Optimal investment under multiple defaults: a BSDE-decomposition approach

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Multiple defaults times and marks

On a probability space \((\Omega, \mathcal{G}, \mathbb{P})\):

- **Reference filtration** \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\): default-free information

Progressive information provided, when they occur, by:

- a family of \(n\) random times \(\tau = (\tau_1, \ldots, \tau_n)\) associated to a family of \(n\) random marks \(\zeta = (\zeta_1, \ldots, \zeta_n)\).

  - \(\tau_i\): default time of name \(i \in \mathbb{I}_n = \{1, \ldots, n\}\).

  - The mark \(\zeta_i\), valued in \(E\) Borel set of \(\mathbb{R}^p\), represents a jump size at \(\tau_i\), which cannot be predicted from the reference filtration, e.g. the loss given default.
The **global market information** is defined by:

\[ G = \mathcal{F} \vee D^1 \vee \ldots \vee D^n, \]

where \( D^i \) is the default filtration generated by the observation of \( \tau_i \) and \( \zeta_i \) when they occur, i.e.

\[ D^i = (D^i_t)_{t \geq 0}, \quad D^i_t = \sigma\{1_{\tau_i \leq s}, s \leq t\} \vee \sigma\{\zeta_i 1_{\tau_i \leq s}, s \leq t\}. \]

\[ \rightarrow G = \mathcal{F} \vee \mathcal{F}^\mu, \text{ where } \mathcal{F}^\mu \text{ is the filtration generated by the jump random measure } \mu(dt, de) \text{ associated to } (\tau_i, \zeta_i). \]
Successive defaults

For simplicity of presentation, we assume that

\[ \tau_1 \leq \ldots \leq \tau_n \]

**Remark.** The general multiple random times case for \((\tau_1, \ldots, \tau_n)\) can be derived from the ordered case by considering the filtration generated by the corresponding ranked times \((\hat{\tau}_1, \ldots, \hat{\tau}_n)\) and the index marks \(\nu_i, \ i = 1, \ldots, n\) so that \((\hat{\tau}_1, \ldots, \hat{\tau}_n) = (\tau_{\nu_1}, \ldots, \tau_{\nu_n})\).

**Notation:** For \(k = 0, \ldots, n\),

\[ \tau_k = (\tau_1, \ldots, \tau_k) \] valued in \(\Delta_k = \{(\theta_1, \ldots, \theta_k) : 0 \leq \theta_1 \leq \ldots \leq \theta_k\}\).

\[ \zeta_k = (\zeta_1, \ldots, \zeta_k) \] valued in \(E^k\),

with the convention \(\tau_0 = \emptyset, \zeta_0 = \emptyset\).
Decomposition of $\mathcal{G}$-adapted and predictable processes

**Lemma**
Any $\mathcal{G}$-adapted process $Y$ is represented as:

$$Y_t = \sum_{k=0}^{n} 1_{\{\tau_k \leq t < \tau_{k+1}\}} Y^k_t(\tau_k, \zeta_k),$$  \hspace{1cm} (1)

where $Y^k_t$ is $\mathcal{F}_t \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable.

**Remarks.** • A similar decomposition result holds for $\mathcal{G}$-predictable processes: $< \leftrightarrow \leq$, and $Y^k$ is $\mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable in (1).
• Extension of Jeulin-Yor result (case of single random time without mark).
• We identify $Y$ with the $n + 1$-tuple $(Y^0, \ldots, Y^n)$. 
• **Portfolio of** $N$ **assets** with $\mathcal{G}$-adapted value process $S$:

$$S_t = \sum_{k=0}^{n} 1\{\tau_k \leq t < \tau_{k+1}\} S^k_t(\tau_k, \zeta_k),$$

where $S^k(\theta_k, e_k)$, $\theta_k = (\theta_1, \ldots, \theta_k) \in \Delta_k$, $e_k = (e_1, \ldots, e_k) \in E^k$, indexed $\mathcal{F}$-adapted process valued in $\mathbb{R}^N_+$, represents the assets value given the past default events $\tau_k = \theta_k$ and marks at default $\zeta_k = e_k$. 
Change of regimes with jumps at defaults

- Dynamics of $S = S^k$ between $\tau_k = \theta_k$ and $\tau_{k+1} = \theta_{k+1}$:

$$dS^k_t(\theta_k, e_k) = S^k_t(\theta_k, e_k) \ast (b^k_t(\theta_k, e_k)dt + \sigma^k_t(\theta_k, e_k)dW_t),$$

where $W$ is a $m$-dimensional $(\mathbb{P}, \mathbb{F})$-Brownian motion, $m \geq N$.

- Jumps at $\tau_{k+1} = \theta_{k+1}$:

$$S^k_{\theta_{k+1}}(\theta_{k+1}, e_{k+1}) = S^k_{\theta^+_{k+1}}(\theta_k, e_k) \ast (1_N + \gamma^k_{\theta_{k+1}}(\theta_k, e_k, e_{k+1})),

\gamma^k \text{ vector-valued in } [-1, \infty)^N.$$
Admissible control strategies

- A trading strategy in the $N$-assets portfolio is a $\mathcal{G}$-predictable process $\pi = (\pi^0, \ldots, \pi^n)$:

  $$\pi^k(\theta_k, e_k) \quad \text{is valued in} \quad A^k \quad \text{closed convex set of} \quad \mathbb{R}^N,$$

denoted $\pi^k \in \mathcal{P}_F(\Delta_k, E^k; A^k)$, and representing the amount invested given the past default events $(\tau_k, \zeta_k) = (\theta_k, e_k)$, $k = 0, \ldots, n$, and until the next default time.

- The set of admissible controls: $A_G = A^0_F \times \ldots \times A^n_F$, where $A^k_F$ includes some integrability conditions.
Wealth process

- Given an admissible trading strategy $\pi = (\pi^k)_{k=0,\ldots,n}$, the controlled wealth process is given by:

$$X_t = \sum_{k=0}^{n} 1_{\{\tau_k \leq t < \tau_{k+1}\}} X^k_t(\tau_k, \zeta_k), \quad t \geq 0,$$

where $X^k$ is the wealth process with an investment $\pi^k$ in the assets of price $S^k$ given the past defaults events $(\tau_k, \zeta_k)$.

- Dynamics between $\tau_k = \theta_k$ and $\tau_{k+1} = \theta_{k+1}$:

$$dX^k_t(\theta_k, e_k) = \pi^k_t(\theta_k, e_k)'(b^k_t(\theta_k, e_k)) dt + \sigma^k_t(\theta_k, e_k) dW_t).$$

- Jumps at default time $\tau_{k+1} = \theta_{k+1}$:

$$X^k_{\theta_{k+1}}(\theta_{k+1}, e_{k+1}) = X^k_{\theta_{k+1}}(\theta_k, e_k) + \pi^k_{\theta_{k+1}}(\theta_k, e_k)' \gamma^k_{\theta_{k+1}}(\theta_k, e_k, e_{k+1}).$$
Value function

- **Value function** of the optimal investment problem:

  \[ V_0(x) = \sup_{\pi \in \mathcal{A}_G} \mathbb{E}\left[U(X_T^x,\pi)\right], \quad x \in \mathbb{R}. \]

  where \( U \) is an utility function.

**Remark.** One can also deal with running gain function, involving e.g. utility from consumption, and utility-based pricing with credit derivative.
Usual global approach

- Write the dynamics of assets and wealth process in the global filtration $\mathcal{G}$
  $\rightarrow$ Jump-Itô controlled process under $\mathcal{G}$ in terms of $W$ and $\mu$
  (random measure associated to $(\tau_k, \zeta_k)_k$).

- Use a martingale representation theorem for $(W, \mu)$ w.r.t. $\mathcal{G}$
  under intensity hypothesis on the default times

  $\blacktriangleright$ Derive the dynamic programming Bellman equation in the $\mathcal{G}$
  filtration
  $\rightarrow$ BSDE with jumps or Integro-Partial-differential equations:
  Ankirchner et al. (09), Lim and Quenez (10), Jeanblanc et al (10).
Our solutions approach

- Find a suitable decomposition of the \( \mathbb{G} \)-control problem on each default scenario → sub-control problems in the \( \mathbb{F} \)-filtration
  - by relying on the \( \mathbb{F} \)-decomposition of \( \mathbb{G} \)-processes,
  - density hypothesis on the defaults

- Backward system of BSDEs in Brownian filtration
  - Get rid of the jump terms and overcome the technical difficulties in BSDEs with jumps
  - Existence, uniqueness and characterization results in a general formulation under weaker conditions
- Explicit description of the optimal strategies and impact of the defaults
Density hypothesis on defaults

- There exists $\alpha_T(\theta, e), \mathcal{F}_T \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$-measurable, s.t.

\[
(DH) \quad \mathbb{P}[(\tau, \zeta) \in d\theta de | \mathcal{F}_T] = \alpha_T(\theta, e)d\theta\eta(de)
\]

where $d\theta = d\theta_1 \ldots d\theta_n$ is the Lebesgue measure on $\mathbb{R}^n$, and
\[
\eta(de) = \eta_1(de_1) \prod_{k=1}^{n-1} \eta_{k+1}(e_k, de_{k+1}).
\]
Comments on density hypothesis

- Standard hypothesis in the theory of initial enlargement of filtrations, see Jacod (85). Insider problems in finance

- Density approach introduced in progressive enlargement of filtrations for credit risk modelling by El Karoui, Jeanblanc, Jiao (09,10) successive defaults without marks:
  - More general setting than intensity approach: one can express the intensity of each default time in terms of the density. Semimartingale invariance property \((H')\) holds and Immersion hypothesis \((H)\) (martingale invariance property) is not required.
Auxiliary survival density

- Let us define $\alpha_k^T$, $\mathcal{F}_T \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable, $k = 0, \ldots, n-1$, by recursive induction from $\alpha_n^T = \alpha_T$,

$$
\alpha_k^T(\theta_k, e_k) = \int_T^\infty \int_E \alpha_{k+1}^T(\theta_k, \theta, e_k, e) d\theta \eta_{k+1}(e_k, de),
$$

so that

$$
\mathbb{P}[\tau_{k+1} > T | \mathcal{F}_T] = \int_{\Delta_k \times E_k} \alpha_k^T(\theta_k, e_k) d\theta_k \eta(de_k),
$$

where $d\theta_k = d\theta_1 \ldots d\theta_k$, $\eta(de_k) = \eta_1(de_1) \ldots \eta_k(e_{k-1}, de_k)$. 

Decomposition result

The value function $V_0$ is obtained by backward induction from the optimization problems in the $\mathbb{F}$-filtration:

$$
V_n(x, \theta, e) = \underset{\pi^n \in A^n_{\mathbb{F}}}{\text{ess sup}} \mathbb{E} \left[ U(X^n_T, x) \alpha_T(\theta, e) \mid \mathcal{F}_{\theta_n} \right]
$$

$$
V_k(x, \theta_k, e_k) = \underset{\pi^k \in A^k_{\mathbb{F}}}{\text{ess sup}} \mathbb{E} \left[ U(X^k_T, x) \alpha^k_T(\theta_k, e_k) \right]
$$

$$
+ \int_{\theta_k}^{T} \int_{E} V_{k+1}(X_{\theta_{k+1}}^k, x + \pi_{\theta_{k+1}}^k \gamma_{\theta_{k+1}}^k(e_{k+1}), \theta_{k+1}, e_{k+1})
\eta_{k+1}(e_k, de_{k+1}) d\theta_{k+1} \mid \mathcal{F}_{\theta_k} \right].
$$
• This $\mathbb{F}$-decomposition of the $\mathbb{G}$-control problem can be viewed as a *nonlinear extension of Dellacherie-Meyer and Jeulin-Yor formula*, which relates linear expectation under $\mathbb{G}$ in terms of linear expectation under $\mathbb{F}$, and is used in option pricing for credit derivatives.

• Each step in the backward induction $\iff$ stochastic control problem in the $\mathbb{F}$-filtration (solved e.g. by dynamic programming and BSDE)
Consider an utility function:

\[ U(x) = - \exp(-px), \quad p > 0, \quad x \in \mathbb{R}. \]

and assume that \( \mathbb{F} = \mathbb{F}^W \) Brownian filtration generated by \( W \).

Then, the value functions \( V_k, k = 0, \ldots, n \), are given by

\[ V_k(x, \theta_k, e_k) = U(x - Y^k_{\theta_k}(\theta_k, e_k)), \]

where \( Y^k, k = 0, \ldots, n \), are characterized by means of a recursive system of (indexed) BSDEs, derived from dynamic programming arguments in the \( \mathbb{F} \)-filtration.
BSDE after $n$ defaults

$$Y_t^n(\theta, e) = \frac{1}{p} \ln \alpha_T(\theta, e) + \int_t^T f^n(r, Z_r^n, \theta, e)dr$$
$$- \int_t^T Z_r^n . dW_r, \quad t \geq \theta_n,$$

with a (quadratic) generator $f^n$:

$$f^n(t, z, \theta, e) = \inf_{\pi \in A^n} \left\{ \frac{p}{2} |z - \sigma_t^n(\theta, e)' \pi|^2 - b^n(\theta, e). \pi \right\}.$$ 

**Remark.** Similar BSDE as in El Karoui, Rouge (00), Hu, Imkeller, Müller (04), Sekine (06), for default-free market
BSDE after $k$ defaults, $k = 0, \ldots, n - 1$

$$Y^k_t(\theta_k, e_k) = \frac{1}{p} \ln \alpha^k_T(\theta_k, e_k)$$

$$+ \int_t^T f^k(r, Y_r^k, Z_r^k, \theta_k, e_k)dr - \int_t^T Z_r^k.dW_r, \quad t \geq \theta_k,$$

with a generator

$$f^k(t, y, z, \theta_k, e_k) = \inf_{\pi \in A^k} \left\{ \frac{p}{2} |z - \sigma_t^k(\theta_k, e_k)'\pi|^2 - b_t^k(\theta_k, e_k).\pi \right\}$$

$$+ \frac{1}{p} U(y) \int_E U(\pi.\gamma_t^k(e_{k+1}) - Y_t^{k+1}(\theta_k, t, e_k, e_{k+1}))$$

$$\eta_{k+1}(de_{k+1}) \right\}.$$
BSDE characterization of the optimal investment problem

**Theorem.** Under standard boundedness conditions on the coefficients of the model \((b, \sigma, \gamma, \alpha)\), there exists a unique solution \((Y, Z) = (Y_0, \ldots, Y^n, Z^0, \ldots, Z^n) \in S^\infty \times L^2\) to the recursive system of quadratic BSDEs. The initial value function is

\[
V_0(x) = U(x - Y_0^0),
\]

and the optimal strategies between \(\tau_k\) and \(\tau_{k+1}\) by

\[
\pi_t^k \in \arg \min_{\pi \in A_k} \left\{ \frac{p}{2} |Z_t^k - (\sigma_t^k)'\pi|^2 - b_t^k . \pi \right. \\
+ \frac{1}{p} U(Y_t^k) \int_E U(\pi . \gamma_t^k(e) - Y_{t+1}^k(t, e)) \eta_{k+1}(e_k, de) \right\}.
\]
Technical remarks

- Existence for the system of recursive BSDEs: quadratic term in $z$ + exponential term in $y$:
  - Kobylanski techniques + approximating sequence + convergence

- Uniqueness: verification arguments + BMO techniques

- We don’t need to assume boundedness condition on the portfolio control set
Default times density

- Two defaultable assets with default times \((\tau_1, \tau_2) \perp \mathbb{F}\).
  
  \(\tau_i \sim \mathcal{E}(a_i)\), and dependence of \((\tau_1, \tau_2)\) via a copula function:

  \[
  \mathbb{P}[\tau_1 \geq \theta_1, \tau_2 \geq \theta_2] = C(\mathbb{P}[\tau_1 \geq \theta_1], \mathbb{P}[\tau_2 \geq \theta_2])
  \]
  
  (Gumbel example) = \(\exp \left( - \left( (a_1 \theta_1)^\beta + (a_2 \theta_2)^\beta \right)^{1/\beta} \right)\),

  \(\beta \geq 1 \leftrightarrow \) nonnegative correlation between \(\tau_1\) and \(\tau_2\).

  - Density of \((\tau_1, \tau_2)\):

    \[
    \alpha^\tau(\theta_1, \theta_2) = a_1 a_2 e^{-a_1 \theta_1 - a_2 \theta_2} \frac{\partial^2 C}{\partial u_1 \partial u_2}(e^{-a_1 \theta_1}, e^{-a_2 \theta_2})
    \]

  - Density of ranked default times and index marks \((\hat{\tau}_1, \hat{\tau}_2, \iota_1, \iota_2)\):

    \[
    \alpha(\hat{\tau}_1, \hat{\tau}_2, i, j) = 1_{\{i=1, j=2\}} \alpha^\tau(\hat{\theta}_1, \hat{\theta}_2) + 1_{\{i=2, j=1\}} \alpha^\tau(\hat{\theta}_2, \hat{\theta}_1).
    \]
Defaultable assets

- Before any default: BS model for the two assets with drift $b^0$, volatility $\sigma^0$, correlation $\rho$.

- At default $\tau_i$ of asset $i = 1, 2$:
  - Asset $i$ drops to zero (no more traded)
  - Asset $j$ jumps by relative size $\gamma \in (-1, \infty)$: $\gamma < 0 \leftrightarrow$ loss, and $\gamma > 0 \leftrightarrow$ gain, and then follows a BS model with coefficients $b^1 = 0.01, \sigma^1 = 0.2$, until its default.

- Investment horizon $T = 1$. 
BSDEs as ODEs (I)

\[ Y^2(\theta, i, j) = \frac{1}{p} \ln \alpha(\theta, i, j), \quad \theta = (\theta_1, \theta_2) \in \Delta_2, \ i, j \in \{1, 2\}, \ i \neq j \]

\[ Y_{t}^{1,i}(\theta_1) = \frac{1}{p} \left[ \beta \ln a_i + (\beta - 1) \ln \theta_1 + \frac{1}{\beta} \ln((a_i \theta_1)^\beta + (a_j t)^\beta) \right. \]

\[ - \left. ((a_i \theta_1)^\beta + (a_j t)^\beta)^{1/\beta} \right] + \int_{t}^{T} f^{1,i}(s, Y_{s}^{1,i}, \theta_1) ds, \]

where

\[ f^{1,i}(t, y, \theta_1) = \inf_{\pi \in \mathbb{R}} \left\{ \frac{p}{2} |s_1^1 \pi|^2 - b_1 \pi + \frac{1}{p} e^{-p(y-\pi)} \alpha(\theta_1, t, i, j) \right\}, \]
\[ Y_t^0 = -\frac{T}{p} (a_1^\beta + a_2^\beta)^{1/\beta} + \int_t^T f^0(s, Y_s^0) ds, \]

where

\[ f^0(t, y) = \inf_{\pi = (\pi^1, \pi^2) \in \mathbb{R}^2} \left\{ \frac{p}{2} \left| (\sigma^0)' \pi \right|^2 - b^0 \cdot \pi + \frac{1}{p} e^{-py} \left[ e^{-p(-\pi^1 + \pi^2\gamma - Y_t^{1,1}(t))} + e^{-p(\pi^1\gamma - \pi^2 - Y_t^{1,2}(t))} \right] \right\}. \]
Value function $V^0(t)$ for different jump sizes

Figure: Value function $V^0(t)$: $a_1 = a_2 = 0.01$, $\beta = 2$
Optimal strategy in function of jump size for various default intensities

Figure: optimal strategy by varying intensity $a_1 = a_2$, and fixed $\beta = 2$
Optimal strategies in both assets by varying jump sizes and default intensities

Table: Optimal strategies $\hat{\pi}^1$ and $\hat{\pi}^2$ before any defaults with various $\gamma$ and default intensities.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$-0.5$</th>
<th>$-0.1$</th>
<th>$0$</th>
<th>$0.5$</th>
<th>$1$</th>
<th>Merton</th>
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<tbody>
<tr>
<td>$a_1 = 0.01, a_2 = 0.1, \beta = 2$</td>
<td>$0.462$</td>
<td>$1.659$</td>
<td>$1.892$</td>
<td>$2.621$</td>
<td>$2.832$</td>
<td>$2$</td>
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<tr>
<td>$\hat{\pi}^1$</td>
<td>$-1.047$</td>
<td>$-0.709$</td>
<td>$-0.498$</td>
<td>$0.623$</td>
<td>$1.168$</td>
<td>$2$</td>
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<tr>
<td>$\hat{\pi}^2$</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>$a_1 = 0.1, a_2 = 0.1, \beta = 2$</td>
<td>$-0.353$</td>
<td>$-0.210$</td>
<td>$-0.147$</td>
<td>$0.556$</td>
<td>$2$</td>
<td>$2$</td>
</tr>
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</tr>
<tr>
<td>$a_1 = 0.3, a_2 = 0.1, \beta = 2$</td>
<td>$-1.723$</td>
<td>$-1.719$</td>
<td>$-1.647$</td>
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</table>
Optimal strategies in both assets by varying correlation parameters

Table: Optimal strategies $\hat{\pi}^1$ and $\hat{\pi}^2$ before any defaults with various $\rho$ and $\beta$. $a_1 = 0.01$, $a_2 = 0.1$

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<thead>
<tr>
<th>$\gamma$</th>
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</tr>
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<td>-0.498</td>
<td>0.623</td>
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<td>$\rho = 0.3, \beta = 1$</td>
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<tr>
<td>$\rho = 0.3, \beta = 2$</td>
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<td>-0.817</td>
<td>-0.626</td>
<td>0.216</td>
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</table>
Concluding remarks (I)

- Beyond the optimal investment problem considered here, we provide a general formulation of stochastic control under progressive enlargement of filtration with multiple random times and marks:
  - Change of regimes in the state process, control set and gain functional after each random time
  - Includes in particular the formulation via jump-diffusion controlled processes

- Recursive decomposition on each default scenario of the $\mathcal{G}$-control problem into $\mathcal{F}$-stochastic control problems by relying on the density hypothesis
Concluding remarks (II)

- $\mathbb{F}$-decomposition method $\rightarrow$ another perspective for the study of controlled diffusion processes with (finite number of) jumps, (quadratic) BSDEs with (finite number of) jumps
  $\rightarrow$ Get rid of the jump terms
    - obtain comparison theorems under weaker conditions
    - Alternative approach for numerical schemes of BSDEs with jumps
  $\rightarrow$ Recent works by Kharroubi and Lim (11 a,b).