Optimal retirement planning under partial information

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Abstract
The present paper analyzes an optimal consumption and investment problem of a retiree with a constant relative risk aversion (CRRA) who faces parameter uncertainty about the financial market. We solve the optimization problem under partial information by making the market observationally complete and consequently applying the martingale method to obtain closed-form solutions to the optimal consumption and investment strategies. Further, we provide some comparative statics and numerical analyses to deeply understand the consumption and investment behavior under partial information. Bearing partial information has little impact on the optimal consumption level, but it makes retirees with a RRA smaller than one invest more riskily, while it makes retirees with a RRA larger than one invest more conservatively.

JEL: C6, G1, D9.

1 Introduction
Decreasing benefits from state pension and increasing longevity (c.f. Bodie et. al. (2004), Cocco and Gomes (2012)) has made individual retirement planning more important than ever. Decisions related to retirement planning are very complex and need
to take account of many aspects: a retiree shall decide whether and how much to invest in private retirement plans like annuities, how to withdraw a stream of retirement income from her accumulated assets for consumption. Too little withdrawal might not ensure her desired living standard. From too much withdrawal, the accumulated funds can be exhausted too soon. In addition, the retiree needs to consider how to optimally invest the residual funds after withdrawals, which better serves to maintain the living standard. The retiree faces in this decision process a stochastic economy and an age-dependent mortality. The present paper takes account of both aspects, and particularly extends the literature of optimal retirement problems addressing stochastic economy to incorporate parameter uncertainty of the financial market.\(^1\) In the finance literature, parameter uncertainty has been frequently addressed: investors do not observe all the parameters driving the financial market and learn about these parameters by using observable data (see Pastor and Veronesi (2009) providing a nice review of works on learning in finance). Apparently, a realistic setting with the parameter uncertainty shall be considered in the retiree’s consumption and investment decision, too. We find that such partial information about the financial market has little impact on the optimal consumption level, but does influence the optimal investment strategies of the retiree. In particular, those retirees with a relative risk aversion level smaller than one hold a higher fraction in equity than in the full information case; and those with a relative risk aversion larger than one, hold a less volatile investment strategy than in the full information case.

Since the paper by Merton (1971) the problem of finding optimal consumption and investment strategy has become one of the most studied optimization problem in economy, finance and insurance. In this paper, we study an optimal consumption and investment problem for a retiree. The retiree buys a lifetime annuity providing a certain future income and invests the remaining available assets on the financial market. We incorporate parameter uncertainty in the optimization problem of the retiree. We assume that the retiree can observe the asset prices on the market, but cannot observe the drift term of the assets (or the market price of risk). This assumption is realistic since the volatility of the stock prices can be estimated very well, whereas the drift is notoriously difficult to estimate (see e.g. Gennette (1986)). With such partial information about the financial market, the retiree aims at increasing the future consumption over a pre-specified minimum consumption level by trading on the financial market. The introduction of a minimum consumption level in the optimization problem takes account of the fact that the retiree cares about sustaining a minimum living standard during retirement. In such a setting incorporating the realistic parameter uncertainty in the consumption and investment problem, the present paper solves the retiree’s optimization problem and compares it to the case with full information about the financial market.

\(^1\)For instance, Farhi and Panageas (2007), Pliska and Ye (2007), Dybvig and Liu (2010, 2011) focus on the aspect of stochastic economy in the literature on optimal retirement decision, just to mention a few.
In the literature on optimal asset allocation with partial information, the optimization problem is frequently solved by the so called primal formulation which requires a Markovian assumption on the state process and leads to an HJB equation for the indirect utility function (see e.g. Gennotte (1986), Rieder and Bäuerle (2005), Brendle (2006)). In this paper, we use a static martingale approach to determine the optimal consumption and investment strategies over a random time horizon in the case of partial information. With full information, this dual method has been studied in e.g. Pliska (1986), Cox and Huang (1989), Huang and Pages (1992) and Karatzas et. al. (1987) and in an incomplete market setting e.g. by He and Pearson (1991a,b) and Karatzas et. al. (1991). With partial information investment problems over a finite time horizon have been solved with the martingale approach by Karatzas and Zhao (2001), Sass and Haussmann (2004) and Björk et al. (2010) among others and with consumption and a stochastic factor by Putschögl and Sass (2008), Lindensjö (2016), Hata and Sheu (2017). A random but bounded time horizon has e.g. been considered in Blanchet-Scalliet et al. (2008). The key to solve the optimization problem under partial information is to make the market an observationally complete market, under which the state price density can be expressed as a functional of the observational magnitude. In the observationally complete market, we can consequently apply the martingale approach for an infinite time horizon to achieve closed-form solutions to the optimal consumption (as a function of the state price) and investment strategies (as a function of wealth). This is the big advantage of the martingale approach in comparison to dynamic programming: it gives rather explicit expressions for the optimal controls. In the present paper, we deal with a retiree’s consumption and investment decision, i.e. we have additionally the challenge to combine the random time horizon (caused by the random death time) with the partial information. To better understand the impact of partial information on the retiree’s optimal consumption and investment, we carry out a series of analytical and numerical studies and compare these optimal solutions to the full information case. Though the explicit optimal strategies are rather complicated, we manage to obtain some interesting comparison results.

The remainder of the paper is organized as follows. Section 2 describes the underlying financial market and formulates and solves the optimal consumption, asset allocation and annuity decision problem under partial information for an individual with a power utility function. Section 3 deals with the case with log utility. In Section 4, we demonstrate some numerical results. Finally, Section 5 provides some concluding remarks and perspectives for further research.

2 Model setup

2.1 Consumption and asset allocation problem

Think of a retiree who is endowed with a wealth level $\hat{x} > 0$ at her retirement date (in our framework at time $t = 0$). She can use this money for consumption and trade in
the financial market while alive. We assume that the time of death \(\tau \geq 0\) of the retiree can be modelled using a deterministic force of mortality process \(\zeta\). Thus we obtain the survival probability at age \(x\) for time \(t \geq 0\) by

\[
P(\tau \geq t) := \exp \left(- \int_0^t \zeta(u)du \right) = t p_x, \tag{2.1}
\]

with \(\zeta(u) \geq 0\). Based on this, we can compute the density function of \(\tau\):

\[
f(t) = - \frac{\partial P(\tau \geq t)}{\partial t} = \zeta(t) \cdot t p_x. \tag{2.2}
\]

Since nowadays retirees are highly encouraged to invest in private retirement plans, we assume that at time \(t = 0\) the retiree spends a part of her initial wealth \(\alpha \hat{x}, 0 \leq \alpha \leq 1\), on a lifetime annuity. From this investment the retiree will receive a lifetime income at the rate \(a(\alpha \hat{x}) > 0\). More precisely, by spending \(\alpha \hat{x}\) on the annuity the retiree buys an annuity which pays \(a(\alpha \hat{x})\) for lifetime. The annuity income \(a(\alpha \hat{x})\) is determined by the annuity provider (we assume that the annuity price is given and the retiree does not have impact on the price.). The remaining money \((1 - \alpha)\hat{x}\) is used by the retiree to set up an investment plan and to increase the consumption which will be guaranteed by the lifetime annuity. Note that buying annuity is a one-shot decision made at \(t = 0\).

We assume that we are in a financial market with two traded assets: one risky and one risk-free asset. Suppose that \((\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, 0 \leq t < \infty\}, \mathbb{P})\) is a filtered probability space. The risk free asset \(B = (B(t))\) evolves according to

\[
 dB(t) = r B(t)dt, \quad t \geq 0
\]

for a deterministic risk free rate \(r \geq 0\).

As motivated in the introduction, we want to take account of some realistic aspects in the individual retirement decision. Delong and Chen (2016) addresses the issue of non-exponential discounting. Here, we suppose that the retiree only has partial information about the risky asset. The asset price dynamics for the risky asset \(S = (S(t))\) is given by

\[
dS(t) = S(t) \left( (r + \theta \sigma)dt + \sigma dW(t) \right), \quad t \geq 0 \tag{2.3}
\]

where \(W = (W(t))\) denotes a standard Brownian motion under the probability measure \(\mathbb{P}\) and \(\sigma > 0\) the volatility of the stock. The stock price itself is observable to the retiree, but the outcome of the market price of risk \(\theta\) is unknown to her. It is a random variable with known initial distribution \(\mathbb{P}(\theta = \vartheta_k) =: p_k > 0, \ k = 1, \ldots, m\) where \(\vartheta_1, \ldots, \vartheta_m\) are the possible values of \(\theta\). This distribution can be seen as a prior belief the retiree has about the distribution of the market price of risk at time \(t = 0\). The distribution of \(\theta\) will be updated by the investor based on her observation. We assume that \(\theta\) and \(W\) are independent. Note that observing the stock price is equivalent to observing the process \(Y\) with \(Y(t) := W(t) + \theta t\), as we can rewrite the stock price as follows:

\[
dS(t) = S(t) \left( r dt + \sigma dY(t) \right). \tag{2.4}
\]
In the considered Bayesian model, the investor draws her inferences about \( \theta \) and updates her belief about \( \theta \) via \( \hat{\theta}(t) = \mathbb{E}[\theta|\mathcal{F}^S_t] \) where \( \mathbb{F}^S = \{ \mathcal{F}^S_t, 0 \leq t < \infty \} \) is the augmented filtration generated by the stock price processes \( S \). This conditional expectation can be computed as follows. First let us consider \( \mathbb{P}(\theta = \vartheta_k|\mathcal{F}^S_t) \). Standard filtering theory implies that \( \mathbb{P}(\theta = \vartheta_k|\mathcal{F}^S_t) = \mathbb{P}(\theta = \vartheta_k|Y(t)). \)

Using Bayes’ rule and the fact that the density of \( W(t) \) is given by \( \varphi_t(x) = (2\pi t)^{-1/2}e^{-x^2/(2t)} \), we obtain that

\[
\mathbb{P}(\theta = \vartheta_k|Y(t) = y) = \frac{\mathbb{P}(\theta = \vartheta_k, Y(t) = y)}{\sum_{i=1}^m \mathbb{P}(\theta = \vartheta_i, Y(t) = y)} = \frac{\mathbb{P}(\theta = \vartheta_k, W(t) = y - \vartheta_k t)}{\sum_{i=1}^m \mathbb{P}(\theta = \vartheta_i, W(t) = y - \vartheta_i t)} = \frac{p_k \varphi_t(y - \vartheta_k t)}{\sum_{i=1}^m p_i \varphi_t(y - \vartheta_i t)} \tag{2.5}
\]

where \( \mathbb{P}(X = x) \) should be understood as the density of \( X \). Inserting now the density and eliminating all common factors finally yields the expression

\[
\mathbb{P}(\theta = \vartheta_k|Y(t)) = \frac{p_k L_t(\vartheta_k, Y(t))}{F(t, Y(t))}
\]

where

\[
F(t, y) := \sum_{k=1}^m L_t(\vartheta_k, y)p_k, \quad L_t(\vartheta_k, y) := \exp(\vartheta_k y - \frac{1}{2} \vartheta_k^2 t).
\]

Hence the conditional expectation is given by

\[
\hat{\theta}(t) := \mathbb{E}[\theta|\mathcal{F}^S_t] = \sum_{k=1}^m \vartheta_k \mathbb{P}(\theta = \vartheta_k|Y(t)) = \sum_{k=1}^m \frac{\vartheta_k p_k L_t(\vartheta_k, Y(t))}{F(t, Y(t))} = \frac{F_y(t, Y(t))}{F(t, Y(t))}.
\]

Based on her own Bayesian updating \( \hat{\theta} \), we can rewrite the stock price evolution

\[
dS(t) = S(t) \left( (r + \sigma \hat{\theta}(t))dt + \sigma d\hat{W}(t) \right) \tag{2.6}
\]

where

\[
d\hat{W}(t) = dW(t) + (\theta - \hat{\theta}(t))dt, \quad \text{or} \quad d\hat{W}(t) = dY(t) - \hat{\theta}(t)dt \tag{2.7}
\]

and thus we ensure that the stock price processes expressed in (2.6) agree with those in (2.3). Note that filtering theory implies that \( \hat{W} = (\hat{W}(t)) \) is an \((\mathbb{F}^S, \mathbb{P})\)-Brownian motion (see Karatzas and Zhao (2001), sec.3 and the references given there). In this sense, the stock price is expressed with all processes being \( \mathbb{F}^S \)-adapted. It is the key result used later in our optimization problem.

Let \( \pi(t) \) denote the amount that the retiree invests in the risky asset and \( (X(t) - \pi(t)) \) the amount invested in the risk-free bank account, and \( c(t) \) the consumption rate.

\(^2\)The process \( p_k(t) := \mathbb{P}(\theta = \vartheta_k|\mathcal{F}^S_t) \) is called Wonham-filter and an SDE can be derived for it, see e.g. Elliott et al. (2008), Karatzas and Zhao (2001).
Since the individual retiree is only able to observe the stock price and not the outcome of \( \theta \), the processes \((\pi(t)), (c(t))\) have to be \( \mathcal{F}^S_t \)-adapted. Thus, she chooses an investment and consumption strategy from the following admissible set
\[
\mathcal{A}(x_0) := \left\{ (\pi, c) \mid X_0 = x_0, (\pi(t)), (c(t)) \text{ are } (\mathcal{F}^S_t) - \text{progressively measurable,} \\
\text{ } c(t) \geq 0, X(t) \geq -\frac{1}{r}a(\alpha \hat{x}) \text{ for all } t \geq 0, \int_0^\infty \pi^2(t)dt < \infty \right\} \tag{2.8}
\]
where \( x_0 = (1 - \alpha)\hat{x} \). An admissible strategy prevents the portfolio value \( X(t) \) from falling below \(-\frac{1}{r}a(\alpha \hat{x})\) to ensure that the total wealth from the portfolio and from the annuity payment is nonnegative. We assume that \( x_0 \geq \frac{1}{r}(\zeta - a(\alpha \hat{x})) \). Based on these assumptions, the wealth process of the retiree \( X = (X(t)) \) satisfies the SDE
\[
dX(t) = \pi(t)\left( (r + \sigma \hat{\theta}(t))dt + \sigma d\hat{W}(t) \right) + (X(t) - \pi(t))r dt - c(t)dt + a(\alpha \hat{x})dt, \\
X(0) = (1 - \alpha)\hat{x} = x_0. \tag{2.9}
\]
We assume that the retiree is interested in maintaining a given living/consumption standard. The retiree needs to decide about how to invest and how much to withdraw from the available funds to consume during the retirement life. At the retirement date \( t = 0 \), we assume that our retiree solves the following optimization problem
\[
\sup_{(\pi, c) \in \mathcal{A}(x_0)} \mathbb{E}_{x_0} \left[ \int_0^\tau e^{-\rho t}\frac{(c(t) - \zeta)^{1-\gamma}}{1 - \gamma} dt \right], \text{ s.t. (2.9)}, \tag{2.10}
\]
where \( \rho \) is the individual’s subjective discount rate and the supremum is taken over all admissible consumption and investment strategies. Recall that \( \tau \) is the time of death. We assume that the retiree is interested in sustaining a certain living standard and gains only utility when her consumption rate is above a minimum consumption \( \zeta \). We can also interpret \( \zeta \) as a habit level, which is usually defined as a function of past consumption rates, see e.g. Munk (2008). Here, we assume that \( \zeta \) is a positive constant. The total utility of the retiree is given by the utility from the excess of the realized consumption over the minimum consumption \( \zeta \). We assume that the individual evaluates her payoff through a power utility with a relative risk aversion coefficient \( \gamma \in (0, \infty), \gamma \neq 1 \). The power utility is abundantly used in both theoretical and empirical research because of its nice analytical tractability. Most importantly, the use of the power utility is also well-motivated economically, since the long-run behavior of the economy suggests that the long run risk aversion cannot strongly depend on wealth, see Campbell and Viceira (2002).

### 2.2 Solution to the optimal consumption and asset allocation problem

We aim to find the optimal self-financing investment strategies and the optimal consumption to maximize the expected utility from the consumption, given \( \alpha \)-fraction...
of wealth is used to buy annuity. The key to solve the optimization problem under partial information is to make the market observationally complete for the investor who does not have full information about the market. Note that due to the relation

$$ Y(t) := W(t) + \int_0^t \theta(s)ds, $$

we can define a new probability measure $Q$, under which $Y(t)$ becomes a standard $(\mathbb{F}^S, Q)$ Brownian motion:

$$ dQ = \exp \left( - \int_0^T \theta(s)dW(s) - \frac{1}{2} \int_0^T \theta^2(s)ds \right) := \nu'(T). $$

(2.11)

Hereby, we have used the fact that $\hat{W} = (\hat{W}(t))$ is a standard Brownian motion under $(\mathbb{F}^S, \mathbb{P})$.

Lemma 2.1. We can express $\nu'(t)$ as a function of the observational magnitude $Y(t)$:

$$ \nu'(t) = F(t, Y(t))^{-1}, \quad t \geq 0. $$

(2.12)

The discounted process $\xi^I(t) := e^{-rt}\nu'(t)$ gives the state price density process in the observationally complete market and satisfies

$$ d\xi^I(t) = \xi^I(t)(-rdt - \hat{\theta}(t)d\hat{W}(t)), \quad \xi^I(0) = 0. $$

The dynamic optimal asset allocation problem (2.10) is now ready to be solved through so-called static martingale approach for infinite time horizon (c.f. Huang and Pages (1992)). Note first that the objective function in (2.10) can be written as follow thanks to Fubini:

$$ \sup_{(\pi, c) \in \mathcal{A}(x_0)} \mathbb{E}_{x_0} \left[ \int_0^\infty e^{-\rho t} \frac{(c(t) - \xi)^{1-\gamma}}{1-\gamma} \mathbb{P}(\tau > t)dt \right] $$

which turns the original problem into one with infinite time horizon and different time preferences. Via the martingale approach, the original optimization problem can then reformulated as

$$ \sup_{(\pi, c) \in \mathcal{A}(x_0)} \mathbb{E}_{x_0} \left[ \int_0^\infty e^{-\rho t} \frac{(c(t) - \xi)^{1-\gamma}}{1-\gamma} \mathbb{P}(\tau > t)dt \right] $$

s.t. $\mathbb{E}_{x_0} \left[ \int_0^\infty \xi^I(t)c(t)dt - \int_0^\infty \xi^I(t)a(\alpha \hat{x})dt \right] \leq x_0. $$

(2.13)

Here, we try to find the solutions to the optimal investment-consumption, which shall satisfy the above budget constraint. The first term in the budget constraint determines the “time-0 price” of the consumption stream $(c(t))_{t \geq 0}$. The term after the negative sign gives the “time-0 price” of the annuity payments corresponding to $\alpha \hat{x}$. It means that the total consumption shall cost today no more than the initial wealth $x_0$ invested and the market value of the future annuity payments.

3We do not use the relation $Y(t) := W(t) + \theta t$ to define the change of measure, as $\theta$ is not observable.
Following the regular Lagrangian Ansatz, we obtain the optimal solution to consumption
\[
e^t \xi^* (t) = c + \left( \lambda^t \xi^t (t) e^{\rho t + \int_0^t \zeta^t (s) ds} \right)^{-1 / \gamma}, \ t \geq 0
\] (2.14)
where $\lambda^t$ is the Lagrangian multiplier which satisfies the budget constraint below:
\[
\mathbb{E}_{x_0} \left[ \int_0^\infty \xi^t (t) (c + \left( \lambda^t \xi^t (t) e^{\rho t + \int_0^t \zeta^t (s) ds} \right)^{-1 / \gamma}) dt - \int_0^\infty \xi^t (t) a(\alpha \hat{x}) dt \right] = x_0
\] (2.15)
As $\mathbb{E}[\xi^t (t)] = e^{-rt}$, we are now ready to write down the critical Lagrangian multiplier:
\[
\lambda^t = \left( \frac{x_0 + (a(\alpha \hat{x}) - c)}{\mathbb{E}_{x_0} \left[ \int_0^\infty \xi^t (t) (\xi^t (t) e^{\rho t + \int_0^t \zeta^t (s) ds} \right)^{-1 / \gamma}) dt] \right)^{-\gamma}.
\] (2.16)

A sufficient condition for the denominator to be finite is that $(1 - \gamma)(r + \frac{\alpha^2}{2\gamma}) < \rho$ for all $k = 1, \ldots, m$. A proof that this really yields the optimal consumption is given in the appendix.

In order to derive the optimal investment strategy at $t \geq 0$, we shall determine the time-$t$ wealth (a detailed calculation can be found in the appendix):
\[
X^t (t) = \mathbb{E}_{x_0} \left[ \int_t^\infty \xi^t (u) \left( e^t (u) - a(\alpha \hat{x}) \right) du \bigg| \mathcal{F}^S_t \right]
= \mathbb{E}_{x_0} \left[ \int_t^\infty \xi^t (u) \left( \lambda^t \xi^t (u) e^{\rho u + \int_0^u \zeta (z) dz} \right)^{-\frac{1}{\gamma}} du \bigg| \mathcal{F}^S_t \right] + \frac{1}{r} (c - a(\alpha \hat{x})).
= \lambda^t \int_t^\infty h^t (u, Y (t)) \left( e^{\rho u + \int_0^u \zeta (z) dz} \right)^{-\frac{1}{\gamma}} du + \frac{1}{r} (c - a(\alpha \hat{x}))
= \kappa^t (t, Y (t)) + \frac{1}{r} (c - a(\alpha \hat{x}))
\]
with
\[
h^t (t, u, Y (t)) := e^{(\frac{1}{\gamma} - 1) ru + rt} \int_{\mathbb{R}} F(u, Y (t) + z)^{1 / \gamma} \varphi_{u-t} (z) dz.
\]
In order to compute the optimal amount invested in the risky asset based on the observation $Y(t) = y$, note first that the first derivative of $h^t$ with respect to $y$ is by dominated convergence given by
\[
h^t_y (t, u, Y (t)) = \frac{1}{\gamma} e^{(\frac{1}{\gamma} - 1) ru + rt} \int_{\mathbb{R}} F_y (u, Y (t) + z) F(u, Y (t) + z)^{1 / \gamma - 1} \varphi_{u-t} (z) dz.
\]
It has the consequence that the first derivative of $\kappa^t_y$ with respect to $y$ is:
\[
\kappa^t_y (t, Y (t)) = \left( \lambda^t \right)^{-1 / \gamma} \int_t^\infty h^t (u, Y (t)) \left( e^{\rho u + \int_0^u \zeta (z) dz} \right)^{-\frac{1}{\gamma}} du.
\] (2.17)
Using the Itô-Doeblin formula we obtain

\[ dX^I(t) = \kappa^I(t, Y(t))dt + \frac{1}{2} \kappa_{yy}^I(t, Y(t))dt + \kappa^I_y(t, Y(t))dY(t). \]

Equating this with

\[ dX^I(t) = rX^I(t)dt + \pi(t)\sigma dY(t) - (c^I(t) - a(\alpha \hat{x}))dt \]

yields \( \pi^I(t) = \frac{1}{\sigma} \kappa^I_y(t, Y(t)). \) We summarize our results in the next theorem:

**Theorem 2.1.** In the model with partial information, the optimal consumption is

\[ c^I(t) = \xi + (\lambda^I(t) \xi(t)e^\rho t + \int_0^t \zeta(z) dz)^{-1/\gamma}, \quad t \geq 0 \]

where \( \lambda^I \) is defined in (2.16) and the optimal amount invested in the risky asset is

\[ \pi^I(t) = \frac{1}{\sigma} \kappa^I_y(t, Y(t)), \quad t \geq 0 \]

where \( \kappa^I_y(t, Y(t)) \) is defined by (2.17).

### 2.3 The case with full information

As we want to compare our partial information case with the case with full information in our numerical section, let us mention the problem and the solution to that case shortly in this place. The traded asset \( S \) follows again Black-Scholes and satisfies:

\[ dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \]

where \( W = (W(t)) \) denotes a standard Brownian motion under a probability measure \( \mathbb{P}. \) With full information means that instantaneous rate of return \( \mu > 0 \) and volatility \( \sigma > 0 \) are both known to the investor. Due to this specification, the considered financial market is a complete market, under which the state price density process in this economy is uniquely defined by

\[ d\xi(t) = -\xi(t)(\rho dt + \eta dW(t)), \quad \xi(0) = 1 \]  

(2.18)

with \( \eta := \frac{\mu - r}{\sigma} \) and the wealth process of the individual is given by

\[ dX(t) = \pi(t)(\mu dt + \sigma dW(t)) + (X(t) - \pi(t))rdt - c(t)dt + a(\alpha \hat{x})dt \]

\[ X(0) = (1 - \alpha)\hat{x} =: x_0 \]  

(2.19)

The filtration \( (\mathcal{F}_t) \) which consists of the augmented Brownian filtration is here the same as the filtration that is generated by the stock price process. On the financial market described above, the optimal consumption can be expressed as follows:

\[ c^*(t) = \xi + (\lambda \xi(t)e^\rho t + \int_0^t \zeta(z) dz)^{-1/\gamma}, \quad t \geq 0 \]
where \( \lambda \) is the critical Lagrangian multiplier which makes the budget constraint binding:

\[
\lambda = \left( \frac{x_0 + (a(\alpha \hat{x}) - \zeta) \frac{1}{\gamma}}{\mathbb{E}_x \left[ \int_0^\infty \xi(t)(\xi(t) e^{\rho t + f_\zeta(z) dz})^{-1/\gamma} dt \right]} \right)^{-\gamma}.
\] (2.20)

We have to assume here that \((1 - \gamma)(r + \frac{\eta^2}{2}) < \rho\) in order to have a finite value for \( \lambda \). The optimal intermediate wealth at \( t \geq 0 \) is given by (again a detailed calculation can be found in the appendix):

\[
X^*(t) = \kappa(t) \xi(t)^{-\frac{1}{\gamma}} + \frac{1}{\gamma} (\zeta - a(\alpha \hat{x}))
\]
with \( \kappa(t) := \lambda^{-\frac{1}{\gamma}} e^{-td} \int_t^\infty e^{u(d - \frac{\eta}{2}) - \frac{1}{2} \int_u^\infty \zeta(z) dz} du \) and \( d = (1 - \frac{1}{\gamma})(-r - \frac{1}{2} \frac{\eta^2}{\gamma}) \).

Using the Itô-Doeblin formula we obtain

\[
dX^*(t) = \kappa(t) \xi(t)^{-\frac{1}{\gamma}} dt - \frac{1}{\gamma} \kappa(t) \xi(t)^{-\frac{1}{\gamma} - 1} d\xi(t) + \frac{1}{2\gamma} (\frac{1}{\gamma} + 1) \kappa(t) \eta^2 \xi(t)^{-\frac{1}{\gamma}} dt
\]
\[
= \kappa(t) \xi(t)^{-\frac{1}{\gamma}} \left( -d + \frac{r}{\gamma} + \frac{1}{2\gamma} (\frac{1}{\gamma} + 1) \eta^2 \right) dt - (c^*(t) - \zeta) dt + \frac{\eta}{\gamma} \kappa(t) \xi(t)^{-\frac{1}{\gamma}} dW(t)
\]
\[
= \left( X^*(t) r - c^*(t) + a(\alpha \hat{x}) \right) dt + \frac{\eta^2}{\gamma} \kappa(t) \xi(t)^{-\frac{1}{\gamma}} dt + \frac{\eta}{\gamma} \kappa(t) \xi(t)^{-\frac{1}{\gamma}} dW(t)
\] (2.21)

In the second step, we have used

\[
\kappa(t) = -d \kappa(t) - (c^*(t) - \zeta) \xi(t)^{\frac{1}{\gamma}}
\]
and in the third step we apply

\[
-d + \frac{r}{\gamma} + \frac{1}{2\gamma} (\frac{1}{\gamma} + 1) \eta^2 = \frac{\eta^2}{\gamma} + r.
\]

Comparing (2.21) with (2.19), we obtain the optimal amount in the risky asset:

\[
\pi^*(t) = \frac{\eta}{\gamma \sigma} X^*(t) - \frac{\eta}{\gamma \sigma} \frac{1}{r} (\zeta - a(\alpha \hat{x})), \quad \eta = \frac{\mu - r}{\sigma}.
\] (2.22)

With the optimal investment strategy, the two wealth processes (2.21) and (2.19) coincide. The first part of the strategy is actually the Merton portfolio which suggests investing a constant fraction in the risky asset. The second part of the strategy is an adjustment term which is highly dependent of the sign of \((\zeta - a(\alpha \hat{x}))\). For \( \zeta > a(\alpha \hat{x}) \), the annuity income is insufficient to provide the minimum subsistence level of the individual. In this case, the individual holds a lower fraction in equity than the Merton portfolio. For \( \zeta < a(\alpha \hat{x}) \), this effect is reversed and the individual holds a higher fraction in equity than the Merton portfolio. However, note that a lower/higher fraction does not imply necessarily a lower/higher amount invested in the equity.
2.4 Comparative Statics

It is possible to compare the optimal consumption and investment strategy in the model with partial information with the respective optimal strategies in the full information model. First it is immediate to see that the formulas for the optimal consumption are the same. Note however, that the state price densities $\xi$ and $\xi^I$ are different and hence also the Lagrange multipliers $\lambda$ and $\lambda^I$. The formulas for the optimal investment though are very different. A naive approach for an investment strategy in the partial information case would be to take formula (2.22) for the full information case and replace the unknown $\mu - r \sigma$ by its estimate $\hat{\theta}(t)$ at time $t \geq 0$. Indeed we can prove the following result:

**Proposition 2.1.** Suppose $r \leq \vartheta_k, k = 1, \ldots, m$. If $\gamma < 1$, the optimal fraction invested in stock in the model with partial information satisfies

$$\pi^I^*(t) \geq \frac{\hat{\theta}(t)}{\gamma \sigma} \left( X^{I^*}(t) - \frac{1}{r} (\xi - a(\alpha \hat{x})) \right), \ t \geq 0. \tag{2.23}$$

If $\gamma > 1$ then

$$\pi^I^*(t) \leq \frac{\hat{\theta}(t)}{\gamma \sigma} \left( X^{I^*}(t) - \frac{1}{r} (\xi - a(\alpha \hat{x})) \right), \ t \geq 0 \tag{2.24}$$

**Proof.** Under the assumption $r \leq \vartheta_k, k = 1, \ldots, m$ and $\gamma < 1$ it has been shown in Rieder and Bäuerle (2005) Theorem 9d) that for any $T > t$

$$\hat{\theta}(t) \leq \frac{\int_{\mathbb{R}} F_y(T, Y(t) + z) F(T, Y(t) + z)^{1/\gamma - 1} \varphi_{T-t}(z) dz}{\int_{\mathbb{R}} F(T, Y(t) + z)^{1/\gamma} \varphi_{T-t}(z) dz}. \tag{2.25}$$

Hence

$$\kappa_y^I(t, Y(t)) = \frac{1}{\gamma} (\lambda^I)^{-1} e^{rt} \int_t^\infty e^{-\xi s + (\gamma - 1)r s - \frac{1}{2} f_0^s \zeta(z') dz'}$$

$$\left( \int_{\mathbb{R}} F(s, Y(t) + z)^{1/\gamma - 1} F_y(s, Y(t) + z) \varphi_{s-t}(z) dz ds \right)$$

$$\geq \frac{\hat{\theta}(t)}{\gamma} (\lambda^I)^{-1} e^{rt} \int_t^\infty e^{-\xi s + (\gamma - 1)r s - \frac{1}{2} f_0^s \zeta(z') dz'} $$

$$\int_{\mathbb{R}} F(s, Y(t) + z)^{1/\gamma} \varphi_{s-t}(z) dz ds$$

$$= \frac{\hat{\theta}(t)}{\gamma} \kappa_y^I(t, Y(t)).$$

If we use this inequality we obtain

$$\pi^I^*(t) = \frac{1}{\sigma} \kappa_y^I(t, Y(t)) \geq \frac{\hat{\theta}(t)}{\gamma \sigma} \left( X^{I^*}(t) - \frac{1}{r} (\xi - a(\alpha \hat{x})) \right)$$

which is exactly the statement. In case $\gamma > 1$, the inequality in (2.25) reverses. \qed
Proposition 2.2. Suppose \( r \leq \vartheta_1 \leq \ldots \leq \vartheta_m \). Then, the optimal amount invested in stock in the model with partial information satisfies
\[
\frac{1}{\gamma} \sigma \left( X^{I^*}(t) - \frac{1}{r}(c - a(\alpha \hat{x})) \right) \leq \pi^{I^*}(t) \leq \frac{1}{\gamma} \sigma \left( X^{I^*}(t) - \frac{1}{r}(c - a(\alpha \hat{x})) \right), \quad t \geq 0.
\]
\[(2.26)\]

Proof. First note that due to the non-negativity of \( p_k \) and the exponential function we obtain
\[
\vartheta_1 F(T, y + z) \leq F_y(T, y + z) \leq \vartheta_m F(T, y + z).
\]
This further implies that
\[
\frac{1}{\gamma} \kappa(t, Y(t)) \leq \kappa(t, Y(t)) \leq \frac{m}{\gamma} \kappa(t, Y(t)).
\]
This inequality then implies the statement in the same way as in the proof of Proposition 2.1.

\[\square\]

3 Logarithmic Utility

In this section we consider the logarithmic utility for consumption,
\[
\sup_{(\pi, c) \in A(X_0)} \mathbb{E}_{x_0} \left[ \int_0^\infty e^{-\rho t} \log \left( c(t) - \xi \right) \mathbb{P}(\tau > t) dt \right]
\]
\[
\text{s.t. } \mathbb{E}_{x_0} \left[ \int_0^\infty \xi^I(t)c(t) dt - \int_0^\infty \xi^I(t)a(\alpha \hat{x}) dt \right] \leq x_0
\]
\[(3.1)\]

We can proceed in the same way as for the power utility and obtain the following solution (for a proof, see appendix):

**Theorem 3.1.** In the model with partial information with logarithmic utility, the optimal consumption is given by
\[
c^{I^*}(t) = \xi + \frac{e^{-\rho t - \int_0^t \xi(z) dz}}{\lambda^I \xi^I(t)}, \quad t \geq 0
\]
where \( \lambda^I \) makes the budget constraint binding:
\[
\lambda^I = \left( x_0 + \frac{1}{r} \left( a(\alpha \hat{x}) - \xi \right) \right)^{-1}.
\]
\[(3.2)\]

The optimal investment strategy is given as
\[
\pi^{I^*}(t, Y(t)) = \frac{\hat{\theta}(t)}{\sigma} \left( X^{I^*}(t) - \frac{1}{r}(c - a(\alpha \hat{x})) \right), \quad t \geq 0.
\]
Note here that \( X^{I^*}(t) \) and \( \hat{\theta}(t) \) are functions of \( Y(t) \) and \( t \) as can be seen from the proof.
Note that in the case of full information we obtain then
\[ \pi^*(t) = \frac{\mu - r}{\sigma^2} \left( X^*(t) - \frac{1}{r} (\xi - a(\alpha \hat{x})) \right), \]
where \( X^*(t) = k(t) \xi(t)^{-1} + 1/r(\xi - a(\alpha \hat{x})) \) with \( \kappa(t) = \lambda^{-1} \int_t^\infty e^{-\rho s - \int_0^s \zeta(z) dz} du \). Hence the solution in the partial information case is obtained by replacing the unknown \( \frac{\mu - r}{\sigma} \) by its estimate \( \hat{\theta}(t) \) at time \( t \). This is the so-called certainty equivalence principle.

**Proposition 3.1.** The optimal consumption and the optimal investment strategies in the model with partial information and power utility converge for \( \gamma \to 1 \) to the solution of the logarithmic utility model.

**Proof.** In case of the optimal consumption the statement is immediate. Note that also the Lagrange multipliers for the power utility problem in (2.16) converge due to dominated convergence against the Lagrange multiplier of the logarithmic utility problem. For the optimal investment strategy note that the expression \( h^I(t, u, Y_t) \) converges for \( \gamma \to 1 \) to \( \frac{1}{\xi(t)} = e^{rt} F(t, Y(t)) \). This is due to dominated convergence and the fact that
\[ \int \exp \left( \vartheta_k z - \frac{1}{2} \vartheta_k^2 (u - t) \right) \varphi_{u-t}(z) dz = 1. \]
Thus it follows that \( \kappa^I(t, Y(t)) \) converges for \( \gamma \to 1 \) to
\[ (\lambda^I)^{-1} e^{rt} F(t, Y(t)) \int_t^\infty e^{-\rho s - \int_0^s \zeta(z) dz} ds, \]
and finally by dominated convergence \( \kappa_y^I(t, Y(t)) \) converges for \( \gamma \to 1 \) to
\begin{align*}
(\lambda^I)^{-1} e^{rt} F_y(t, Y(t)) \int_t^\infty e^{-\rho s - \int_0^s \zeta(z) dz} ds \\
= \frac{F_y(t, Y(t))}{F(t, Y(t))} (\lambda^I)^{-1} e^{rt} F(t, Y(t)) \int_t^\infty e^{-\rho s - \int_0^s \zeta(z) dz} ds \\
= \hat{\theta}(t) \left( X^{I^*}(t) - \frac{1}{r} (\xi - a(\alpha \hat{x})) \right)
\end{align*}
which implies the result. \( \square \)

4 Numerical analyses

In the following, we give some numerical examples to examine how partial information affects the individual’s optimal consumption, investment and annuity decision. In the classical finite horizon investment problem, studies like this have been carried out in e.g. Rieder and Bäuerle (2005), Longo and Mainini (2016) and Bäuerle and Grether (2017). We assume that the individual investor has 100 money unit available at retirement,
which can be used to buy annuity, to consume and to invest. For our analysis, we will assume a given \( \alpha = 0.2 \), i.e. initially the individual uses 20 money unit to buy annuity.

In the present paper, we only deal with post-retirement mortality, which is typically at and above age 60 (or 65). Throughout the paper, we assume that at \( t = 0 \), our investor is 65 years old. Richards (2012) has studied a large number of analytic mortality laws and found that the quality of a given law depends heavily on the age interval it is used for. According to Richards (2012), the Gompertz-law (Gompertz (1825)) works best for ages 60-90 and is hence used in this paper as it describes the most important age-group for our analysis. We use the parameterization for the mortality intensity given by

\[
\zeta_{x+t} = \frac{1}{b} e^{(x+t-m)/b},
\]

where \( b > 0 \) is the dispersion coefficient and \( m > 0 \) the modal age at death, following Gumbel (1958) and Milevsky and Salisbury (2015). The survival probabilities can then be derived to

\[
p_x := e^{-\int_0^t \zeta_{x+s} ds} = e^{e^{x-m} - e^{t/b}}.
\]

We follow Milevsky and Salisbury (2015) and set the Gompertz parameters to \( m = 88.721 \) and \( b = 10 \). For most of the numerical analyses, we assume that the individual and the insurer use the same survival distribution. In other words, for the given interest rate (2%), and the survival probability, we obtain \( a(\alpha \hat{x}) \approx 1.22 \). If we assume that the insurer uses another life table for pricing purpose, we can e.g. use \( m_1 > m \) to obtain a higher survival probability for prudence reasons.

The remaining parameters are fixed as follows:

\[
\begin{align*}
\mu &= 0.05, \\
r &= 0.02, \\
\sigma &= 0.15, \\
\xi &= 1, \\
\rho &= 0.02, \\
\vartheta_1 &= 0.15, \\
\vartheta_2 &= 0.20, \\
\vartheta_3 &= 0.25, \\
p_1 &= 1/3, \\
p_2 &= 1/3, \\
p_3 &= 1/3.
\end{align*}
\]

In case of of full information, we obtain \((\mu - r)/\sigma = 0.2\). By assuming the above parameters for \( \vartheta_i \) and \( p_i \), we obtain \( E[\theta] = 0.2 \) for the case with partial information, which allows a reasonable comparison between these two cases. Furthermore, the condition \( r \leq \vartheta_k, \forall k \) used in Proposition 2.1 is fulfilled by our parameter choices. Note also that \( x_0 \geq \frac{1}{\vartheta} (\xi - a(\alpha \hat{x})) \) and the other conditions which guarantee finite Lagrange multipliers are satisfied. In addition, as the state price densities in the two cases are different, we will express optimal consumption and equity holding as a function of the stock price. In the case with partial information, we have expressed \( \xi(t) \) as a function of \( Y(t) \). Due to the one-to-one relation between \( Y(t) \) and \( S(t) \), we obtain a one-to-one between the \( \xi(t) \) and \( S(t) \). In case with full information, the one-to-one between \( \xi(t) \) and \( S(t) \) is apparent.

Figures 1 and 2 plot the optimal consumption rate \( c^*(t) \) (full information case) and \( c^{I*}(t) \) (partial information case) at \( t = 10 \) as a function of the stock price \( S(t) \) for two levels of relative risk aversion, \( \gamma = 0.7 \) and \( \gamma = 3 \), using our base case parameters.
Both under full and partial information, the optimal consumption increases in the stock price. In other words, when the financial market performs better, the investor becomes more wealthy and chooses to consume more. The optimal consumption rate for the more risk-averse investor ($\gamma = 3$) increases in the stock price in a much less volatile trend and it remains in a relatively stable interval (here, roughly between $[2, 7]$, c.f. Figure 2). In contrast, the optimal consumption rate for the less risk-averse investor ($\gamma = 0.7$) shows a much bigger increase as the stock price goes up. Finally, the optimal consumption is an increasing and convex function of the stock price for $\gamma = 0.7$; and an increasing and concave function for $\gamma = 3$. Partial information does not seem to have a substantial impact on the optimal consumption. The two optimal consumption curves are almost overlapping.

Figures 3 and 4 exhibit the optimal fraction invested in stock $\pi^*(t)/X^*(t)$ (full information case) and $\pi^{I*}(t)/X^{I*}(t)$ (partial information case) for $t = 10$, as a function of the stock price, for two levels of relative risk aversion, $\gamma = 0.7$ and $\gamma = 3$. For comparison, we also demonstrate the Merton portfolio (which is obtained by setting $\alpha = 0$ and $\zeta = 0$ from our full information case solution). We observe the following: (a) From (2.22) and Proposition 2.1, we learn that the optimal fraction invested in the stock in our model is higher than $\hat{\theta}(t)$ at least for small wealth, if the annuity payment

---

4 Generally, we cannot look at the optimal fraction because for the optimal wealth equal to 0, the fraction is not defined. In the considered numerical example, the optimal wealth is unequal to 0.
\[
\begin{aligned}
\gamma &= 0.7 \\
\gamma &= 1 \\
\gamma &= 1.5 \\
\gamma &= 3 \\
\gamma &= 5
\end{aligned}
\]

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Table 1: Comparison statistics for optimal amount invested in stock, using our base case parameters (particularly $S(0) = 1$, $S(t) = 1.2$ for $t = 0, 5, 10, 15, 20$). $\gamma = 1$ is referred to the log utility case.
Figure 2: Optimal consumption rate $c^*(t)$ and $c^{I*}(t)$ at $t = 10$ as a function of stock price $S(t)$, using our base case parameters and $\gamma = 3$.

is beyond the minimum consumption level (which is the case here). This is verified for both $\gamma = 0.7$ and $\gamma = 3$. In our numerical example we can see that the optimal fraction is also higher than the Merton ratio. (b) As expected, the optimal fraction invested in the stock for $\gamma = 0.7$ is substantially higher than for $\gamma = 3$. For $\gamma = 0.7$, the optimal fraction held in the risky asset is far beyond 100%, whereas for $\gamma = 3$, this fraction is a bit higher than 50%. Moreover, the investor with a relative risk aversion of 0.5 chooses to invest a slightly higher fraction in the equity in the partial information case than in the full information case. For the investor with a relative risk aversion of 3, partial information leads to the consequence that the investor holds a lower (higher) fraction in equity in bad (good) economic state than in full information case. In total, the optimal equity ratio displays almost a constant fraction. In other words, partial information makes the less risky investor (here $\gamma = 0.7$) invest more riskily, while this does not hold for the more risk-averse investor.

Table 1 provides some comparison statistics for the optimal amount invested in the risky asset. We observe the following: (a) The results stated in Proposition 2.1 is verified: for $\gamma < 1$, we do observe that $\pi^{I*}(t) \geq \frac{1}{\gamma} \beta(t) (X^{I*}(t) - \frac{1}{r}(\bar{c} - a(\alpha \hat{x}))$, while the reversed effect holds for all the $\gamma$ values higher than 1. For log utility case, we observe the equality of these two magnitudes. (b) The argument stated in Proposition 2.2 holds
true, i.e.
\[
\frac{1}{\gamma} \frac{\theta(t)}{\sigma} (X^{I^*}(t) - \frac{1}{r} (c - a(\alpha x))) \leq \pi^{I^*}(t) \leq \frac{1}{\gamma} \frac{\theta_{m}(t)}{\sigma} (X^{I^*}(t) - \frac{1}{r} (c - a(\alpha x)))
\]
for all the degrees of relative risk aversion. (c) For an investor with a given risk aversion, less and less will be invested in the stock as time goes by. It is an intuitive result as more fund values have been used for consumption. Moreover it seems that $\pi^{I^*}(t)$ and $\frac{1}{\gamma} \frac{\theta(t)}{\sigma} X^{I^*}(t) - \frac{1}{\gamma} \frac{\theta(t)}{\sigma} y(t, T)$ converge to the same value. (d) Due to the fact that the continuous annuity rate is higher than the minimum consumption level, the optimal fraction invested in the stock is higher than the Merton ratio, as shown in Figures 3 and 4. However, the optimal amount invested in the stock $\pi^*(t)$ under full information does not have to be higher than the Merton case. (e) As already seen in Figures 3 and 4, we do not generally observe the strategies under the partial information are less (or more) risky than those under full information, if we consider the fraction invested in the stock. As to the amount, full information leads to a bit smaller amount invested in the stock.
Figure 4: Optimal fraction in the stock $\pi^*(t)/X^*(t)$ and $\pi^{I*}(t)/X^{I*}(t)$ at $t = 10$ as a function of stock price $S(t)$, using our base case parameters and $\gamma = 3$.

5 Conclusion

The present paper studies how the parameter uncertainty with an unknown drift process influences the optimal retirement decisions of an individual. Using the static martingale approach, we analytically determine the optimal consumption and investment strategies. We find that partial information has little impact on the optimal consumption level, but it can have substantial influence on the optimal investment strategies. In the future research, we can consider another important aspect of the retirement decision: the optimal retirement time problem. In the existing literature (see e.g. Choi and Shim (2006) and Choi et. al. (2008), Farhi and Panageas (2007), Dybvig and Liu (2010) and Lee and Shin (2015)), this problem has been solved for complete information and a constant force of mortality rate.

6 Appendix

In this appendix we collect some proofs and lengthy calculations.
6.1 Proof of Lemma 2.1

Proof. Note first that (2.11) implies
\[ d\nu^I(t) = -\nu^I(t)\dot{\theta}(t)d\hat{W}(t), \quad \text{with} \quad \nu^I(0) = 1. \]  
(6.1)

In order to achieve a more explicit representation of \( \nu^I(t) \) in terms of \( Y(t) \), let us now define \( Z(t) := F(t, Y(t)) \). Then by the Itô-Doeblin formula and using the definition of \( F \) we obtain:
\[
dZ(t) = F_t(t, Y(t))dt + F_y(t, Y(t))dY(t) + \frac{1}{2}F_{yy}(t, Y(t))dt =
\sum_{k=1}^m L(\vartheta_k, Y(t))p_k(-\frac{1}{2}\vartheta_k^2)dt + \sum_{k=1}^m L(\vartheta_k, Y(t))p_k\vartheta_kdY(t) + \frac{1}{2} \sum_{k=1}^m L(\vartheta_k, Y(t))p_k\vartheta_k^2dt
\]
\[= Z(t) \sum_{k=1}^m \frac{L_t(\vartheta_k, Y(t))p_k\vartheta_k}{F(t, Y(t))}dY(t) = Z(t)\dot{\theta}(t)dY(t). \]

Defining \( \nu^I(t) := Z(t)^{-1} \) and using the Itô-Doeblin formula, we can check that
\[ d\nu^I(t) = -\nu^I(t)\dot{\theta}(t)dY(t) + \nu^I(t)\theta(t)^2dt = -\nu^I(t)\dot{\theta}(t)d\hat{W}(t), \]
which is identical to (6.1). In other words, the solution to (6.1) can be expressed by \( \nu^I(t) = F(t, Y(t))^{-1} \) since stochastic exponentials are unique up to indistinguishable processes.

6.2 Proof of the optimality of \( c^I^* \)

Proof. Given \( \lambda^I > 0 \) as in (2.16), note \( c^I^*(t) \) given in (2.14) is indeed the optimal solution to Problem (3.1). This can be seen as follows. For any concave function \( U \) and two points \( c(t) - \xi \) and \( c^I^*(t) - \xi \), we have
\[ U(c(t) - \xi) \leq U(c^I^*(t) - \xi) + U'(c^I^*(t) - \xi)(c(t) - c^I^*(t)). \]

Inserting \( c^I^* \) and \( U' \) yields
\[ U(c(t) - \xi) \leq U(c^I^*(t) - \xi) + \lambda^I(t)e^{\rho t + \int_0^t \xi(z)dz}(c(t) - c^I^*(t)). \]

This then implies for \( c^I^* \) and any other feasible consumption plan \( c \)
\[
E_{\pi_0}\left[ \int_0^\infty e^{-\rho t}(c(t) - \xi)^{(1-\gamma)}(1-\gamma)\pi(t)dt \right] 
\leq E_{\pi_0}\left[ \int_0^\infty e^{-\rho t}(c^I^*(t) - \xi)^{(1-\gamma)}\pi(t)dt \right]
+ \lambda^I E_{\pi_0}\left[ \int_0^\infty \xi^I(t)((c(t) - a(\alpha \dot{x}) - (c^I^*(t) - a(\alpha \dot{x}))dt \right].
\]

Since \( c \) is a feasible consumption and \( c^I^* \) meets the budget constraint with equality, the optimality of \( c^I^* \) follows.
6.3 Detailed calculations of Section 2.2

Note that we obtain:

\[
\mathbb{E} \left[ \left( \frac{\xi^I(u)}{\xi^I(t)} \right)^{1-1/\gamma} \bigg| \mathcal{F}_t \right] = \frac{1}{\xi^I(t)} \mathbb{E} \left[ \left( \frac{\xi^I(u)}{\xi^I(t)} \right)^{1-1/\gamma} Y(t) \bigg| \mathcal{F}_t \right] = \frac{1}{\xi^I(t)} \mathbb{E} \left[ (e^{-ru} F(u, Y(u))^{-1})^{1-1/\gamma} Y(t) \bigg| \mathcal{F}_t \right]
\]

\[
= e^{(1/\gamma-1)ru} \xi^I(t) \mathbb{E}^Q \left[ \frac{F(u, Y(u))}{F(t, Y(t))} (F(u, Y(u)))^{1-1/\gamma} Y(t) \bigg| \mathcal{F}_t \right]
\]

\[
= e^{(1/\gamma-1)ru} \xi^I(t) F(t, Y(t)) \int_{\mathbb{R}} F(u, Y(t) + z)^{1/\gamma} \varphi_{u-t}(z) dz
\]

\[
= e^{(1/\gamma-1)ru + rt} \int_{\mathbb{R}} F(u, Y(t) + z)^{1/\gamma} \varphi_{u-t}(z) dz =: h^I(t, u, Y(t)).
\]

Recall that \(Y\) is a standard Brownian motion under \(Q\), which implies that for \(0 \leq t < u\) \(Y(u) - Y(t)\) is independent of \(Y(t)\) and normally distributed with mean 0 and variance \(u - t\) under \(Q\).

6.4 Detailed calculations of Section 2.3

We obtain here

\[
X^*(t) = \mathbb{E}_{x_0} \left[ \int_t^\infty \frac{\xi(u)}{\xi(t)} \left( e^* (u) - a(\alpha \hat{x}) \right) du \bigg| \mathcal{F}_t \right]
\]

\[
= \mathbb{E}_{x_0} \left[ \int_t^\infty \frac{\xi(u)}{\xi(t)} \left( \lambda \xi(u) e^{\rho u + \int_0^u \zeta(z) dz} \right) \left( 1 - \frac{1}{\gamma} \right) du \bigg| \mathcal{F}_t \right] + (\zeta - a(\alpha \hat{x})) \mathbb{E} \left[ \int_t^\infty \frac{\xi(u)}{\xi(t)} ds \bigg| \mathcal{F}_t \right]
\]

\[
= \lambda^{-\frac{1}{\gamma}} \xi(t)^{-\frac{1}{\gamma}} \int_t^\infty \mathbb{E} \left[ \left( \frac{\xi(u)}{\xi(t)} \right)^{1-\frac{1}{\gamma}} \bigg| \mathcal{F}_t \right] e^{-\xi(u) \left( \frac{1}{\gamma} \right) \int_0^u \zeta(z) dz} du + \frac{1}{r} (\zeta - a(\alpha \hat{x}))
\]

An easy computation now yields that

\[
\mathbb{E} \left[ \left( \frac{\xi(u)}{\xi(t)} \right)^{1-\frac{1}{\gamma}} \bigg| \mathcal{F}_t \right] = e^{(u-t)d}
\]

with \(d = (1 - \frac{1}{\gamma})(-r - \frac{1}{2} \frac{\eta^2}{\gamma})\).
6.5 Proof of Theorem 3.1

Proof. We proceed as before. Let \( X^I(t) \) be the time-\( t \) value of consumption minus annuity rate which we can compute as follows

\[
X^I(t) = \mathbb{E}_{x_0} \left[ \int_t^\infty \frac{\xi^I(u)}{\xi^I(t)} \left( c^* - a(\alpha \hat{x}) \right) du \bigg| \mathcal{F}_t^S \right]
\]

\[
= (\lambda^I)^{-1} \int_t^\infty e^{-\rho s - \int_0^s \zeta(z) dz} ds \xi^I(t)^{-1} + \frac{1}{r}(\zeta - a(\alpha \hat{x}))
\]

\[
= \kappa^I(t) \xi^I(t)^{-1} + \frac{1}{r}(\zeta - a(\alpha \hat{x}))
\]

\[
= \kappa^I(t) F(t, Y_t) e^{rt} + \frac{1}{r}(\zeta - a(\alpha \hat{x})).
\]

where \( \kappa^I(t) := (\lambda^I)^{-1} \int_t^\infty e^{-\rho s - \int_0^s \zeta(z) dz} ds \). Using the Itô-Doeblin formula we obtain

\[
dX^I(t) = rF(t, Y(t)) e^{rt} \kappa^I(t) dt + F(t, Y(t)) e^{rt} \kappa^I(t) dt + (\zeta - c^*(t)) dt
\]

\[
+ \frac{1}{2} F_{yy}(t, Y(t)) e^{rt} \kappa^I(t) dt + F_y(t, Y(t)) e^{rt} \kappa^I(t) dY(t).
\]

Equating this with the wealth equation implies

\[
\pi^I(t) = \frac{1}{\sigma} F_y(t, Y(t)) e^{rt} \kappa^I(t) = \frac{1}{\sigma} F(t, Y(t)) e^{rt} \kappa^I(t) \frac{F_y(t, Y(t))}{F(t, Y(t))}
\]

\[
= \frac{\hat{\theta}(t)}{\sigma} \left( X^I(t) - \frac{1}{r}(\zeta - a(\alpha \hat{x})) \right)
\]

which concludes the proof. \( \square \)

References


He, H. and Pearson, N.D. (1991a). Consumption and portfolio policies with incom-
plete markets and short-sales constraints: the finite-dimensional Case. Mathematical
Finance 1, pp. 1-10.

He, H. and Pearson, N.D. (1991b). Consumption and portfolio policies with incom-
plete markets and short-sales constraints: the infinite-dimensional case. Journal of
Economic Theory 54, 239-250.

an infinite horizon: existence and convergence. The Annals of Applied Probability,
36-64.

Karatzas, I., Lehoczky, J.P and Shreve, S.E. (1987). Optimal portfolio and consump-
tion decisions for a small investor on a finite horizon. SIAM Journal of Control and
Optimization 25, 1557-1586.

methods of utility maximization in an incomplete market. SIAM Journal of Control
and Optimization 29, 702-730.

Mathematical Finance: Option pricing, interest rates and risk management, 632669,
Cambridge Univ. Press, Cambridge.

retirement choice problem with disutility and subsistence consumption constraints: A
dynamic programming approach. Journal of Mathematical Analysis and Applications
428(2), 762-771.


Longo, M. and Mainini, A. (2016). Learning and Portfolio Decisions for CRRA In-


Mathematics and Economics 64, 91-105.

Munk, C., 2008. Portfolio and consumption choice with stochastic investment opportu-
nities and habit formation in preferences. Journal of Economic Dynamics & Control
32, 3560-3589.

14646.


