \geq 0\) is called a saddle-point of the Lagrange-function \(L_{x_0}(\pi, \lambda)\) if
\[
\sup_{\lambda \geq 0} L_{x_0}(\pi^*, \lambda) = L_{x_0}(\pi^*, \lambda^*) = \inf_{\pi} L_{x_0}(\pi, \lambda^*).
\]

Lemma 3.1. Let \((\pi^*, \lambda^*)\) be a saddle-point of \(L_{x_0}(\pi, \lambda)\). Then the value of \((MV)\) is given by
\[
\inf_{\pi} \sup_{\lambda \geq 0} L_{x_0}(\pi, \lambda) = \sup_{\lambda \geq 0} \inf_{\pi} L_{x_0}(\pi, \lambda) = L_{x_0}(\pi^*, \lambda^*)
\]
and \(\pi^*\) is optimal for \((MV)\).
Proof. Obviously the value of (MV) is equal to \( \inf_{\pi} \sup_{\lambda \geq 0} L_{x_0}(\pi, \lambda) \) and
\[
\inf_{\pi} \sup_{\lambda \geq 0} L_{x_0}(\pi, \lambda) \geq \sup_{\lambda \geq 0} \inf_{\pi} L_{x_0}(\pi, \lambda).
\]

For the reverse inequality we obtain
\[
\inf_{\pi} \sup_{\lambda \geq 0} L_{x_0}(\pi, \lambda) \leq \sup_{\lambda \geq 0} L_{x_0}(\pi^*, \lambda) = L_{x_0}(\pi^*, \lambda^*)
\]
\[
= \inf_{\pi} L_{x_0}(\pi, \lambda^*) \leq \sup_{\lambda \geq 0} \inf_{\pi} L_{x_0}(\pi, \lambda),
\]
and the first statement follows. Further from the definition of a saddle-point we obtain for all \( \lambda \geq 0 \)
\[
\lambda^*(\mu - \mathbb{E}_{x_0}[X_T^{\pi^*}]) \geq \lambda(\mu - \mathbb{E}_{x_0}[X_T^{\pi^*}]),
\]
and hence \( \mathbb{E}_{x_0}[X_T^{\pi^*}] \geq \mu \). Then we conclude \( L_{x_0}(\pi^*, \lambda^*) = \text{Var}_{x_0}[X_T^{\pi^*}] \) and \( \pi^* \) is optimal for (MV).

From Lemma 3.1 we see that it is sufficient to look for a saddle point \((\pi^*, \lambda^*)\) of \( L_{x_0}(\pi, \lambda) \). It is not difficult to see that the pair \((\pi^*, \lambda^*)\) is a saddle-point if \( \lambda^* > 0 \) and \( \pi^* = \pi^*(\lambda^*) \) satisfy
\[
\pi^* \text{ is optimal for } P(\lambda^*) \text{ and } \mathbb{E}_{x_0}[X_T^{\pi^*}] = \mu.
\]
Here, \( P(\lambda) \) denotes the so-called Lagrange-problem for the parameter \( \lambda > 0 \)
\[
P(\lambda) \begin{cases} 
L_{x_0}(\pi, \lambda) \to \min \\
\pi \text{ is an admissible investment strategy.}
\end{cases}
\]
Note that the problem \( P(\lambda) \) is not a standard stochastic control problem. We embed the problem \( P(\lambda) \) into a tractable auxiliary problem \( QP(\eta) \) that turns out to be a stochastic LQ-problem. For \( \eta \in \mathbb{R} \) define
\[
QP(\eta) \begin{cases} 
\mathbb{E}_{x_0}[(X_T^\eta - \eta)^2] \to \min \\
\pi \text{ is an admissible investment strategy.}
\end{cases}
\]
The following result shows the relationship between the problems \( P(\lambda) \) and \( QP(\eta) \).

Lemma 3.2. If \( \pi^* \) is optimal for \( P(\lambda) \), then \( \pi^* \) is optimal for \( QP(\eta) \) with \( \eta := \mathbb{E}_{x_0}[X_T^{\pi^*}] + \lambda \).

Proof. Suppose \( \pi^* \) is not optimal for \( QP(\eta) \) with \( \eta := \mathbb{E}_{x_0}[X_T^{\pi^*}] + \lambda \). Then there exists an admissible \( \pi \) such that
\[
\mathbb{E}_{x_0}[(X_T^\eta)^2] - 2\eta \mathbb{E}_{x_0}[X_T^\eta] < \mathbb{E}_{x_0}[(X_T^{\pi^*})^2] - 2\eta \mathbb{E}_{x_0}[X_T^{\pi^*}].
\]
Define the function \( U : \mathbb{R}^2 \to \mathbb{R} \) by
\[
U(x, y) := y - x^2 + 2\lambda(\mu - x).
\]
Then \( U \) is concave and \( U(x, y) = L_{x_0}(\pi, \lambda) \) for \( x := \mathbb{E}_{x_0}[X_T^\eta] \) and \( y := \mathbb{E}_{x_0}[(X_T^\eta)^2] \). Moreover, we set \( x^* := \mathbb{E}_{x_0}[X_T^{\pi^*}] \) and \( y^* := \mathbb{E}_{x_0}[(X_T^{\pi^*})^2] \). The concavity of \( U \) implies
\[
U(x, y) \leq U(x^*, y^*) - 2(\lambda + x^*)(x - x^*) + y - y^*
\]
\[
= U(x^*, y^*) - 2\eta(x - x^*) + y - y^* < U(x^*, y^*),
\]
where the last inequality is due to our assumption \( y - 2\eta x < y^* - 2\eta x^* \). Hence \( \pi^* \) is not optimal for \( P(\lambda) \), leading to a contradiction.

The implication of Lemma 3.2 is that any optimal solution of \( P(\lambda) \) (as long as it exists) can be found by solving problem \( QP(\eta) \).
4. Solution of MV for a Classical Adapted Investor

We will first solve problem $QP(\eta)$ which is a classical stochastic control problem with no running cost and terminal cost $(x - \eta)^2$. Let us denote

$$V(t, x) := \inf_{\pi} E_t, x \left[(X^t_T - \eta)^2\right]$$

where as usual $E_{t, x}$ is the conditional expectation given $X^t_T = x$. In view of the generator of the wealth process, the corresponding Hamilton-Jacobi-Bellman (HJB) equation reads (note that with a slight abuse of notation we name the action again $\pi$)

$$0 = \inf_{\pi \in \mathbb{R}^d} \left\{ v_t + v_x (r x + \alpha + (b-r1)^\top \pi) + \frac{1}{2} v_{xx} (\pi^\top \Sigma \pi + \beta^2 + 2 \beta \pi^\top \sigma \rho) \right\}$$

$$2(x - \eta)^2 = v(T, x),$$

where we denote by $v_t$ and $v_x$ the partial derivatives. Since this is a standard LQ-problem a solution of the HJB equation can easily be found by using the Ansatz $v(t, x) = A(t) + B(t)x + C(t)x^2$. Plugging this form into the HJB equation yields

$$0 = \inf_{\pi \in \mathbb{R}^d} \left\{ \dot{A}(t) + \dot{B}(t)x + \dot{C}(t)x^2 + (B(t) + 2C(t)x)(r x + \alpha + (b-r1)^\top \pi) + C(t)(\pi^\top \Sigma \pi + \beta^2 + 2 \beta \pi^\top \sigma \rho) \right\}$$

where $\dot{A}(t)$ denotes the derivative w.r.t. time. The minimum point of this equation is given by

$$\pi^*(t, x) = -\Sigma^{-1}(b-r1) \left( \frac{B(t)}{2C(t)} + x \right) - \beta \Sigma^{-1} \sigma \rho.$$  

Inserting the minimum point into the HJB equation and collecting the terms without $x$, the terms with $x$ and the terms with $x^2$ yields the following ordinary differential equations for $B(t)$, $C(t)$ and $A(t)$:

$$\dot{C}(t) = -C(t)(2r - (b-r1)^\top \Sigma^{-1}(b-r1))$$

$$\dot{B}(t) = -B(t)(r - (b-r1)^\top \Sigma^{-1}(b-r1)) - 2C(t)(\alpha - \beta(b-r1)^\top \Sigma^{-1} \sigma \rho)$$

$$\dot{A}(t) = -B(t)(\alpha - \beta(b-r1)^\top \Sigma^{-1} \sigma \rho) - C(t)\beta^2(1 - \rho^\top \sigma^\top \Sigma^{-1} \sigma \rho) + (b-r1)^\top \Sigma^{-1}(b-r1) \frac{B(t)^2}{4C(t)}$$

with boundary condition $C(T) = 1, B(T) = -2\eta, A(T) = \eta^2$. The differential equation for $A(t)$ involves only $B(t)$ and $C(t)$ on the right-hand side. Since we are only interested in the optimal investment strategy $\pi^*$, the interesting quantity is $h(t) := \frac{B(t)}{C(t)}$. For $h(t)$ we obtain the differential equation

$$\dot{h}(t) = \frac{\dot{B}(t)C(t) - B(t)\dot{C}(t)}{C^2(t)} = h(t)r - 2\delta r$$

with $\delta r := \alpha - \beta(b-r1)^\top \Sigma^{-1} \sigma \rho$ and boundary condition $h(T) = -2\eta$. A solution is given by

$$h(t) = 2\delta - 2(\delta + \eta) e^{-r(T-t)}.$$

Plugging this expression into (4.3) yields

$$\pi^*(t, x) = -\Sigma^{-1}(b-r1) \left( \delta - (\delta + \eta) e^{-r(T-t)} + x \right) - \beta \Sigma^{-1} \sigma \rho.$$  

Altogether we obtain the following result with a standard verification argument (cf. for example [17], [10]):

**Theorem 4.1.** The value function of problem $QP(\eta)$ is given by $V(t, x) = A(t) + B(t)x + C(t)x^2$ with $A, B, C$ being solutions of (4.6), (4.5) and (4.4) respectively and the optimal investment strategy $(\pi^*_t)$ is determined via (4.8) by $\pi^*_t := \pi^*(t, X^t_T)$ where $(X^t_T)$ is the corresponding optimal wealth process solving (2.5) with $\pi^*$. 

Finally we want to solve problem (MV). Thus we have to compute $\mathbb{E}_{x_0}[X_T^\pi]$, the expected terminal wealth under the optimal strategy $\pi^*$ for $QP(\eta)$. We obtain:

$$
\mathbb{E}_{x_0}[X_T^\pi] = x_0 + \int_0^t r \mathbb{E}_{x_0}[X_s^\pi] + \alpha + (b - r1)\mathbb{E}[\pi_s^\pi] ds
$$

$$
= x_0 + \int_0^t r \mathbb{E}_{x_0}[X_s^\pi] + \delta r - a (\delta - (\delta + \eta)e^{-r(T-t)} + \mathbb{E}_{x_0}[X_s^\pi]) ds.
$$

with $a := (b - r1)\Sigma^{-1}(b - r1)$. Thus, $\mathbb{E}_{x_0}[X_T^\pi]$ follows from solving the ordinary differential equation

$$
\dot{h}(t) = h(t)(r-a) + \delta r - \delta a + a(\delta + \eta)e^{-r(T-t)}
$$

and we get

$$
\mathbb{E}_{x_0}[X_T^\pi] = x_0 e^{-T(a-r)} - \delta e^{-Ta} (1 - e^{rT}) + \eta (1 - e^{-Ta}).
$$

From $\mathbb{E}_{x_0}[X_T^\pi] = \mu$ and $\eta = \lambda^* + \mu$ we conclude

$$
\lambda^* = \frac{e^{-Ta}}{1 - e^{-Ta}} \left( \mu - x_0 e^{Tr} - \delta (e^{rT} - 1) \right)
$$

which is positive due to (2.7). Hence, we obtain the following result:

**Theorem 4.2.** The optimal investment strategy $\pi^*$ for problem (MV) is determined by (4.8) with $\eta = \mu + \lambda^*$ and $\lambda^*$ given by (4.9).

5. MV Problem for an Investor with Deterministic Investment Strategies

In this section we assume now that the investment strategy has to be pre-determined, i.e. that the process $\pi$ is $\mathbb{F}_0$-measurable which means it is deterministic and only a function of time. Thus, the fund manager of the insurance company has to explain at $t=0$ the investment strategy for the time horizon $[0,T]$ without using further knowledge about the evolution of the processes. This seems at least sometimes to be more realistic than the adaptive strategy (4.8). A similar situation has been considered in [8] where the authors motivate such a strategy by pension funds often being managed by time-dependent investment strategies only. Hence we consider

$$
\begin{align*}
(MVD) \quad & \text{Var}_{x_0}[X_T^\pi] \to \min \\
& \mathbb{E}_{x_0}[X_T^\pi] \geq \mu \\
& \pi \text{ is a deterministic investment strategy.}
\end{align*}
$$

This is now the same problem over a smaller class of investment strategies. We consider first problem $PD(\lambda)$:

$$
\begin{align*}
PD(\lambda) \quad & \text{Var}_{x_0}[X_T^\pi] + 2\lambda (\mu - \mathbb{E}_{x_0}[X_T^\pi]) \to \min \\
& \pi \text{ is a deterministic investment strategy.}
\end{align*}
$$

Here it is not necessary to consider the artificial problem $QP(\eta)$. $PD(\lambda)$ can be transformed into a deterministic control problem as follows. To this end note that the stochastic differential equation (2.5) for the wealth can easily be solved. When we denote by $\tilde{X}_t^\pi = \frac{X_t^\pi}{S_0(t)}$ the discounted wealth process, then we obtain

$$
\tilde{X}_t^\pi = x_0 + \int_0^t e^{-rs}(\alpha + (b - r1)\pi_s) ds + \int_0^t e^{-rs} \sqrt{\pi_s^\pi \Sigma \pi_s + \beta^2 + 2\beta \pi_s^\pi \sigma \rho d\tilde{W}_s}. \quad (5.1)
$$
For a deterministic process \( \pi \), the second integral is obviously a true martingale and we obtain

\[
E_{x_0}[\tilde{X}_t] = x_0 + \int_0^t e^{-rs} (\alpha + (b - r_1^\top) \pi_s) ds =: x(t)
\]

\[
\text{Var}_{x_0}[\tilde{X}_t] = \int_0^t e^{-2rs} (\pi_s^\top \Sigma \pi_s + \beta^2 + 2\beta \pi_s^\top \sigma \rho) ds =: y(t).
\]

Note that \( x(t) \) and \( y(t) \) both depend on \( \pi \). Thus, the target function of PD(\( \lambda \)) can be written as

\[
\text{Var}_{x_0}[X^\pi_T] + 2\lambda (\mu - E_{x_0}[X^\pi_T]) = e^{2rT} \text{Var}_{x_0}[\tilde{X}_T] + 2\lambda (\mu - e^{rT} E_{x_0}[\tilde{X}_T])
\]

The deterministic control problem is then:

\[
PD(\lambda) \begin{cases} 
  e^{2rT} y(T) + 2\lambda (\mu - e^{rT} x(T)) \to \min \\
  \dot{x}(t) = e^{-rt} (\alpha + (b - r_1^\top) \pi_t) \\
  \dot{y}(t) = e^{-2rt} (\pi_t^\top \Sigma \pi_t + \beta^2 + 2\beta \pi_t^\top \sigma \rho) \\
  \pi_t \in \mathbb{R}^d.
\end{cases}
\] (5.2)

The value function of this problem is

\[
V(t, x, y) := \inf_{\pi} \{ e^{2rT} y(T) + 2\lambda (\mu - e^{rT} x(T)) \}.
\]

Obviously the related HJB equation is

\[
0 = \inf_{\pi \in \mathbb{R}^d} \left\{ v_t + v_x e^{-rt} (\alpha + (b - r_1^\top) \pi) + v_y e^{-2rt} (\pi^\top \Sigma \pi + \beta^2 + 2\beta \pi^\top \sigma \rho) \right\} \quad (5.3)
\]

\[
v(T, x, y) = e^{2rT} y + 2\lambda (\mu - e^{rT} x).
\]

In order to find a solution, we consider now the Ansatz

\[
v(t, x, y) = e^{2rT} (y + g(t)) + 2\lambda (\mu - e^{rT} (x + f(t)))
\]

with \( f(T) = g(T) = 0 \). Thus, we obtain:

\[
\begin{align*}
v_t & = e^{2rT} \hat{g}(t) - 2\lambda e^{rT} \hat{f}(t), \\
v_x & = -2\lambda e^{rT}, \\
v_y & = e^{2rT}.
\end{align*}
\]

The minimizer of (5.2) is determined by

\[
\pi_t^* = -\Sigma^{-1} (b - r_1) \frac{v_x}{v_y} e^{rt} - \beta \Sigma^{-1} \sigma \rho
\]

\[
= \Sigma^{-1} (b - r_1) \lambda e^{-r(T-t)} - \beta \Sigma^{-1} \sigma \rho. \quad (5.4)
\]

Plugging this into the HJB equation (5.2) yields:

\[
0 = e^{2rT} \hat{g}(t) - 2\lambda e^{rT} \hat{f}(t) - 2\lambda e^{rT} \hat{x}^*(t) + e^{2rT} \hat{y}^*(t).
\]

Note that this equation is satisfied when \( \hat{f}(t) = -\hat{x}^*(t) \) and \( \hat{g}(t) = -\hat{y}^*(t) \), thus

\[
f(t) = \int_t^T e^{-rs} (\alpha + (b - r_1^\top) \pi_s^*) ds \quad (5.5)
\]

\[
g(t) = \int_t^T e^{-2rs} ((\pi_s^*)^\top \Sigma \pi_s^* + \beta^2 + 2\beta (\pi_s^*)^\top \sigma \rho) ds. \quad (5.6)
\]

Note that under the control \( \pi^* \) and corresponding state trajectories \( x^*, y^* \) it holds that \( x^*(t) + f(t) = E_{x_0}[\tilde{X}_T] \) and \( y^*(t) + g(t) = \text{Var}_{x_0}[\tilde{X}_T] \) for all \( t \in [0, T] \). We summarize our results in the following theorem. A verification is straightforward.
Theorem 5.1. The value function of problem PD(\(\lambda\)) is given by
\[
V(t,x,y) = e^{2rT}(y + g(t)) + 2\lambda(\mu - e^{rT}(x + f(t)))
\]
with \(f, g\) being solutions of (5.5) and (5.6) respectively. The optimal investment strategy (\(\pi^*_1\)) is given by (5.4).

Finally we solve problem (MVD). First note that
\[
E_{x_0}[X^*_T] = e^{rT}x_0 + \delta(e^{rT} - 1) + a\lambda T.
\]
From \(E_{x_0}[X^*_T] = \mu\) we obtain
\[
\lambda^* = (aT)^{-1}\left(\mu - e^{rT}x_0 - \delta(e^{rT} - 1)\right),
\]
which is positive due to condition (2.7). Thus we obtain the following result:

Theorem 5.2. The optimal investment strategy \(\pi^*\) for problem (MVD) is determined by (5.4) with \(\lambda^*\) given by (5.7).

Remark 5.3. In this setting it is also easy to determine the strategy with the minimum achievable variance. In case the financial market is not perfectly correlated with the risk reserve this minimal variance is not zero. For an arbitrary deterministic investment strategy we obtain:
\[
\text{Var}_{x_0}[X^*_T] = e^{2rT} \int_0^T e^{-2rs}(\pi^*_s\Sigma \pi^*_s + \beta^2 + 2\beta \pi^*_s \sigma \rho)ds.
\]
Minimizing this expression in \(\pi_s\) yields the minimum variance investment strategy \(\hat{\pi}_t \equiv -\beta\Sigma^{-1}\sigma \rho\) with corresponding minimal variance
\[
\text{Var}_{x_0}[\hat{X}_T] = \frac{1}{2r}\beta^2(1 - \rho^\top \sigma^\top \Sigma^{-1} \sigma \rho)(e^{2rT} - 1)
\]
and expectation
\[
E_{x_0}[\hat{X}_T] = x_0 e^{rT} + \delta(e^{rT} - 1).
\]
Thus, in case \(\mu \leq x_0 e^{rT} + \delta(e^{rT} - 1)\) problem (MVD) is trivial because then \(\hat{\pi}\) satisfies the constraint \(E_{x_0}[\hat{X}_T] \geq \mu\) and yields the minimal possible variance. As a result condition (2.7) is reasonable.

Remark 5.4. Of course for a given expected return of \(\mu\), when we minimize the variance over the smaller set of deterministic investment strategies, the variance will be not smaller than in the classical stochastic case. The corresponding formulas for the variance can be computed from our findings above. However, in order to keep the comparison simple, we restrict the comparison here to the case \(\alpha = \beta = 0\), i.e. no additional insurance business. Here we obtain the corresponding formula for the stochastic case from [19]:
\[
\text{Var}_{x_0}[X^*_T] = \frac{1}{e^{rT} - 1} (\mu - e^{rT}x_0)^2
\]
and for the deterministic case we have
\[
\text{Var}_{x_0}[X^*_T] = \frac{1}{e^{rT}} (\mu - e^{rT}x_0)^2.
\]
From these expressions we can conclude that the difference in variance depends severely on the time horizon. This is of course expected.
6. More General Mean-Risk Problems and Other Optimization Criteria

In this section we will briefly discuss some other optimality criteria for the investment problem with deterministic investment strategies. Of course when the solution of the classical stochastic control problem with adapted investment strategies yields an optimal strategy which is itself deterministic, then this strategy is also optimal in the smaller class of deterministic strategies. A situation like this can occur when we consider the probability of ruin or the expected exponential utility as a target function. We discuss these cases below. But we start this section with the observation that in the Mean-Variance framework our optimal deterministic investment strategy is not only optimal w.r.t. to minimizing the variance.

6.1. More General Mean-Risk Problems. The variance or standard deviation is of course just one way to measure risk. Suppose now that \( \varrho \) is an arbitrary, law invariant and positive homogeneous risk measure, i.e. \( \varrho(\lambda X) = \lambda \varrho(X) \) for all \( \lambda > 0 \). We claim now that the problem

\[
(MRD) \begin{cases}
\varrho(X_{T}^\pi - \mathbb{E}_{x_{0}}[X_{T}^\pi]) \to \min \\
\mathbb{E}_{x_{0}}[X_{T}^\pi] \geq \mu \\
\pi \text{ is a deterministic investment strategy.}
\end{cases}
\]

has the same solution as (MVD) which is obtained when we use the standard deviation \( \varrho(X) = \sqrt{\text{Var}[X]} \).

Theorem 6.1. The optimal investment strategy for (MRD) coincides with the optimal investment strategy for (MVD).

Proof. First note that in both cases because of \( \text{Var}_{x_{0}}[X_{T}^\pi] = e^{2rT} \text{Var}_{x_{0}}[\tilde{X}_{T}^\pi] \) and the fact that \( \varrho(X_{T}^\pi - \mathbb{E}_{x_{0}}[X_{T}^\pi]) = e^{rT} \varrho(\tilde{X}_{T}^\pi - \mathbb{E}_{x_{0}}[\tilde{X}_{T}^\pi]) \) we can minimize the target function with \( X_{T}^\pi \) replaced by \( \tilde{X}_{T}^\pi \) and the side constraint \( \mathbb{E}_{x_{0}}[X_{T}^\pi] \geq \mu e^{-rT} \). Now due to (5.1) we see that for deterministic investment strategies, \( \tilde{X}_{T}^\pi \) has a normal distribution \( N(m_{\pi}, s_{\pi}^{2}) \) with parameters

\[
m_{\pi} = x_{0} + \int_{0}^{T} e^{-rs} (\alpha + (b - r1)^{\top} \pi_{s}) \, ds \\
s_{\pi}^{2} = \int_{0}^{T} e^{-2rs} (\pi_{s}^{\top} \Sigma \pi_{s} + \beta^{2} + 2\beta \rho \pi_{s}^{\top} \sigma) \, ds.
\]

Hence in distribution \( X_{T}^\pi \overset{d}{=} m_{\pi} + s_{\pi} Z \) where \( Z \) is a standard normal random variable. The optimization problem (MRD) can thus be written as

\[
(MRD) \begin{cases}
\varrho(s_{\pi} Z) \to \min \\
m_{\pi} \geq \mu e^{-rT} \\
\pi \text{ is a deterministic investment strategy.}
\end{cases}
\]

Since \( \varrho \) is positive homogeneous we obtain \( \varrho(s_{\pi} Z) = s_{\pi} \varrho(Z) \) which means that we have to minimize the standard deviation of \( X_{T}^\pi \). Hence the statement follows. \( \square \)

As a consequence, the optimal investment strategy we obtained is very robust w.r.t. the choice of risk measure. Indeed it does not depend on the precise risk measure as long as we agree to take a law invariant and positive homogeneous one.

6.2. Maximizing Exponential Utility of Terminal Wealth. In this subsection we consider the problem of maximizing \( \mathbb{E}_{x_{0}}[-\frac{1}{\gamma} e^{-\gamma X_{T}^\pi}] \) with \( \gamma > 0 \). For the classical stochastic case and
only one stock this has been done in [6]. We consider now directly the multi-asset model in the framework of deterministic strategies, i.e. we consider

$$\begin{align*}
\mathbb{E}_{x_0}\left[-\frac{1}{\gamma}e^{-\gamma X_T^\pi}\right] & \to \max \\
\pi & \text{ is a deterministic investment strategy.}
\end{align*}$$

It turns out that the solution of this problem is very easy. We know already for deterministic $\pi$ that $X_T^\pi$ has a normal distribution $N(m_\pi, s_\pi^2)$ with parameters

$$\begin{align*}
m_\pi &= e^{rT}(x_0 + \int_0^T e^{-rs} \left( \alpha + (b - r1)^\top \pi_t \right) ds) \\
s_\pi^2 &= e^{2rT} \left( \int_0^T e^{-2rs} \left( \pi_t^\top \Sigma \pi_t + \beta^2 + 2\beta \pi_t^\top \sigma \right) ds. \right)
\end{align*}$$

Hence we can write $X_T^\pi = m_\pi + s_\pi Z$ and the target function thus reduces to

$$\begin{align*}
\mathbb{E}_{x_0}\left[-\frac{1}{\gamma}e^{-\gamma s_\pi Z}\right] = -\frac{1}{\gamma}e^{-\gamma m_\pi + \frac{1}{2}\gamma^2 s_\pi^2}.
\end{align*}$$

Obviously, the problem is equivalent to minimize $-\frac{2}{\gamma}m_\pi + s_\pi^2$ and we end up with the following deterministic control problem:

$$\begin{align*}
e^{2rT} y(T) - \frac{2}{\gamma} e^{rT} x(T) & \to \min \\
\dot{x}(t) &= e^{-rt} \left( \alpha + (b - r1)^\top \pi_t \right) \\
\dot{y}(t) &= e^{-2rt} \left( \pi_t^\top \Sigma \pi_t + \beta^2 + 2\beta \pi_t^\top \sigma \right) \\
\pi_t & \in \mathbb{R}^d
\end{align*}$$

But this is equivalent to problem $PD(\lambda)$ with $\lambda = \frac{1}{\gamma}$ and we know from (5.4) that the optimal investment strategy is given by

$$\pi_t^* = \Sigma^{-1}(b - r1) \frac{1}{\gamma} e^{-r(T-t)} - \beta \Sigma^{-1} \sigma.$$

Thus due to (5.7) there is a one-to-one relation between optimal mean-variance strategies and optimal strategies for the problem with exponential utility function. For an early discussion about the relation of expected utility and mean-variance, see e.g. [14, 7]. Note that the optimal investment strategy we have computed here is also optimal within the larger class of adapted strategies.

6.3. Minimizing the Probability of Ruin. Another popular ‘risk measure’ in the actuarial sciences is the probability of ruin of a controlled risk reserve. When we consider the classical situation of $(\mathbb{F}_t)$-adapted investment strategies, it is very easy to find the one which minimizes the probability of ruin of process (2.5). According to [3], the optimal feedback function $\pi^*(x)$ is obtained by maximizing the ratio of mean over variance of the process, i.e.

$$\begin{align*}
\frac{rx + \alpha + (b - r1)^\top \pi}{\pi^\top \Sigma \pi + \beta^2 + 2\beta \pi^\top \sigma \rho}.
\end{align*}$$

Obviously, when $r = 0$, then the maximizer $\pi^*$ is independent from $x$ and deterministic. If further $d = 1$ and $\alpha = 0$, then $\pi^* = \frac{\beta}{\sigma}$.

7. Problems with Lévy Processes

The standard model for the risk reserve process of an insurance company is the so-called Cramér-Lundberg model. It assumes that the risk reserve process follows a Lévy process given
as the difference of the premium income process and the claims which have been paid out so far. More precisely it is usually assumed that
\[ Y_t = x_0 + ct - \sum_{k=1}^{N_t} U_k \] (7.1)
where \( c > 0 \) is the premium income rate, \( N = (N_t) \) is a Poisson process with parameter \( \nu > 0 \) which counts the number of claims and \( U_1, U_2, \ldots \) are independent and identically distributed random variables, representing the claim sizes. We denote \( m := \mathbb{E} U \) and \( m_2 := \mathbb{E} U^2 \). The process in (2.1) can be seen as a diffusion approximation of the process (7.1) when claims are small and frequent. The mean-variance problems we have considered in sections 4 and 5 can be dealt with in a Lévy framework along the same lines. The solution of the classical problem (MV) may be derived from [11]. Here we concentrate on the problem (MVD) with deterministic investment strategies. We assume that
\[ \mu > x_0 e^{rT} + (e^{rT} - 1) \frac{c - \nu m}{r}. \] (7.2)

In order to have an elegant notation we write \((Y_t)\) with the help of its Poisson random measure 
\[ M([0,t] \times B), t \geq 0, B \in \mathcal{B}(\mathbb{R}+) \]
which is the sum of all claims taking values in the set \( B \) up to time \( t \). Hence we can write
\[ Y_t = x_0 + ct - \int_{[0,t]} \int_{\mathbb{R}+} y M(ds, dy). \] (7.3)

For simplicity we leave the financial market as in the sections before, though one might also allow for jumps there. We again assume here that admissible trading strategies \( \pi = (\pi_t) \) are \( \mathbb{F}_0 \)-measurable. The corresponding wealth of the insurance company follows the stochastic differential equation
\[ dX^\pi_t = \left[ rX^\pi_t + (b - r1)^\top \pi_t \right] dt + dY_t + \pi_t^\top \sigma dW_t \] (7.4)
\[ X^\pi_0 = x_0. \]

Now we consider the deterministic Mean-Variance problem
\[
(MVD) \quad \begin{cases} 
\text{Var}_{x_0}[X^\pi_T] \to \min \\
\mathbb{E}_{x_0}[X^\pi_T] \geq \mu \\
\pi \text{ is a deterministic investment strategy.}
\end{cases}
\]

As in section 5 we start with problem \( PD(\lambda) \):
\[
PD(\lambda) \quad \begin{cases} 
\text{Var}_{x_0}[X^\pi_T] + 2\lambda(\mu - \mathbb{E}_{x_0}[X^\pi_T]) \to \min \\
\pi \text{ is a deterministic investment strategy.}
\end{cases}
\]
Next we compute \( \mathbb{E}_{x_0}[X^\pi_T] \). To this end note that \( Y_t - (c - \nu m)t \) is a martingale. Thus, we obtain
\[ \mathbb{E}_{x_0}[X^\pi_t] = x_0 + \int_0^t r \mathbb{E}_{x_0}[X^\pi_s] + (b - r1)^\top \pi_s ds + (c - \nu m)t. \]

This is an ordinary differential equation for \( f(t) := \mathbb{E}_{x_0}[X^\pi_t] \) of the form
\[ \dot{f}(t) = rf(t) + (b - r1)^\top \pi_t + c - \nu m \]
with boundary condition \( f(0) = x_0 \). The solution is given by
\[ f(t)e^{-rt} = x_0 + \int_0^t e^{-rs}((b - r1)^\top \pi_s + c - \nu m) ds =: x(t). \]
In order to compute the variance we need the second moment of $X_t^\pi$. Using partial integration we get

\[(X_t^\pi)^2 = x_0^2 + 2 \int_0^t X_s^\pi dX_s^\pi + [X^\pi, X^\pi](t)\]

\[= x_0^2 + 2 \int_0^t r(X_s^\pi)^2 + X_s^\pi (b - r1)^\top \pi_s ds + 2 \int_0^t X_s^\pi \pi_s^\top \sigma dW_s\]

\[+ 2 \int_0^t cX_s^\pi ds - \int_0^t \int_{\mathbb{R}^+} X_s^\pi yM(ds, dy) + \int_0^t \pi_s^\top \Sigma \pi_s ds + \int_0^t \int_{\mathbb{R}^+} y^2 M(ds, dy).\]

Taking the expectation yields

\[\mathbb{E}_{x_0}[(X_t^\pi)^2] = x_0^2 + 2 \int_0^t \left[r \mathbb{E}_{x_0}((X_s^\pi)^2) + \mathbb{E}_{x_0}[X_s^\pi ((b - r1)^\top \pi_s + c)]\right] ds\]

\[+ \int_0^t \pi_s^\top \Sigma \pi_s ds + m_2 \nu t - 2 \int_0^t \mathbb{E}_{x_0}[X_s^\pi] \nu m ds\]

which is an ordinary differential equation for $g(t) := \mathbb{E}_{x_0}[(X_t^\pi)^2]$ of the form

\[\dot{g}(t) = 2rg(t) + 2f(t)((b - r1)^\top \pi_t + c - \nu m) + \pi_t^\top \Sigma \pi_t + m_2 \nu.\]

with boundary condition given by $g(0) = x_0^2$. When we define the variance as a function of time $h(t) := \text{Var}_{x_0}[X_t^\pi] = \mathbb{E}_{x_0}[(X_t^\pi)^2] - (\mathbb{E}_{x_0}[X_t^\pi])^2 = g(t) - f^2(t)$ it follows

\[\dot{h}(t) = \dot{g}(t) - 2f(t)f_t = 2rh(t) + \pi_t^\top \Sigma \pi_t + m_2 \nu\]

with boundary condition $h(0) = 0$. Thus, we get

\[h(t)e^{-2rt} = \int_0^t e^{-2rs}(\pi_s^\top \Sigma \pi_s + m_2 \nu) ds =: y(t).\]

The target function of $PD(\lambda)$ can be written as

\[\text{Var}_{x_0}[X_T^\pi] + 2\lambda(\mu - \mathbb{E}_{x_0}[X_T^\pi]) = e^{2rT}y(T) + 2\lambda(\mu - e^{rT}x(T))\]

and we obtain the deterministic control problem:

\[\begin{array}{ll}
PD(\lambda) & \{ e^{2rT}y(T) + 2\lambda(\mu - e^{rT}x(T)) \rightarrow \min \\
\dot{x}(t) & = e^{-rt}(c - \nu \pi_t + (b - r1)^\top \pi_t) \\
\dot{\pi_t} & = e^{-2rt}(\pi_t^\top \Sigma \pi_t + m_2 \nu) \\
\pi_t & \in \mathbb{R}^d
\end{array}\]

which is the same as in section 5 where we have to set $\rho := (0, \ldots, 0)^\top$ and replace $\alpha$ by $c - \nu m$ and $\beta^2$ by $m_2 \nu$. The HJB equation is then

\[0 = \inf_{\pi \in \mathbb{R}^d} \left\{ v_t + v_x e^{-rt}(c - \nu \pi + (b - r1)^\top \pi) + v_y e^{-2rt}(\pi^\top \Sigma \pi + m_2 \nu) \right\} \quad (7.5)\]

\[v(T, x, y) = e^{2rT}y + 2\lambda(\mu - e^{rT}x). \quad (7.6)\]

We use again the Ansatz

\[v(t, x, y) = e^{2rT}(y + g(t)) + 2\lambda(\mu - e^{rT}(x + f(t))). \quad (7.7)\]

The minimizer of (7.5) is determined by

\[\pi_t^\ast = \Sigma^{-1}(b - r1) \lambda e^{-r(T-t)} \quad (7.8)\]
and the value function is given by (7.7) with

\[ f(t) = \int_t^T e^{-rs}(c - \nu m + (b - r1)^\top \pi_s^*)\,ds \]  
\[ g(t) = \int_t^T e^{-2rs}((\pi_s^*)^\top \Sigma \pi_s^* + m_2\nu)\,ds. \]  

We summarize our results in the following theorem. A verification is straightforward.

**Theorem 7.1.** The value function of problem PD(\(\lambda\)) is given by

\[ V(t, x, y) = e^{2rT}(y + g(t)) + 2\lambda(\mu - e^{rT}(x + f(t))) \]

with \(f, g\) being solutions of (7.9) and (7.10) respectively. The optimal investment strategy \((\pi_t^*)\) is given by (7.8).

Finally we solve the problem (MVD). Note that

\[ \mathbb{E}_{x_0}[X_T^\pi] = e^{rT}x_0 + aT + c - \nu m \left( e^{rT} - 1 \right) \]

From \(\mathbb{E}_{x_0}[X_T^\pi] = \mu\) we obtain

\[ \lambda^* = (aT)^{-1}\left( \mu - e^{rT}x_0 + \frac{c - \nu m}{r}(1 - e^{rT}) \right), \]  

which is positive due to condition (7.2). Thus we obtain the following result:

**Theorem 7.2.** The optimal investment strategy \(\pi^*_T\) for problem (MVD) is determined by (7.8) with \(\lambda^*\) given by (7.11).

As a result we see that the optimal control depends only on the drift of the risk reserve (here \(c - \nu m\)) and it is not important whether the process has jumps or not.

8. Conclusion

We have shown that stochastic control problems with deterministic investment strategies lead to deterministic control problems which are in general easier to solve. In particular in the case of a Brownian setting, the terminal wealth has a normal distribution under any admissible deterministic investment strategy. This leads to some very favorable properties like insensitivity of the optimal control w.r.t. to a class of target functions. Moreover there are some interesting links between these problems. Optimal deterministic investment strategies for Mean-Variance problems for example correspond to optimal investment strategies for an insurance company with exponential utility. Finally we also show that the current approach works in the setting of Lévy processes.

**References**


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