Measure-valued derivatives and applications

Georg Ch. Pflug and Philipp Thoma (PhD student)

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Our basic problem is to solve

$$\min_{\theta \in \Theta} \mathbb{E}[\mathcal{H}(X_\theta)]$$

where

- $X_\theta(\cdot)$ is a Markovian process, with a transition law which depends on a parameter $\theta$
- the functional $\mathcal{H}(X_\theta)$ may be
  - $\mathcal{H}(X_\theta) = h(X_\theta(T))$
  - $\mathcal{H}(X_\theta) = h(X_\theta(\infty))$
  - $\mathcal{H}(X_\theta) = \int_0^T h(X_\theta(t)) \, dt$
  - $\mathcal{H}(X_\theta) = \int_0^\tau h(X_\theta(t)) \, dt$, where $\tau$ is some stopping time.
Examples

- Markov Systems (queueing, service, manufacturing)
  Suppose that $\theta$ denotes the parameters of a Markov System (queueing, inventory, renewal). Let $X_\theta(t)$ be the state of the system at time $t$ and $X_\theta(\infty)$ the steady state (if exists). Then
    - $\mathbb{E}[h(X_\theta(T)))]$ is the performance of the system at time $T$
    - $\int_0^T \mathbb{E}[h(X_\theta(t))] \, dt$ is the expected integrated transient behavior
    - $\mathbb{E}[h(X_\theta(\infty)))]$ is the expected stationary behavior.

- Finance
  Let $X_\theta(t)$ describe the evolution of an underlying asset, for a parameter vector $\theta$. Let $h$ be the payoff function of a contingent contract. Then $\mathbb{E}[h(X_\theta(T)))]$ is the value of the contingent contract (European type). If $\tau$ is a stopping time, then $\mathbb{E}[h(X_\theta(\tau)))]$ is the value of the American type contingent contract. We want to estimate the sensitivity of the price w.r.t. the parameter $\theta$ (the Greeks).
Finite differences: The Kiefer-Wolfowitz procedure

\[ \theta_{n+1} = \theta_n + a_n \frac{h(X_{\theta_n}+c_n) - h(X_{\theta_n}-c_n)}{2c_n}. \]

Small \( c_n \) ⇒ small bias and large variance
Large \( c_n \) ⇒ large bias and small variance
Differentiability of \( h \) is required.

Left: Differences \( c_n \to 0 \), Right: No convergence \( c_n \to 0 \)
We are interested in finding the value of

$$\frac{\partial}{\partial \theta} \mathbb{E}[h(X_\theta)]$$

and - more generally -

$$\frac{\partial^k}{\partial \theta^k} \mathbb{E}[h(X_\theta)],$$

as well as a way to estimate it based on sampling. We formulate the problem in terms of the distribution $\mu_\theta$ of $X_\theta$. 
The two paradigms for derivatives

Let \( F_\theta(x) \) be the distribution function of \( \mu_\theta \)

<table>
<thead>
<tr>
<th>Measure derivatives</th>
<th>Pathwise derivatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta \mapsto F_\theta(\cdot) )</td>
<td>( \theta \mapsto F_\theta^{-1}(\cdot) )</td>
</tr>
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</table>

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<tr>
<th>Measure-valued derivatives</th>
<th>Finite differences (FD)</th>
<th>Infinitesimal perturb. (IPA)</th>
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<tr>
<td>Score-function method</td>
<td></td>
<td>Malliavin calculus (part. int.)</td>
</tr>
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</table>
Let \((R, d)\) be a metric space. To the family of Borel probabilities on \((R, d)\), we associate a "dual function space" \(F\) such as

- the space of all bounded, continuous functions
- the space of all continuous functions \(h\), such that \(|h(u)| \leq K_1 + K_2 d^p(u, u_0)\)

**Definition.** The family of probability measures \((\mu_\theta)_{\theta \in \Theta \subseteq \mathbb{R}}\) on \(R\) is **weakly differentiable w.r.t. the dual space** \(F\), if there is a finite signed measure \(\mu'_\theta\) such that for all \(h \in F\)

\[
\frac{1}{s} \left[ \int h(w) \, d\mu_{\theta+s}(w) - \int h(w) \, d\mu_\theta(w) \right] \to \int h(w) \, d\mu'_\theta(w)
\]
as \(s \to 0\). (Heidergott, Vasquez-Abad, Leahu, Xi-Ren Cao, G.P., ...)

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Measure-valued derivatives and applications
Any finite signed measure may be decomposed into its positive and negative part (Jordan decomposition). Since $\int 1\,d\mu_\theta = 1$, we have that $\int 1\,d\mu'_\theta = 0$, i.e. the positive and the negative part have the same mass. Thus we may decompose the derivative object $\mu'_\theta$

$$\mu'_\theta = c_\theta (\dot{\mu}_\theta^+ - \dot{\mu}_\theta^-)$$

where $\dot{\mu}_\theta^+$ and $\dot{\mu}_\theta^-$ are probability measures. The representation as a multiple of the difference of two probability measures $\mu'_\theta = c(\mu_1 - \mu_2)$ is not unique, however the constant $c$ is minimal if the two parts $\mu_1$ and $\mu_2$ are orthogonal, i.e. if the decomposition is the Jordan decomposition.
Any triplet \((c_\theta, \dot{\mu}_\theta^+, \dot{\mu}_\theta^-)\), such that for \(h \in \mathbb{F}\)

\[
\frac{1}{s} \left[ \int h(w) \, d\mu_{\theta+s}(w) - \int h(w) \, d\mu_\theta(w) \right] \rightarrow c_\theta \left[ \int h(w) \, d\dot{\mu}_\theta^+(w) - \int h(w) \, d\dot{\mu}_\theta^-(w) \right],
\]

for \(s \to 0\), is called a measure valued derivative triplet.
The Gamma($a$, $b$) distribution has density

$$
\frac{1}{b^a \Gamma(a)} x^{a-1} \exp\left(-\frac{x}{b}\right)
$$

- If $X \sim Gamma(1/2, 2\sigma^2)$ (i.e. $\chi^2(1)$), then $\sqrt{X}$ is distributed according to the positive part of a $N(0, \sigma^2)$ distribution.
- If $X \sim Gamma(1, 2\sigma^2)$, (i.e. $\chi^2(2)$), then $\sqrt{X}$ is distributed according to a Raleigh distribution, which is a Weibull distribution with exponent 2.
- If $X \sim Gamma(3/2, 2\sigma^2)$, (i.e. $\chi^2(3)$), then $\sqrt{X}$ is distributed according to a Maxwell distribution.
Example: Gradient w.r.t. the variance parameter

The family $N(0, \theta^2)$

The derivative w.r.t. $\theta$ as a signed measure
The Jordan–Hahn decomposition of the signed measure representing the derivative

The decomposition of the derivative in a Maxwell part and a normal part
Examples for measure valued derivatives

<table>
<thead>
<tr>
<th>Distribution $\mu_\theta$ ((\theta) varies)</th>
<th>Constant $c_\theta$</th>
<th>Positive part of the derivative: $\dot{\mu}_\theta^+$</th>
<th>Negative part of the derivative: $\dot{\mu}_\theta^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson((\theta))</td>
<td>1</td>
<td>Poisson((\theta)) + 1</td>
<td>Poisson((\theta))</td>
</tr>
<tr>
<td>Normal((\theta, \sigma^2))</td>
<td>1/(\sigma\sqrt{2\pi})</td>
<td>(\theta + \text{Raleigh}(\frac{1}{2\sigma^2}))</td>
<td>(\theta - \text{Raleigh}(\frac{1}{2\sigma^2}))</td>
</tr>
<tr>
<td>Normal((m, \theta^2))</td>
<td>1/(\theta)</td>
<td>ds-Maxwell((m, \theta^2))</td>
<td>Normal((m, \theta^2))</td>
</tr>
<tr>
<td>Exponential((\theta))</td>
<td>1/(\theta)</td>
<td>Exponential((\theta))</td>
<td>(\theta^{-1}) Erlang(2)</td>
</tr>
<tr>
<td>Gamma((a, \theta))</td>
<td>a/(\theta)</td>
<td>Gamma((a, \theta))</td>
<td>Gamma((a + 1, \theta))</td>
</tr>
<tr>
<td>Weibull((\alpha, \theta))</td>
<td>1/(\theta)</td>
<td>Weibull((\alpha, \theta))</td>
<td>[Gamma(2, (\theta))]^{1/(\alpha)}</td>
</tr>
</tbody>
</table>
Higher derivatives

The Normal family $N(\theta, \sigma^2)$ has first derivative

$$\left[ \frac{1}{\sigma \sqrt{2\pi}}, \theta + \text{Raleigh}(\frac{1}{2\sigma^2}), \theta - \text{Raleigh}(\frac{1}{2\sigma^2}) \right]$$

and second derivative

$$\left[ \frac{1}{\sigma^2}, \text{ds-Maxwell}(\theta, \sigma^2), \text{Normal}(\theta, \sigma^2) \right].$$
Use of weak derivatives in sensitivity estimation

If $\dot{X}_\theta^+$ resp. $\dot{X}_\theta^-$ are distributed according to $\dot{\mu}_\theta^+$ resp. $\dot{\mu}_\theta^-$, then

$$c_\theta [h(\dot{X}_\theta^+) - h(\dot{X}_\theta^-)]$$

is a consistent estimate of $\frac{\partial}{\partial \theta} \mathbb{E}[h(X_\theta)]$.

(Academic) Example. Let $H(\theta) = \mathbb{E}[\cos(X_\theta)]$, where $X_\theta$ is a Normal($0, \theta^2$) variable. Then

$$\frac{1}{\theta} [\cos(\dot{X}_\theta^+) - \cos(\dot{X}_\theta^-)]$$

where

$$\dot{X}_\theta^+ \sim \text{double-sided-Maxwell}(0, \theta^2) \text{ and } \dot{X}_\theta^- \sim \text{Normal}(0, \theta^2)$$

is an unbiased estimate for $\frac{\partial}{\partial \theta} H(\theta)$.

Notice that no infinitesimal quantities appear and that we do not have to know the derivative of $\cos$. 
A *coupling* of two probability measures $\mu_1$ and $\mu_2$ w.r.t. $h$ is a probability measure $\bar{\mu}$ on $R \times R$ with given marginals $\mu_i$, which minimizes the expectation of the criterion function $d(u, v)$:

Minimize $\int d(u, v) \bar{\mu}(du, dv)$

subject to

$\text{proj}_1 \bar{\mu} = \mu_1$,

$\text{proj}_2 \bar{\mu} = \mu_2$,

$\bar{\mu}$ is a probability on $R \times R$

The solution may not be unique. We denote the solution (or the set of solutions) by

$\mu_1 \overset{d}{\circlearrowright} \mu_2$

and call it ”$\mu_1$ and $\mu_2$ coupled over $d$”.
<table>
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<tr>
<th>Distribution $\mu_\theta$ ($\theta$ varies)</th>
<th>Constant $c_\theta$</th>
<th>Positive part of the derivative: $\dot{\mu}_\theta^+$</th>
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<td>Poisson($\theta$)</td>
</tr>
</tbody>
</table>

Coupling over the Euclidean distance $d(u, v) = |u - v|$ leads to taking $\dot{X}^+ = X_\theta + 1$, $\dot{X}^- = X_\theta$, i.e.: For any integrable (summable) cost function $h$ and any Poisson variable $X_\theta \sim \text{Poisson}(\theta)$ we have

$$\frac{\partial}{\partial \theta} \mathbb{E}[h(X_\theta)] = \mathbb{E}[h(X_\theta + 1)] - \mathbb{E}[h(X_\theta)]$$

with very low variance.
If $X_\theta$ has density $f_\theta(x)$, then under appropriate conditions

$$\frac{\partial}{\partial \theta} \mathbb{E}[h(X_\theta)] = \mathbb{E} \left[ h(X_\theta) \frac{\dot{f}_\theta(X_\theta)}{f_\theta(X_\theta)} \right]$$

where

$$\frac{\dot{f}_\theta(x)}{f_\theta(X_\theta)} = \frac{\partial}{\partial \theta} \log f_\theta(x)$$

is the score function.

- The score function methods requires stronger assumptions than the MVD-method.
- When using an appropriate coupling, the MVD-method leads to smaller variances. In a stochastic process situation, the score function martingale has typically large variance.
A comparison

Estimation of sensitivity of $\theta \mapsto \mathbb{E}[\sqrt{X_\theta}]$, where $X_\theta \sim \text{Exponential}(\theta)$. Then the variances are

<table>
<thead>
<tr>
<th>Method</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerical differences</td>
<td>1069.9</td>
</tr>
<tr>
<td>Score function method</td>
<td>2.81</td>
</tr>
<tr>
<td>MVD with coupling</td>
<td>0.022</td>
</tr>
</tbody>
</table>
Let \((P_\theta)_{\theta \in \Theta}\) be a family of Markov transitions on the metric state space \((R, r)\).

**Definition.** A (regular) finite signed transition operator \(T(w, A)\) is a mapping \(R \times \mathcal{B} \to \mathbb{R}\) with the property

- \(w \mapsto T(w, A)\) is measurable for each Borel set \(A\),
- \(A \mapsto T(w, A)\) is a finite signed measure for each \(w\).

We introduce the following notations

- \(1_\mu\) is the transition \(T(w, A) = \mu(A)\)
- \(\mu T\) is the measure \((\mu T)(A) = \int T(w, A) \, d\mu(w)\).
- \(T h\) is the function \((T h)(u) = \int h(w) T(u, dw)\).
Definition. The Markov transition $\mathbb{P}_\theta(\cdot, \cdot)$ is called (uniformly) weakly differentiable, if there is a signed transition $\mathbb{P}_\theta'$ such that for all continuous functions $h$ satisfying $|h(u)| \leq K(1 + \|u\|^p)$ for all $u$ and some constant $K$ and every point mass $\delta_w$ (i.e. the probability distribution concentrated on the point $w$)

$$\frac{1}{s} |\delta_w \mathbb{P}_{\theta+s} h - \delta_w \mathbb{P}_\theta h - s \cdot \delta_w \mathbb{P}_\theta' h| \to 0$$

as $s \to 0$, uniformly in $w$. 
Every finite signed transition may be decomposed as

\[ T(w, A) = c(w)[P_1(w, A) - P_2(w, A)], \]

where \( P_1 \) and \( P_2 \) are regular Markov transitions. Again, we may select convenient decompositions: We choose two Markov transitions \( \dot{P}^+ \) and \( \dot{P}^- \) and a measurable function \( c_\theta(w) \) such that

\[ P'_\theta(w, A) = c_\theta(w)[\dot{P}^+\theta(w, A) - \dot{P}^-\theta(w, A)]. \]
The Leibniz formula and its sampling counterpart

\[(P^n_{\theta})' = \sum_{i=1}^{n} P_{\theta}^{i-1}P'_{\theta}P_{\theta}^{n-i}.\]

Estimation of \(\gamma(P^n_{\theta})'h:\)

1. Sample a random uniform time \(\tau\) in \(\{1, \ldots, n\}\).
2. Sample \(X_\theta(0)\) from the starting distribution \(\gamma\).
3. Sample \(\tau - 1\) steps with transition \(P_{\theta}\), giving \(X_\theta(1), \ldots, X_\theta(\tau - 1)\)
4. Sample one transition step from \(X_\theta(\tau - 1)\) with transition \(P^+_{\theta}\) and one with transition \(P^-_{\theta}\), giving \(\dot{X}^+_{\theta}(\tau)\) resp. \(\dot{X}^-_{\theta}(\tau)\).
5. Continue the processes \(\dot{X}^+_{\theta}(t)\) resp. \(\dot{X}^-_{\theta}(t)\), \(t = \tau + 1, \ldots, n\) using transition \(P_{\theta}\) and a coupling technique.
6. The final estimate is

\[nc(X_\theta(\tau - 1))[h(\dot{X}^+_{\theta}(t)) - h(\dot{X}^-_{\theta}(t))].\]
Alternatively, instead of sampling the stopping time \( \tau \), one may take the sum over all \( i \) without further sampling. We call this method exact measure valued derivation.
Gradients of stationary distributions

Let

\[ \pi_\theta P_\theta = \pi_\theta \]

for all \( \theta \) be the (unique) stationary distribution of \( P_\theta \). Then \( \theta \mapsto \pi_\theta \) is weakly differentiable and

\[ \pi'_\theta = \pi_\theta \cdot P'_\theta \cdot S_\theta \]

where \( S_\theta \) is the inverse Poisson operator, satisfying

\[ S_\theta (I - P_\theta + 1 \cdot \pi_\theta) = I, \]

with \( I \) being the identity operator. The von Neumann series for \( S_\theta \) is

\[ S_\theta = \sum_{m=0}^{\infty} (P_\theta^m - 1 \cdot \pi_\theta). \]
1. Start with an arbitrary starting distribution
2. Do $m$ steps with transition $\mathbb{P}_\theta$ to get $X^{(m)}_\theta$
3. Make one transition with $\mathbb{P}^+_\theta$ and one transition with $\mathbb{P}^-_\theta$ to get $\dot{X}^+_\theta(0)$ resp. $\dot{X}^-_\theta(0)$
4. Starting with $\dot{X}^+_\theta(0)$ resp. $\dot{X}^-_\theta(0)$ do $n$ steps with transition $\mathbb{P}_\theta$ to get sample $\dot{X}^+_\theta(n)$ resp. $\dot{X}^-_\theta(n)$. These two processes should be coupled.
5. The final estimate is
   \[ c(X^{(m)}_\theta)[h(\dot{X}^+_\theta(n)) - h(\dot{X}^-_\theta(n))]. \]
Consider two Markov transition operators \( P \) and \( Q \) and set
\[
P_\theta = (1 - \theta)P + \theta Q.
\]
Let \( \pi_\theta \) be the stationary distribution of \( P_\theta \), i.e. \( \pi_\theta P_\theta = \pi_\theta \).

Introduce the deviation matrix
\[
D = \sum_{j=0}^{\infty} (P - 1 \pi_0)^j.
\]

Then - at least for small \( \theta \) -
\[
\pi_\theta = \pi_0 \sum_{j=0}^{\infty} \theta^j [(Q - P)D]^j
\]

and therefore the expected costs for a cost function \( h \) (not depending on \( \theta \)) are
\[
E_{\pi_\theta}[h(X(\infty))] = \langle \pi_\theta, h \rangle = \pi_0 \sum_{j=0}^{\infty} \theta^j [(Q - P)D]^j h.
\]
Example: the MM1 queue

One queue with arrival intensity $\lambda$

One server with service intensity $\theta$, which is the decision variable

Intensity matrix:

$$
\begin{pmatrix}
-\lambda & \lambda & 0 & 0 & \cdots \\
\theta & -(\lambda + \theta) & \lambda & 0 & \cdots \\
0 & \theta & -(\lambda + \theta) & \lambda & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

Costs: Probability that more than 3 customers are in the waiting queue.
Taylor expansion up to first order
Left: The stationary distribution $\pi_\theta$. Right: The probability that more than 3 customers wait

Taylor expansion up to fifth order
Lévy Processes and the estimation of Greeks

$X(t)$ is a Lévy Process, if it has independent, stationary increments. The pertaining price process is

$$S(t) = S_0 \exp(X(t)).$$

We typically make an Esscher transform to find a measure $Q$ under which

$$\exp(-rt)S(t)$$

is a martingale. We consider especially the following Lévy processes

- The Brownian motion
- The Poisson process
- The Compound Poisson process
- The Gamma process
- The Variance Gamma process
Measure-valued Differentiation at Random Point

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Measure-valued derivatives and applications
Brownian motion

Left: Poisson process
Right: Compound Poisson process
Left: Gamma process  
Right: Variance Gamma process
The geometric Brownian Motion (GBM)

Under the martingale measure, the GBM motion is

$$d\log S(t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW(t)$$

We aim at calculating the $\rho$, i.e. the sensitivity w.r.t. $r$ for the two types of options

- The plain vanilla call option with payoff function

  $$g(S(T)) = e^{-rT}[S(T) - K]^+.$$

- The digital option with payoff function

  $$g(S(T)) = e^{-rT}1_{\{S(T) > K\}}.$$
Pathwise derivative

\[ dS_{t_i}^r = f_r(S_{t_i}^r)dt + \sigma_r(S_{t_i}^r)dW_{t_i} \]

\[ D_{t_{i+1}}^r = D_{t_i}^r + \left[ \dot{f}_r(S_{t_i}^r) + f'_r(S_{t_i}^r)D_{t_i}^r \right] h + \left[ \dot{\sigma}_r(S_{t_i}^r) + \sigma'_r(S_{t_i}^r)D_{t_i}^r \right] \sqrt{hZ} \]

\[ \dot{f}_r(S_t^r) = S_t^r, \quad f'_r(S_t^r) = \left( r - \frac{1}{2} \sigma^2 \right), \quad \dot{\sigma}_r(S_{t_i}^r) = 0, \quad \sigma'_r(S_{t_i}^r) = \sigma \]

and therefore

\[ D_{t_{i+1}}^r = D_{t_i}^r + \left[ S_{t_i}^r + \left( r - \frac{1}{2} \sigma^2 \right)D_{t_i}^r \right] \frac{1}{n} + \left[ \sigma D_{t_i}^r \right] \sqrt{\frac{1}{n}Z} \]

\[ \frac{\partial \mathbb{E}(g(S_T))}{\partial r} = \mathbb{E}(g'(S_T)D_T^r) \]
We consider the payoff function $g(S_T) = e^{-rT}[S(T) - K]^+$. For the derivative we have to simulate

$$\frac{\partial \mathbb{E}[g(S_T)]}{\partial r} = c_r \left( \mathbb{E}[g(S_T^+)] - \mathbb{E}[g(S_T^-)] \right).$$
### GBM: Plain vanilla call option

<table>
<thead>
<tr>
<th></th>
<th>$\rho$</th>
<th>Variance</th>
<th>Computational time in sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>IPA</td>
<td>92.9442</td>
<td>7.3883e+002</td>
<td>0.3112</td>
</tr>
<tr>
<td>FD</td>
<td>96.9025</td>
<td>1.8287e+006</td>
<td>0.0035</td>
</tr>
<tr>
<td>MVD</td>
<td>93.0916</td>
<td>2.0330e+003</td>
<td>0.1693</td>
</tr>
<tr>
<td>MVD e</td>
<td>93.0983</td>
<td>6.9878e+002</td>
<td>0.6116</td>
</tr>
<tr>
<td></td>
<td>$\rho$</td>
<td>variance</td>
<td>computational time in sec</td>
</tr>
<tr>
<td>---</td>
<td>-------</td>
<td>-----------------</td>
<td>----------------------------</td>
</tr>
<tr>
<td>FD</td>
<td>0.8867</td>
<td>9.4862e+002</td>
<td>0.0037</td>
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<tr>
<td>MVD</td>
<td>1.3934</td>
<td>48.4024</td>
<td>0.1683</td>
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<tr>
<td>MVDe</td>
<td>1.3979</td>
<td>23.7785</td>
<td>0.6089</td>
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</tbody>
</table>
**Poisson: Plain vanilla option**

<table>
<thead>
<tr>
<th></th>
<th>Sensitivity w.r.t. $\lambda$</th>
<th>variance</th>
<th>comp. time in sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>FD</td>
<td>6.2473</td>
<td>$8.9401 \times 10^4$</td>
<td>0.0362</td>
</tr>
<tr>
<td>MVD</td>
<td>0.5768</td>
<td>5.7689</td>
<td>2.0508</td>
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<tr>
<td>MVDe</td>
<td>0.6044</td>
<td>1.6241</td>
<td>63.1382</td>
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</table>
### Poisson: Digital option

<table>
<thead>
<tr>
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<th>Sensitivity w.r.t. $\lambda$</th>
<th>Variance</th>
<th>Comp. time in sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>FD</td>
<td>-1.0496</td>
<td>4.0443e+003</td>
<td>0.0382</td>
</tr>
<tr>
<td>MVD</td>
<td>0.1109</td>
<td>0.2505</td>
<td>2.0537</td>
</tr>
<tr>
<td>MVDe</td>
<td>0.1106</td>
<td>0.1115</td>
<td>63.0970</td>
</tr>
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### Gamma: Plain vanilla

<table>
<thead>
<tr>
<th></th>
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<th>comp. time in sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>FD</td>
<td>2.4686</td>
<td>1.9599e+044</td>
<td>0.0193</td>
</tr>
<tr>
<td>MVD</td>
<td>-0.0512</td>
<td>3.7718</td>
<td>1.6280</td>
</tr>
<tr>
<td>MVDe</td>
<td>-0.0308</td>
<td>1.3068</td>
<td>48.6192</td>
</tr>
</tbody>
</table>
## Gamma: Digital

<table>
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<tbody>
<tr>
<td>FD</td>
<td>1.7554</td>
<td>4.0315e003</td>
<td>0.0197</td>
</tr>
<tr>
<td>MVD</td>
<td>0.0116</td>
<td>0.9859</td>
<td>1.6098</td>
</tr>
<tr>
<td>MVD_e</td>
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Georg Ch. Pflug and Philipp Thoma (PhD student)  
Measure-valued derivatives and applications