Dependence Estimation for High Frequency Sampled Multivariate CARMA Models

Vicky Fasen

The paper considers high frequency sampled multivariate continuous-time ARMA (MCARMA) models, and derives the asymptotic behavior of the sample autocovariance function to a normal random matrix. Moreover, we obtain the asymptotic behavior of the cross-covariances between different components of the model. We will see that the limit distribution of the sample autocovariance function has a similar structure in the continuous-time and in the discrete-time model. As special case we consider a CARMA (one-dimensional MCARMA) process. For a CARMA process we prove Bartlett’s formula for the sample autocorrelation function. Bartlett’s formula has the same form in both models, only the sums in the discrete-time model are exchanged by integrals in the continuous-time model. Finally, we present limit results for multivariate MA processes as well which are not known in this generality in the multivariate setting yet.

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1 Introduction

The paper considers multivariate ARMA (autoregressive moving average) models in continuous time and their dependence estimation. Multivariate time series have the advantage that they are able to model dependence between different time series in the most generality. A classical dependence measure for a multivariate stationary process \((Y_t)_{t \in \mathbb{R}}\) is the autocovariance function. The autocovariance function is defined as

\[
\Gamma_Y(h) = \mathbb{E}((Y_0 - \mathbb{E}(Y_0))(Y_h - \mathbb{E}(Y_h))^T), \quad h \in \mathbb{R}.
\]

An estimator for the autocovariance function is the sample autocovariance function.

One of the most known multivariate time series models is the VARMA\((p,q)\) (vector autoregressive moving average) process \((p,q \in \mathbb{N}_0)\) defined to be the stationary solution to a \(d\)-dimensional difference equation of the form

\[
P(B)Y_k = Q(B)\xi_k,
\]

where \(B\) is the backshift operator satisfying \(BY_k = Y_{k-1}\). \((\xi_k)_{k \in \mathbb{Z}}\) is a sequence of independent and identically distributed (iid) random vectors in \(\mathbb{R}^m\).

\[
P(z) := I_d z^p + P_1 z^{p-1} + \ldots + P_{p-1} z + P_p
\]

with \(P_1, \ldots, P_p \in \mathbb{R}^{d \times d}\) is the autoregressive polynomial and

\[
Q(z) := Q_0 z^q + Q_1 z^{q-1} + \ldots + Q_{q-1} z + Q_q
\]
with \( Q_0, \ldots, Q_p \in \mathbb{R}^{d \times m} \) is the moving average polynomial. In this article we always assume that \( \|z\|_2^2 < \infty \). If \( \det(P(z)) \neq 0 \) for all \( z \in \mathbb{C} \) with \( |z| \leq 1 \), then (1.1) has exactly one solution which has the moving average representation \( Y_t = \sum_{j=0}^{\infty} C_j \varepsilon_{t-j} \), where the \( C_j \) are uniquely determined by \( C(z) = \sum_{j=0}^{\infty} C_j z^{-j} = P(z)^{-1} Q(z) \) for \( |z| \leq 1 \) (cf. Brockwell and Davis (1991), Theorem 11.3.1). The interest in the asymptotic properties of the sample autocovariance and the autocorrelation function of ARMA (one-dimensional VARMA) processes has a long history starting with Bartlett (1955) and continuing with Anderson and Walker (1964), Hannan (1976), Cavazos-Cadena (1994) to name a few (cf. Brockwell and Davis (1991)). However, for VARMA processes the multivariate nature of the covariance matrix \( \Gamma(h) \) is a challenge, and hence, for a very long time people looked only at special cases like the asymptotic behavior of cross-covariances of bivariate Gaussian MA processes or independent MA processes (cf. Brockwell and Davis (1991), Fuller (1996)). Quite recently Su and Lund (2012) developed the asymptotic behavior of multivariate MA processes in a more general setup.

Multivariate continuous-time ARMA (MCARMA) processes \( Y = (Y_t)_{t \in \mathbb{R}} \) are the continuous-time versions of VARMA processes. The driving force of a MCARMA process is a Lévy process \( (L_t)_{t \geq 0} \). A Lévy process \( (L_t)_{t \geq 0} \) is defined to satisfy \( L_0 = 0 \) a.s. \( (L_t)_{t \geq 0} \) has independent and stationary increments and the paths of \( (L_t)_{t \geq 0} \) are stochastically continuous. An extension of a Lévy process \( (L_t)_{t \geq 0} \) is the stationary solution to the stochastic differential equation

\[
\frac{dY_t}{dt} = \alpha Y_t + \gamma Y_t \cdot \Lambda_t, \quad Y_0 = 0.
\]

where \( \alpha \) is a real number, \( \gamma \) is a positive semi-definite matrix in \( \mathbb{R}^{m \times m} \) and \( \Lambda_t \) is a measure on \( \mathbb{R}^{m} \) (\( \mathcal{F}(\mathbb{R}^{m}) \)), called Lévy measure, which satisfies \( \int_{\mathbb{R}^{m}} \min \{ \|x\|^2_2, 1 \} \text{d} \nu(x) < \infty \) and \( \nu(\{0\}) = 0 \). The triplet \( (\gamma, \Sigma, \nu) \) is called the characteristic triplet, because it characterizes completely the distribution of the Lévy process. For more details on Lévy processes we refer to the excellent monograph of Sato (1999).

Let \( L = (L_t)_{t \in \mathbb{R}} \) be a two-sided \( \mathbb{R}^m \)-valued Lévy process and let \( p > q \) be positive integers. Then the \( d \)-dimensional MCARMA \((p,q)\) process can be interpreted as the stationary solution to the stochastic differential equation

\[
P(D)Y_t = Q(D)DL_t \quad \text{for } t \in \mathbb{R},
\]

where \( D \) is the differential operator, and \( P, Q \) are given as in (1.2) respectively (1.3). By this representation we see the analogy to VARMA processes: the backshift operator \( B \) is replaced by the differential operator \( D \) and the id sequence \( \{\varepsilon_t\} \) by the Lévy process \( L \) which has independent and stationary increments. However, this is not the formal definition of a MCARMA process since a Lévy process is not differentiable; see Section C. The formal definition of MCARMA processes was first given in Marquardt and Stelzer (2007). Although the history of Gaussian CARMA (the one-dimensional MCARMA) processes is very old (cf. Doob (1944)) the interest in Lévy driven CARMA processes grew quickly in the last decade; see Brockwell (2005) for an overview. The well-known multivariate Ornstein-Uhlenbeck process is a typical example of a MCARMA process. CARMA processes are important for stochastic modeling in many areas of applications as, e.g., signal processing and control (cf. Garnier and Wang (2008), Larsson et al. (2006)), econometrics (cf. Berghström (1990)), and financial mathematics (cf. Andersen et al., 2014, Benth et al., 2014). Most of the literature restricts attention to CARMA processes which are easier to handle. Their exist only a few references which look at MCARMA processes such as Schlemm and Stelzer (2012), dealing with quasi-maximum-likelihood estimation, and Brockwell and Schlemm (2013), dealing with recovery of the driving Lévy process when the CARMA process is sampled on a discrete-time grid. From Fasen (2014) we already know that the sample mean and the sample covariance of a high frequency sampled CARMA process are consistent estimators for the expectation respectively the covariance.

In recent years interest in the modeling of high frequency data, as they occur in finance and turbulence, has increased rapidly (cf. Barndorff-Nielsen et al. (2013), Breyermann et al. (2003), Todorov (2009)). The estimation of the periodogram, normalized periodogram, smoothed periodogram and parameter estima-
tion in a high frequency sampled CARMA model is the topic of [Brockwell et al. (2013)]#1. Moreover, [Brockwell et al. (2013)]#1 develop a method to estimate the kernel function of high frequency sampled MA processes, and [Ferrazzano and Fuchs (2013a,b)]#2 estimate the increments of the driving Lévy process in high frequency sampled MA models.

The content of this paper is the asymptotic behavior of the sample autocovariance function of a high frequency sampled MCARMA process. The idea is that we have data \( Y_{\Delta n}, \ldots, Y_{n\Delta n} \) at hand where \( \Delta n \to 0 \) and \( n\Delta n \to \infty \) as \( n \to \infty \). We investigate the asymptotic behavior of the sample autocovariance function

\[
\hat{\Gamma}_n(h) = \frac{1}{n} \sum_{k=1}^{n-h/\Delta n} (Y_{k\Delta n} - \overline{Y}_n)(Y_{k\Delta n+h} - \overline{Y}_n)^T \quad \text{for } h \in \{0, \Delta n, \ldots, (n-1)\Delta n\},
\]

where \( \overline{Y}_n = n^{-1} \sum_{k=1}^{n} Y_{k\Delta n} \) is the sample mean, at different lags \( h \). To be more precise we study the joint asymptotic behavior of \( (\hat{\Gamma}_n(h))_{h \in \mathcal{H}} \) for some finite set \( \mathcal{H} \subseteq \bigcap_{n \geq m_0} \{0, \Delta n, 2\Delta n, \ldots, (n-1)\Delta n\}, m_0 \in \mathbb{N} \). We show that the sample autocovariance function is a consistent and an asymptotically normally distributed estimator for the autocovariance function. We present a very general representation of the limit random matrix which helps to understand the dependence between the components of the process quite well as, e.g., cross-covariances. A challenge is, on the one hand, the multivariate structure of the covariances which requires a basic knowledge of matrix calculations, and which would not be necessary if we restrict our attention only to CARMA processes; but also for CARMA processes the results are new. Without much effort we obtain likewise the analogous results for the sample autocovariance function of a multivariate MA process in discrete time extending the work of [Su and Lund (2012)]#3. The structure of the limit distributions of the sample autocovariance functions of multivariate MA and MCARMA processes are of an analogous form. Investigating the sample autocovariance and autocorrelation function in the one-dimensional models show that Bartlett’s formula for the autocovariance and the autocorrelation function have the same structure in the continuous-time and in the discrete-time model; only sums in the discrete-time model are integrals in the continuous-time model and the moments of the white noise are the moments of the Lévy process.

The paper is structured on the following way. We start with the formal definition of a MCARMA process and a short motivation for the asymptotic behavior of the sample autocovariance function of a high frequency sampled MCARMA process in Section 3. For the proof of the asymptotic behavior of the sample autocovariance function we require some preliminary limit results which are the topic of Section 3. This section is divided in two parts. The first part, Section 3.1 contains limit results for the high frequency sampled MCARMA processes, and the second part, Section 3.2 contains the proofs. The main section of this paper is Section 4, where we give the asymptotic behavior of the sample autocovariance function of high frequency sampled MCARMA processes including the asymptotic behavior of cross-covariances between the components of a MCARMA process and Bartlett’s formula for a CARMA process. Again a subsection contains the proofs. All presented estimators are consistent and asymptotically normally distributed. Finally, in Section 5 we introduce similar results for multivariate MA processes and compare both models.

**Notation**

We use the notation \( \Longrightarrow \) for weak convergence and \( \overset{p}{\longrightarrow} \) for convergence in probability. For two random vectors \( \mathbf{X}, \mathbf{Y} \) the notation \( \mathbf{X} \overset{d}{=} \mathbf{Y} \) means equality in distribution. The Euclidean norm in \( \mathbb{R}^d \) is denoted by \( \| \cdot \| \) and the corresponding operator norm for matrices by \( \| \cdot \| \), which is submultiplicative. Recall that two norms on a finite-dimensional linear space are always equivalent and hence, our results remain true if we replace the Euclidean norm by any other norm. For \( \mathbf{A} \in \mathbb{R}^{d \times m} \) the vec-operator \( \text{vec}(\mathbf{A}) \) is a vector in \( \mathbb{R}^{dm} \) which is obtained by stacking the columns of \( \mathbf{A} \). The Kronecker product of two matrices \( \mathbf{A} \in \mathbb{R}^{d \times m}, \mathbf{B} \in \mathbb{R}^{l \times k} \) is denoted by

\[
\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix}
A_{1,1} \mathbf{B} & A_{1,2} \mathbf{B} & \cdots & A_{1,m} \mathbf{B} \\
\vdots & \vdots & \ddots & \vdots \\
A_{d,1} \mathbf{B} & A_{d,2} \mathbf{B} & \cdots & A_{d,m} \mathbf{B}
\end{pmatrix} \in \mathbb{R}^{dl \times mk},
\]
where $A_{ij}$ denotes the entry of $A$ in the $i$-th row and in the $j$-th column. The matrix $0_{d \times m}$ is the zero matrix in $\mathbb{R}^{d \times m}$, $I_d$ is the identity matrix in $\mathbb{R}^{d \times d}$ and $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^d$. The representation $\text{diag}(u_1, \ldots, u_d)$ denotes a diagonal matrix in $\mathbb{R}^{d \times d}$ with diagonal entries $u_1, \ldots, u_d$. For some matrix $\Sigma \in \mathbb{R}^{d \times d}$ the representation $\Sigma = \Sigma^{1/2} \Sigma^{1/2}$ means there exists a matrix $A \in \mathbb{R}^{d \times d}$ such that $\Sigma = A \cdot A^T$ and $\Sigma^{1/2} := A$. For a vector $x \in \mathbb{R}^d$ we write $x^T$ for its transpose and for $x \in \mathbb{R}$ we write $[x] = \sup\{k \in \mathbb{Z} : k \leq x\}$ and $[x] = \inf\{k \in \mathbb{Z} : x \leq k\}$. The space $(\mathbb{D}[0,T],\mathbb{R}^d)$ denotes the space of all càdlàg (continue à droite et limitée à gauche = right continuous, with left limits) functions on $[0,T]$ ($T > 0$) with values in $\mathbb{R}^d$ equipped with the Skorokhod $J_1$ topology.

**Matrix calculation**

We would like to repeat some calculation rules for Kronecker products which are used throughout the paper; for details we refer to Bernstein (2009). Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ be vectors and $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times l}$, $C \in \mathbb{R}^{l \times k}$, $D \in \mathbb{R}^{k \times n}$ be matrices. Then

\[
\begin{align*}
xy^T &= x \otimes y^T = y^T \otimes x, \\
\text{vec}(ABC) &= (C^T \otimes A) \text{vec}(B), \\
\text{vec}(AB^T) &= A^T \otimes B^T, \\
(A \otimes C)(B \otimes D) &= (AB) \otimes (CD).
\end{align*}
\] (1.6)

The matrix $P_{m,m} = \sum_{i,j=1}^{m} e_i e_i^T \otimes e_j e_j^T \in \mathbb{R}^{m^2 \times m^2}$ is the Kronecker permutation matrix. For $A, B \in \mathbb{R}^{m \times m}$ and $x, y \in \mathbb{R}^m$ it has the property

\[
P_{m,m}^2 = I_m, \quad P_{m,m} \cdot (x \otimes y) = y \otimes x, \quad P_{m,m} \cdot (A \otimes B) = (B \otimes A)P_{m,m},
\] (1.7)

see Bernstein (2009), Fact 7.4.30, where several properties of the Kronecker permutation matrix are listed.

**2 MCARMA processes**

In this section we present some background on multivariate continuous-time ARMA (MCARMA) processes. Since a Lévy process is not differentiable, the differential equation (1.4) cannot be used as definition of a MACARMA process. However, it can be interpreted to be equivalent to the following definition, see Marquardt and Stelzer (2007).

**Definition 2.1.** Let $(\mathbf{L}_t)_{t \in \mathbb{R}} = (\mathbf{L}_1(t), \ldots, \mathbf{L}_m(t))_{t \in \mathbb{R}}$ be an $\mathbb{R}^m$-valued Lévy process and let the polynomials $P(z), Q(z)$ be defined as in (1.2) and (1.3) with $p, q \in \mathbb{N}_0$, $q < p$, and $Q_0 \neq 0_{d \times m}$. Moreover, define

\[
\Lambda = \begin{pmatrix}
0_{d \times d} & I_d & 0_{d \times d} & \cdots & 0_{d \times d} \\
0_{d \times d} & 0_{d \times d} & I_d & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0_{d \times d} & \cdots & 0_{d \times d} & I_d & 0_{d \times d} \\
-\mathbf{P}_p & -\mathbf{P}_{p-1} & \cdots & \cdots & -\mathbf{P}_1
\end{pmatrix} \in \mathbb{R}^{pd \times pd},
\]

\[
\mathbf{E} = (I_d, 0_{d \times d}, \ldots, 0_{d \times d}) \in \mathbb{R}^{d \times pd} \quad \text{and} \quad \mathbf{B} = (\mathbf{B}_1^T \cdots \mathbf{B}_p^T)^T \in \mathbb{R}^{pd \times m}
\]

with

\[
\mathbf{B}_j := \cdots := \mathbf{B}_{p-1} := 0_{d \times m} \quad \text{and} \quad \mathbf{B}_{p-j} := -\sum_{i=1}^{p-j-1} \mathbf{P}_i \mathbf{B}_{p-j-i} + \mathbf{Q}_{q-j} \quad \text{for} \quad j = 0, \ldots, q.
\]

Assume $\{z \in \mathbb{C} : \det(P(z)) = 0\} \subseteq \{z \in \mathbb{C} : \det(-\Lambda - 2\mathbf{I}_{pd}) = 0\} \subseteq (-\infty, 0) + i\mathbb{R}$. Furthermore, the Lévy measure $\nu_k$ of $\mathbf{L}$ satisfies $\int_{|x| > 1} \log |x| \, \nu_k(dx) < \infty$. Then the $\mathbb{R}^d$-valued causal MCARMA$(p, q)$ process $(\mathbf{Y}_t)_{t \in \mathbb{R}}$ is defined by the state-space equation

\[
\mathbf{Y}_t = \mathbf{E} \mathbf{Z}_t \quad \text{for} \quad t \in \mathbb{R},
\] (2.1)
where
\[ Z_t = \int_{t}^{\infty} e^{-\Lambda(t-s)} B \, dL_s \quad \text{for } t \in \mathbb{R}. \]  
(2.2)

is the stationary unique solution to the \(pd\)-dimensional stochastic differential equation
\[ dZ_t = -\Lambda Z_t \, dt + B \, dL_t. \]  
The function \( f(t) = E e^{-\Lambda t} B_1(0,t) \) for \( t \in \mathbb{R} \) is called the kernel function.

It is well known that the stationary Ornstein-Uhlenbeck process \( Z \) given in (2.2) observed at the time-grid \( \Delta_n \mathbb{Z} = \{ \ldots, -2\Delta_n, -\Delta_n, 0, \Delta_n, 2\Delta_n, \ldots \} \) with \( \Delta_n \) a positive constant has a representation as a MA process
\[ Z_{k\Delta_n} = \sum_{j=0}^{\infty} e^{-\Lambda \Delta_n j} \xi_{n,k-j} \quad \text{for } k \in \mathbb{Z}, \]
where \( (\xi_{n,k})_{k \in \mathbb{N}} \) is a sequence of iid random vectors in \( \mathbb{R}^{pd} \) with
\[ \xi_{n,k} = \int_{(k-1)\Delta_n}^{k\Delta_n} e^{-\Lambda(k\Delta_n-t)} B \, dL_t \quad \text{for } k \in \mathbb{Z}, n \in \mathbb{N}. \]  
(2.3)

To derive the asymptotic behavior of the sample autocovariance function \( \hat{\Gamma}_n(h) \) as given in (1.5) we have to prove several intermediate steps. First, let us define
\[ \hat{\Gamma}_n(h) = \frac{1}{n} \sum_{k=1}^{n} Y_{k\Delta_n} Y_{k\Delta_n+h}^T. \]  
(2.4)

We will show that \( \sqrt{n\Delta_n} (\hat{\Gamma}_n(h) - \Gamma(h)) = o_p(1) \) so that it is sufficient to investigate the asymptotic behavior of \( \hat{\Gamma}_n(h) \). By the Beveridge-Nelson decomposition we are able to show that
\[ \sqrt{n\Delta_n} (\hat{\Gamma}_n(h) - \Gamma(h)) = \Delta_n \sum_{j=0}^{\infty} \mathop{E} e^{-\Lambda \Delta_n j} \left( \frac{1}{\sqrt{n\Delta_n}} \sum_{k=1}^{n} \xi_{n,k} \xi_{n,k}^T - \mathbb{E} (\xi_{n,k}^{(T)} \xi_{n,k}) \right) e^{-\Lambda (\Delta_n + h) T} E^T \]
\[ + \Delta_n \sum_{j=0}^{\infty} E^T e^{-\Lambda \Delta_n j} \left( \frac{1}{\sqrt{n\Delta_n}} \sum_{k=1}^{n} \sum_{r=1}^{\infty} \xi_{n,k} \xi_{n,k+r}^T e^{-\Lambda (h+\Delta_n r)} + \sum_{r=1}^{\infty} \sum_{r=h/\Delta_n}^{\infty} \xi_{n,k} \xi_{n,k+r}^T e^{-\Lambda (h-\Delta_n r)} + \sum_{r=h/\Delta_n}^{\infty} \int_{r+1}^{\infty} e^{-\Lambda \Delta_n (h-r)} \xi_{n,k} \xi_{n,k+r}^T \right) e^{-\Lambda \Delta_n / E^T} + o_p(1). \]  
(2.5)

This representation is not obvious and will first be developed on pp. [19]. From this we see that we have to understand the joint limit behavior of the four terms in the brackets in (2.5), and this is what we will do in the next section.

3 Limit results for processes with finite fourth moments

3.1 Models in continuous time

The main ingredient to derive the asymptotic behavior of the sample autocovariance function for high frequency sampled MCARMA processes is the following joint limit result of the four terms in the brackets in (2.5).

**Proposition 3.1.** Let \( \{Y_t\}_{t \geq 0} \) be a MCARMA process as defined in (2.1) with \( \mathbb{E} \|L_1\|^4 < \infty \), \( \mathbb{E}(L_1) = 0_m \) and \( \mathbb{E}(L_1 L_1^T) = \Sigma = \Sigma^{1/2} \cdot \Sigma^{1/2} \in \mathbb{R}^{m \times m} \). The sequence \( (\xi_{n,k}) \) is defined as in (2.3) and
\[ Y := \int_{\mathbb{R}^{m}} xx^T \otimes xx^T v(dx). \]  
5
Suppose \((\Delta_n)_{n \in \mathbb{N}}\) is a sequence of positive constants with \(\Delta_n \downarrow 0\) and \(\lim_{n \to \infty} n\Delta_n = \infty\). We assume there exists a sequence of positive constants \(l_n \to \infty\) with \(n/l_n \to \infty\) and \(l_n\Delta_n \to \infty\). Let \(\mathcal{H} \subseteq [0, \infty)\) be a finite set. Then as \(n \to \infty\),

\[
\left(\frac{1}{\sqrt{n\Delta_n}} \sum_{r=1}^{n} \left[ \xi_{n,k}^T e^{-\Lambda(h+\Delta_n r)} \right] \right) \xrightarrow{\mathcal{H}} 0_{(pd)^2 \times (pd)^2}.
\]

(b) When \(L\) has independent components then \(W^*(Y)\) reduces to a much simpler random matrix. Define

\[
\theta_i := \int_0^1 v_i(\omega) d\omega = E(L_i(1)^4) - 3E(L_i(1)^2), \quad i = 1, \ldots, m,
\]

where \(L_i\) is the \(i\)-th component of \(L\) with Lévy measure \(v_i\). Then \(Y = \sum_{i=1}^m \left[ (e_i e_i^T \otimes e_i e_i^T) \cdot \theta_i \right] \), and thus,

\[
W^*(Y) \overset{d}{=} \sum_{i=1}^m \sqrt{\theta_i} N_i e_i e_i^T = \text{diag}(\sqrt{\theta_1 N_1}, \ldots, \sqrt{\theta_m N_m}),
\]

where \(N_1, \ldots, N_m\) are iid \(\mathcal{N}(0, 1)\)-distributed. In particular, we obtain in the one-dimensional case as \(n \to \infty\),

\[
\left(\frac{1}{\sqrt{n\Delta_n}} \sum_{k=1}^n \left[ \xi_{n,k}^2 - E(\xi_{n,1}^2) \right] \right) \xrightarrow{\mathcal{H}} \mathcal{N}(0, E(L_1^4) - 3E(L_1^2)). \tag{3.1}
\]

However, if \((\xi_k)_{k \in \mathbb{N}}\) is an iid sequence with \(E(\xi_k) = 0\) and \(|E(\xi_k)|^4 < \infty\) then obviously by the classical central limit theorem of Lindeberg-Lévy as \(n \to \infty\),

\[
\left(\frac{1}{\sqrt{n\Delta_n}} \sum_{k=1}^n \left[ \xi_k^2 - E(\xi_k^2) \right] \right) \xrightarrow{\mathcal{H}} \mathcal{N}(0, E(\xi_k^4) - E(\xi_k^2)^2). \tag{3.2}
\]

The limit in (3.1) has a factor 3 which does not appear in (3.2). \(\square\)

The proof of Proposition 3.1 is based on some limit results which are interesting on their own. The main task is to derive Proposition 3.2.

**Proposition 3.2.** Let \((\xi_{n,k})_{k \in \mathbb{N}}\) be a sequence of iid random vectors in \(\mathbb{R}^d\) with \(E||\xi_{n,k}||^4 < \infty\), \(E(\xi_{n,1}) = 0_{pd}\) and \(E(\xi_{n,1}^2 e_i e_i^T) = \Sigma_n = \Sigma^{1/2} \cdot \Lambda_n^{1/2} \in \mathbb{R}^{pd \times pd}\) for any \(n \in \mathbb{N}\). Suppose \((\Delta_n)_{n \in \mathbb{N}}\) is a sequence of positive constants with \(\Delta_n \downarrow 0\) and \(\lim_{n \to \infty} n\Delta_n = \infty\). We assume that there exists a sequence of positive constants \(l_n \to \infty\) with \(n/l_n \to \infty\) and \(l_n\Delta_n \to \infty\). Moreover, \(\Delta_n^{-1} \Sigma_n \to \Sigma = \Sigma^{1/2} \cdot \Sigma^{1/2} \in \mathbb{R}^{pd \times pd}\) as...
\[
\lim_{n \to \infty} \Delta_n^{-1} \mathbb{E}(\xi_{n,1}^T \otimes \xi_{n,1}^T) = Y \in \mathbb{R}^{(pd)^2 \times (pd)^2}.
\]

(3.3)

Define for \( t \geq n \), \( n \in \mathbb{N} \),

\[
S_n^{(1)}(t) = \sum_{j=1}^n \sum_{k=j}^n \xi_{n,k}^T \xi_{n,k-j}^T, \quad S_n^{(2)}(t) = \sum_{j=1}^n \sum_{k=j}^n \xi_{n,k}^T \xi_{n,k+j}^T \quad \text{and} \quad S_n^{(3)} = \sum_{k=1}^n \xi_{n,k}^T \xi_{n,k}^T - \sum_n.
\]

Then as \( n \to \infty \),

\[
\frac{1}{\sqrt{n\Delta_n}} \mathbb{E}(S_n(t)) = \left( \frac{1}{\sqrt{n\Delta_n}} S_n^{(1)}(t), \frac{1}{\sqrt{n\Delta_n}} S_n^{(2)}(t), \frac{1}{\sqrt{n\Delta_n}} S_n^{(3)} \right)_{t \in [0,T]}
\]

in \( D([0,T], \mathbb{R}^{pd \times pd} \times \mathbb{R}^{pd \times pd} \times \mathbb{R}^{pd \times pd}) \) where \( \mathbf{W}^*(Y) \) is an \( \mathbb{R}^{pd \times pd} \)-valued normal random matrix with \( \text{vec} (\mathbf{W}^*(Y)) \sim \mathcal{N}(0_{(pd)^2}, Y) \) independent from the \( \mathbb{R}^{pd \times pd} \)-valued standard Brownian motion \( (\mathbf{W}_t)_{t \geq 0} \).

A conclusion of Proposition 3.2 and a continuous mapping theorem is Proposition 3.3. Proposition 3.1 can be seen as special case of Proposition 3.3, we have only to check that the assumptions are satisfied.

**Proposition 3.3.** Let the assumptions of Proposition 3.2 hold. Suppose \( \mathbf{g}_t : [0, \infty) \to \mathbb{R}^{pd \times pd} \) \((l = 1, \ldots, M)\) are maps with finite variation and \( \int_0^\infty \| \mathbf{g}_t(s) \|^2 \, ds < \infty \). Then as \( n \to \infty \),

\[
\frac{1}{\sqrt{n\Delta_n}} \int_0^\infty \mathbf{g}_t(s) S_n^{(1)}(ds), \mathbf{g}_t(s) S_n^{(2)}(ds), \mathbf{g}_t(s) S_n^{(3)}(ds) \right)_{t = 1, \ldots, M}
\]

in \( W([0,T], \mathbb{R}^{pd \times pd} \times \mathbb{R}^{pd \times pd} \times \mathbb{R}^{pd \times pd}) \).

Now for the proof of Proposition 3.1, we have mainly to check that the assumptions of Proposition 3.3 are satisfied, in particular that (3.3) holds. [Asmussen and Rosiński (2001)]. Lemma 3.1, already derived the limit behavior \( \lim_{n \to \infty} \Delta_n^{-1} \mathbb{E}(S_n) \) for an one-dimensional Lévy process. We have to extend this result to a multivariate Lévy process and use it to show (3.3).

**Lemma 3.1.**

(a) Let \( \mathbf{L} = (L_t)_{t \geq 0} \) be a Lévy process with \( \mathbb{E}\| L_1 \|^4 < \infty, \mathbb{E}(L_1) = 0_{m} \), and \((\Delta_n)_{n \in \mathbb{N}}\) be a sequence of positive constants with \( \lim_{n \to \infty} \Delta_n = 0 \). Then

\[
\lim_{n \to \infty} \Delta_n^{-1} \mathbb{E}(L_{\Delta_n} L_{\Delta_n}^T) = \int_{\mathbb{R}^m} xx^T \otimes xx^T \nu(dx).
\]

(b) Let the assumptions of Proposition 3.1 hold. Then

\[
\lim_{n \to \infty} \Delta_n^{-1} \mathbb{E}(\xi_{n,1}^T \xi_{n,1}) = B \otimes B \left( \int_{\mathbb{R}^m} xx^T \otimes xx^T \nu(dx) \right) B^T \otimes B^T.
\]

3.2 Proofs

3.2.1 Auxiliary results for the proof of Proposition 3.2

For the proof of Proposition 3.2, we derive some auxiliary results. First, we want to characterize the limit process \((\Sigma_1^{1/2} \mathbf{W}_t, \Sigma_1^{1/2} \mathbf{W}_t^T, \Sigma_2^{1/2} \Sigma_2^{1/2} \mathbf{W}_t^T, \Sigma_2^{1/2} \Sigma_2^{1/2} \mathbf{W}_t)_{t \geq 0} \).
Lemma 3.2. Let \( P_{pd, pd} = \sum_{i,j=1}^{pd} e_i e_j^T \otimes e_i e_j^T \in \mathbb{R}^{(pd)^2 \times (pd)^2} \) be the Kronecker permutation matrix and
\[
\Sigma^* := \begin{pmatrix}
\Sigma^{1/2} \otimes \Sigma^{1/2} & 0_{(pd)^2 \times (pd)^2} \\
0_{(pd)^2 \times (pd)^2} & \Sigma^{1/2} \otimes \Sigma^{1/2}
\end{pmatrix}.
\]
Then \( \text{vec}(\Sigma^{1/2} W_i \Sigma^{1/2 T}, \Sigma^{1/2} W_i^T \Sigma^{1/2 T}) )_{i \geq 0} \) is a Brownian motion with covariance matrix \( \Sigma^* \).

Proof.
The reason is that since \( \text{vec}(\Sigma^{1/2} W_i \Sigma^{1/2 T}) = \Sigma^{1/2} \otimes \Sigma^{1/2} \text{vec}(W_i) \) we have
\[
\text{vec}(\Sigma^{1/2} W_i \Sigma^{1/2 T}, \Sigma^{1/2} W_i^T \Sigma^{1/2 T}) = \begin{pmatrix}
\Sigma^{1/2} \otimes \Sigma^{1/2} & 0_{(pd)^2 \times (pd)^2} \\
0_{(pd)^2 \times (pd)^2} & \Sigma^{1/2} \otimes \Sigma^{1/2}
\end{pmatrix} \text{vec}(W_i, W_i^T).
\]
Some straightforward calculations give
\[
\mathbb{E}(\text{vec}(W_i) \otimes \text{vec}(W_i^T)^T) = \mathbb{E}(W_i \otimes W_i^T) = \sum_{i,j=1}^{pd} e_i e_j^T \otimes e_j e_i^T = P_{pd, pd},
\]
and thus,
\[
\mathbb{E}(\text{vec}(\Sigma^{1/2} W_i \Sigma^{1/2 T}, \Sigma^{1/2} W_i^T \Sigma^{1/2 T}) \cdot \text{vec}(\Sigma^{1/2} W_i \Sigma^{1/2 T}, \Sigma^{1/2} W_i^T \Sigma^{1/2 T})^T) = \Sigma^*.
\]
The stationary and independent increment property of the Brownian motion \((W_i)_{i \geq 0}\) transfers to
\( (\text{vec}(\Sigma^{1/2} W_i \Sigma^{1/2 T}, \Sigma^{1/2} W_i^T \Sigma^{1/2 T}) )_{i \geq 0} \) such that the conclusion follows.

Next we prove the convergence of \( S_n(3) \) alone which is more or less straightforward.

Lemma 3.3. \( \frac{1}{\sqrt{n \Lambda_n}} \sum_{k=1}^{n} \left[ \xi_{n,k} \otimes \xi_{n,k} - \text{vec}(\Sigma_n) \right] \overset{d}{\to} \mathcal{N}(0_{(pd)^2}, \Upsilon) \) as \( n \to \infty \).

Proof.
By assumption \((\xi_{n,k} \otimes \xi_{n,k})_{k \in \mathbb{N}}\) is a sequence of iid random vectors with \( \mathbb{E}(\xi_{n,k} \otimes \xi_{n,k}) = \text{vec}(\Sigma_n) \) and
\[
\lim_{n \to \infty} \frac{1}{\Lambda_n} \mathbb{E}(\text{vec}(\Sigma_n) - \text{vec}(\Sigma_n T) - \text{vec}(\Sigma_n)^T) = \Upsilon,
\]
where we used that \( \lim_{n \to \infty} \frac{1}{\Lambda_n} \text{vec}(\Sigma_n) \rightarrow 0_{(pd)^2 \times (pd)^2} \). It remains to show the Lindeberg-condition so that we can apply the central limit theorem of Lindeberg-Feller (see [Jacod and Shiryaev, 2003, Theorem VII.5.2]). Let \( \varepsilon > 0 \). As in [Fasen, 2014, Proposition A.1(d)] using [Asmussen and Rosiński, 2001, Lemma 3.1] it is possible to prove that
\[
\lim_{n \to \infty} \frac{1}{\Lambda_n} \mathbb{E}(\left\| \xi_{n,1} \right\|^4, \left\{ \left\| \xi_{n,1} \right\| > \varepsilon \sqrt{\Lambda_n} \right\}) = 0.
\]
Thus, the central limit theorem of Lindeberg-Feller gives the desired weak convergence as \( n \to \infty \).

Now we are able to prove the convergence of the two-dimensional distribution in Proposition 3.2 before we prove the convergence of the stochastic process.

Lemma 3.4. Let \( 0 \leq s < t < \infty \). Then as \( n \to \infty \),
\[
\frac{1}{\sqrt{n \Lambda_n}} (S_n^{(1)}(s), S_n^{(1)}(t) - S_n^{(1)}(s), S_n^{(2)}(s) + S_n^{(2)}(t) - S_n^{(2)}(s), S_n^{(3)}(s)) \overset{d}{\to} (\Sigma^{1/2} W_s \Sigma^{1/2 T}, \Sigma^{1/2} (W_t - W_s) \Sigma^{1/2 T}, \Sigma^{1/2} W_s^T \Sigma^{1/2 T}, \Sigma^{1/2} (W_t - W_s) \Sigma^{1/2 T}, W^*(\Upsilon)).
\]
Proof.

The proof uses Cramér-Wold theorem. Thus, let \( c_1, c_2 \in \mathbb{R}^{2(pd)^2}, \), \( c_3 \in \mathbb{R}^{(pd)^2} \) and define
\[
S_n^* := c_1^T \text{vec}(S_n^{(1)}(s), S_n^{(2)}(s)) + c_2^T \text{vec}(S_n^{(1)}(t), S_n^{(2)}(t) - S_n^{(2)}(s)) + c_3^T \text{vec}(S_n^{(3)})
\]
\[
= \sum_{k=1}^n \left[ \sum_{j=1}^{[t/\Delta_n]} c_1^T \text{vec}(\xi_{n,k-j}; \xi_{n,k+j})^T + \sum_{j=[t/\Delta_n]+1}^{[t/\Delta_n]+l_{n/\Delta_n}} c_2^T \text{vec}(\xi_{n,k-j}; \xi_{n,k+j})^T + c_3^T \text{vec}(\xi_{n,k} \xi_{n,k}^T - \Sigma_n) \right] =: \sum_{k=1}^n Z_{n,k},
\]
with
\[
Z_{n,k} := \sum_{j=1}^{[s/\Delta_n]} c_1^T \text{vec}(\xi_{n,k-j}; \xi_{n,k+j})^T + \sum_{j=[t/\Delta_n]+1}^{[t/\Delta_n]+l_{n/\Delta_n}} c_2^T \text{vec}(\xi_{n,k-j}; \xi_{n,k+j})^T + c_3^T \text{vec}(\xi_{n,k} \xi_{n,k}^T - \Sigma_n).
\]
Moreover, define \( Z_{n,k,j} := \text{vec}(\xi_{n,k-j}, \xi_{n,k+j})^T \) such that
\[
Z_{n,k} = \sum_{j=1}^{[s/\Delta_n]} c_1^T Z_{n,k,j} + \sum_{j=[t/\Delta_n]+1}^{[t/\Delta_n]+l_{n/\Delta_n}} c_2^T Z_{n,k,j} + c_3^T \text{vec}(\xi_{n,k} \xi_{n,k}^T - \Sigma_n).
\]
We will prove that \( \frac{1}{\sqrt{n \Delta_n^1}} S_n^* \) converges weakly to a normal distribution with mean 0 and variance
\[
s c_1^T \Sigma^* c_1 + (t-s)c_2^T \Sigma^* c_2 + c_3^T \Sigma c_3 =: \Sigma(c_1, c_2, c_3),
\]
where \( \Sigma^* \) is as in (3.6). We take the sequence \( (l_n) \) where \( l_n \rightarrow \infty, n/l_n \rightarrow \infty \) and \( l_n \Delta_n \rightarrow \infty \) and assume for the ease of notation that \( l_n \) and \( t \Delta_n^{-1} \) are integers. Write
\[
S_n^* = \sum_{k=1}^n \left[ \sum_{i=1}^{[n/\Delta_n]} \sum_{k=(i-1)\Delta_n+1}^{i\Delta_n-1} Z_{n,k}^* \right] + \sum_{k=1}^{[t/\Delta_n]+l_{n/\Delta_n}} Z_{n,k}^* + \sum_{k=1}^n \sum_{i=1}^{[n/\Delta_n]-1} Z_{n,k}^* + \sum_{k=[n/\Delta_n]+1}^n Z_{n,k}^* =: S_{1,n}^* + S_{2,n}^*.
\]
The proof is divided in two parts. On the one hand, we have to show that \( \frac{1}{\sqrt{n \Delta_n^1}} S_{1,n}^* \stackrel{d}{\rightarrow} \mathcal{N}(0, \Sigma(c_1, c_2, c_3)) \) and on the other hand, that \( \frac{1}{\sqrt{n \Delta_n^2^2}} S_{2,n}^* \rightarrow 0 \). We derive the weak convergence of the first term with the central limit theorem of Lindeberg-Feller. Therefore, we require some auxiliary results: the asymptotic behavior of the covariance matrix of \( S_{1,n}^* \) (Lemma 3.5) and the Lindeberg condition (Lemma 3.6).

Lemma 3.5. Let \( \Sigma(c_1, c_2, c_3) \) be given as in (3.8) and \( S_{1,n}^* \) as in (3.9). Then
\[
\lim_{n \rightarrow \infty} (n \Delta_n)^{-1} \mathbb{E}(S_{1,n}^* S_{1,n}^{*T}) = \Sigma(c_1, c_2, c_3).
\]

Proof.

We start by calculating the asymptotic covariance matrix of
\[
Z_{n,k,j} = \left( \xi_{n,k-j}, \xi_{n,k+j} \right) \otimes \xi_{n,k} \quad \text{where} \quad Z_{n,k,j} Z_{n,k,j}^* = \left( \begin{array}{cc} \xi_{n,k-j} \xi_{n,k-j}^T \xi_{n,k-j} \xi_{n,k-j}^T & \xi_{n,k-j} \xi_{n,k-j}^T \xi_{n,k-j} \xi_{n,k-j}^T \\ \xi_{n,k+j} \xi_{n,k+j}^T \xi_{n,k+j} \xi_{n,k+j}^T & \xi_{n,k+j} \xi_{n,k+j}^T \xi_{n,k+j} \xi_{n,k+j}^T \end{array} \right) \otimes \xi_{n,k} \xi_{n,k}^T.
\]
Having in mind that \((\xi_{n,k})_{k \in \mathbb{N}}\) is an iid sequence with \(\mathbb{E}(\xi_{n,k}) = 0\), we get on the one hand,

\[
\Delta_n^{-2} \mathbb{E}(Z_{n,k} Z_{n,k}^T) \xrightarrow{n \to \infty} \left(\begin{array}{c}
\Sigma_1^{-1} \\
\Sigma_2^{-1}
\end{array}\right) =: \Sigma^*_1,
\]

and on the other hand,

\[
\Delta_n^{-2} \mathbb{E}(Z_{n,k} Z_{n,k+j,j}^T) = \Delta_n^{-2} \left(\begin{array}{c}
\Sigma_1^{-1} \\
\Sigma_2^{-1}
\end{array}\right) =: \Sigma^*_2,
\]

compare (3.5). Moreover, \(\mathbb{E}(Z_{n,k} Z_{n,m,l}^T) = 0\) if \((m, l) \notin \{(k, j), (k + j, j), (k - j, j)\}\). Finally, by (3.7) and (3.9) we obtain with

\[
(n \Delta_n)^{-1} \mathbb{E}(S_1 n^{-1} n_h)
\]

the desired result. 

**Lemma 3.6.** The Lindeberg condition

\[
\lim_{n \to \infty} \frac{1}{n \Delta_n} \sum_{i=1}^{n \Delta_n} \mathbb{E} \left( \frac{d_{n,i} - \Delta_n^{-1} d_{n,i}}{\Delta_n} \right)^2 = 0, \quad \epsilon > 0,
\]

is satisfied.

**Proof.**

In connection to (3.7) let us define for \(n, i \in \mathbb{N},\)

\[
\tilde{Z}_{n,i} = \sum_{k=(i-1) \Delta_n + \Delta_n^{-1}}^{d_{n,i} - \Delta_n^{-1}} \frac{c^T}{\Delta_n} (\xi_{n,k} \xi_{n,k}^T - \Sigma_n) \quad \text{and} \quad Z_{n,i}^* = \left(\begin{array}{c}
\sum_{k=(i-1) \Delta_n + \Delta_n^{-1}}^{d_{n,i} - \Delta_n^{-1}} Z_{n,k}
\end{array}\right)^\top - \tilde{Z}_{n,i}.
\]

Then

\[
\mathbb{E} \left( \frac{d_{n,i} - \Delta_n^{-1} d_{n,i}}{\Delta_n} \right)^2 = 2 \mathbb{E}(\tilde{Z}_{n,1}^2) + \mathbb{E}(\tilde{Z}_{n,1}^2 | \{Z_{n,1} > \epsilon / 2 \sqrt{n \Delta_n}\}).
\]

(3.10)
By the central limit result in Lemma 3.3, \( \frac{1}{n} \) converges as \( n \to \infty \) and \( \text{Example 28.4} \) the Lindeberg-condition

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \left| Z_{n,1} \right|^2 I_{\left| Z_{n,1} \right| > \varepsilon / \sqrt{n \Delta_n}} \right) = 0
\]

holds. For the second term in (3.10) we use the Liapunov condition. Therefore, note that

\[
\frac{1}{(n^2 \Delta_n)^2} \mathbb{E} \left[ \left| Z_{n,1}^* \right|^4 \right] \leq \text{const.}
\]

To see this, define the vectors \( c_i^{(1)} = (c_{i,1}, \ldots, c_{i,pd^2}) \in \mathbb{R}^{pd^2} \) which contains the first \( pd^2 \)-components of the vector \( c_i \) respectively, \( c_i^{(2)} = (c_{i,(pd^2+1)}, \ldots, c_{i,2(pd^2)}) \in \mathbb{R}^{pd^2} \) which contains the last \( pd^2 \)-components of the vector \( c_i \) (\( i = 1, 2 \)). Moreover, for \( n, k \in \mathbb{N} \),

\[
Z_{n,k}^{(1)} := \sum_{j=1}^{[\sqrt{n} \Delta_n]} c_1^{(1)} T \cdot (\xi_{n,k} \otimes \xi_{n,k-j}^T), \quad Z_{n,k}^{(2)} := \sum_{j=1}^{[\sqrt{n} \Delta_n]} c_2^{(2)} T \cdot (\xi_{n,k} \otimes \xi_{n,k-j}^T),
\]

Since by assumption \((\xi_{n,k})_{k \in \mathbb{N}}\) is an iid sequence with \( \mathbb{E}(\xi_{n,k}) = 0 \) \( pd \), \( \mathbb{E}(\xi_{n,k}^2) < \text{const.} \cdot \Delta_n \), and \( \mathbb{E}(\xi_{n,k}^4) < \text{const.} \cdot \Delta_n \), the sequence of random variables \((Z_{n,k})_{k \in \mathbb{N}}\) is an uncorrelated sequence with \( \mathbb{E}(Z_{n,k}^2) = 0 \), \( \mathbb{E}((Z_{n,k}^2)^2) \leq \text{const.} \cdot \Delta_n \), and \( \mathbb{E}((Z_{n,k}^4)^2) \leq \text{const.} \cdot \Delta_n \) (\( i = 1, \ldots, 4 \)). Thus,

\[
\mathbb{E}((Z_{n,k}^4)^2) \leq \text{const.} \cdot \Delta_n \mathbb{E}((Z_{n,k}^2)^4) \leq \text{const.} \cdot \Delta_n \mathbb{E}((Z_{n,k}^2)^2) \leq \text{const.} \cdot \Delta_n \mathbb{E}((Z_{n,k}^2)^2).
\]

In total we receive

\[
\limsup_{n \to \infty} \frac{1}{n \Delta_n} \mathbb{E} \left[ \left| Z_{n,1}^* \right|^2 I_{\left| Z_{n,1}^* \right| > \varepsilon / \sqrt{n \Delta_n}} \right] \leq \limsup_{n \to \infty} \frac{1}{n \Delta_n} \mathbb{E} \left[ \left| Z_{n,1}^* \right|^4 \right] = 0. \tag{3.13}
\]

Finally, (3.10)–(3.13) result in the Lindeberg condition. \( \square \)

Thus, by Lemma 3.5 and Lemma 3.6 the assumptions of the central limit theorem of Lindeberg-Feller (see Jacob and Shiryaev, 2002, Theorem VI.5.5.2) are satisfied and we can conclude the weak convergence

\[
\frac{1}{\sqrt{n \Delta_n}} S_{1,n}^* \xrightarrow{\mathcal{D}} \mathcal{N} \left( 0, \Sigma(c_1, c_2, c_3) \right) \quad \text{as} \quad n \to \infty.
\]

Moreover, since \((Z_{n,k}^{(1)})_{k \in \mathbb{N}}\) as given in (3.12) and \((Z_{n,k}^{(5)})_{k \in \mathbb{N}} := (c_3^T \text{vec}(\xi_{n,k} \xi_{n,k}^T - \Sigma_n))_{k \in \mathbb{N}}\) are uncorrelated sequences, \( \mathbb{E}((Z_{n,k}^{(5)})^2) < \text{const.} \cdot \Delta_n \) and \( I_n \Delta_n \to \infty \), we obtain with Markov’s inequality

\[
P \left( \frac{1}{\sqrt{n \Delta_n}} |S_{2,n}^*| > \varepsilon \right) \leq \frac{\text{const.}}{\varepsilon^2} \frac{1}{n \Delta_n} \sum_{i=1}^{\left\lfloor \Delta_n^2 \right\rfloor} \left[ \sum_{k=1}^{n/l_n} \left| \sum_{i=1}^{n/l_n} c_{i,k} \right| n/l_n - \Delta_n^{-1} \right] \leq \frac{\text{const.}}{n \Delta_n} \frac{n}{l_n \Delta_n} \Delta_n = \frac{1}{l_n \Delta_n} \to 0,
\]

such that \( \frac{1}{\sqrt{n \Delta_n}} S_{2,n}^* \xrightarrow{p} 0 \). This in combination with (3.12) and (3.14) result in

\[
\frac{1}{\sqrt{n \Delta_n}} S_n^* \xrightarrow{\mathcal{D}} \mathcal{N} \left( 0, \Sigma(c_1, c_2, c_3) \right) \quad \text{as} \quad n \to \infty,
\]

and, in particular, the convergence of the two-dimensional distribution. However, we have to be sure that the
limit distribution is as stated. From Lemma 3.2 we already know that 
\( sc^T \Sigma^* c_1 + (t-s)c^T \Sigma^* c_2 \) is the covariance matrix of the normally distributed random variable 
\[ c^T \text{vec}(\Sigma^{1/2} W c_1) + c^T \text{vec}(\Sigma^{1/2} W c_2) + c^T \text{vec}(W^T) =: N^*(c_1, c_2). \]

This means \( \frac{1}{\sqrt{n \Delta n}} \rightarrow N^*(c_1, c_2) + c^T \text{vec}(W^T) \) as \( n \rightarrow \infty \), and the Cramér-Wold technique gives the converges of the two-dimensional distribution as stated.

To prove the tightness of \( \left( \frac{1}{\sqrt{n \Delta n}} S_n(t) \right)_{t \in [0, T]} \) we use the following criteria so that we can apply [Billingsley, 1999, Theorem 13.5].

**Lemma 3.7.** There exists a constant \( K > 0 \) such that for any \( 0 \leq r < s < t \leq T \):
\[
\frac{1}{(n \Delta n)^2} \mathbb{E}((S_n(t) - S_n(s))^2) \leq K(t - r)^2.
\]

**Proof.**
Without loss of generality \( p = 1 \) and \( d = 1 \), otherwise prove the statement componentwise. Therefore, we define
\[
V_{n,k}^{(1)}(u_1, u_2) := \sum_{j=|u_1/\Delta n|+1}^{u_2/\Delta n} \xi_{n,k-j} \quad \text{and} \quad V_{n,k}^{(2)}(u_1, u_2) := \sum_{j=|u_1/\Delta n|+1}^{u_2/\Delta n} \xi_{n,k+j}
\]
for \( 0 \leq u_1 < u_2 < \infty \), such that
\[
S_n^{(i)}(t) = \sum_{k=1}^{n} \xi_{n,k} V_{n,k}(0,t).
\]

Note \( \mathbb{E}(V_{n,k}^{(i)}(u_1, u_2)^2) \leq \text{const.} (u_2 - u_1) \) and \( \mathbb{E}(V_{n,k}^{(i)}(u_1, u_2)^4) \leq \text{const.} (u_2 - u_1) \). Moreover,
\[
\mathbb{E}((S_n^{(i)}(t) - S_n^{(i)}(s))^2) \leq \mathbb{E}(\xi_{n,k}^4) \mathbb{E}(V_{n,k}^{(i)}(s,t)^2) \mathbb{E}(V_{n,k}^{(i)}(t,s)^2) \leq (t-s)^2 \Delta_n^2 \quad \text{(3.15)}.
\]

We investigate the different summands. First, \( \mathbb{E}(V_{n,k}^{(i)}(s,t)^2 V_{n,k}^{(i)}(t,s)^2) \leq \mathbb{E}(\xi_{n,k}^4) \mathbb{E}(V_{n,k}^{(i)}(s,t)^2) \mathbb{E}(V_{n,k}^{(i)}(t,s)^2) \leq (t-s)^2 \Delta_n^2 \) (3.16).

Next for \( |k_1 - k_2| > (t-r)/\Delta_n \),
\[
\mathbb{E}(\xi_{n,k}^2 V_{n,k_1}^{(i)}(s,t)^2 V_{n,k_2}^{(i)}(t,r)^2) \leq \text{const.} (t-r)^2 \Delta_n^2 \quad \text{(3.17)}.
\]

and \( |k_1 - k_2| \leq (t-r)/\Delta_n, k_1 \neq k_2 \),
\[
\mathbb{E}(\xi_{n,k_1}^2 V_{n,k_1}^{(i)}(s,t)^2 V_{n,k_2}^{(i)}(t,r)^2) \leq \text{const.} [(t-r) + (t-r)^2] \Delta_n^2 \quad \text{(3.18)}.
\]

A conclusion of (3.15)-(3.18) is that
\[
\mathbb{E}((S_n^{(i)}(t) - S_n^{(i)}(s))^2) \leq \text{const.} (n \Delta n)^2 (t-r)^2 \quad \text{(3.19)}.
\]
Finding an upper bound for $\mathbb{E}((S_n^{(1)}(t) - S_n^{(1)}(s))^2(S_n^{(2)}(s) - S_n^{(2)}(r))^2)$ is alike. Similar but more technical and tedious calculations as above yield

$$
\mathbb{E}((S_n^{(1)}(t) - S_n^{(1)}(s))^2(S_n^{(2)}(s) - S_n^{(2)}(r))^2) \leq \text{const.} \ (n\Delta_n)^2(t-r)^2. \quad (3.20)
$$

Then (3.19) and (3.20) result in the statement.

\[ \square \]

### 3.2.2 Proofs of the results in Section 3.1

**Proof of Proposition 3.2.**

In Lemma 3.4 we have already proved the convergence of the two-dimensional distribution. The convergence of the finite-dimensional distribution is an obvious extension. Hence, the weak convergence is a consequence of Lemma 3.7 and Jacod and Shiryaev (2002), Theorem VI.4.1 (cf. Billingsley (1999), Theorem 13.5).

**Proof of Proposition 3.3.**

Let $T > 0$. Define $S_n^{(i)}(t) := 0_{pd \times pd}$ and $g_i(t) := 0_{pd \times pd}$ for $t \leq 0$, $i = 1, 2$, and $S^{(1)}(t) := \Sigma^{1/2}W_t\Sigma^{1/2}T$ and $S^{(2)}(t) := \Sigma^{1/2}W_t\Sigma^{1/2}T$ for $t \geq 0$. Moreover, for $i = 1, 2, t = 1, \ldots, M$, we have

$$
\int_0^T g_i(s) S_n^{(i)}(ds) = -\left[\int_0^T S_n^{(i)}(s) d g_i(s)\right]^T + g_i(T) S_n^{(i)}(T).
$$

Applying Proposition 3.2 (Jacod and Shiryaev, 2002, Theorem VI.6.22) and partial integration (cf. Protter (2005), Corollary II.6.2) we obtain as $n \to \infty$,

$$
\left(\frac{1}{\sqrt{n\Delta_n}} \int_0^T g_i(s) S_n^{(i)}(ds)\right)^T, \frac{1}{\sqrt{n\Delta_n}} \int_0^T g_i(s) S_n^{(i)}(ds)\right)_{l=1, \ldots, M, i=1,2}

\overset{D}{\to} \left(-\left[\int_0^T S_n^{(i)}(s) dg_i(s)\right]^T + g_i(T)S_n^{(i)}(T), -\left[\int_0^T S_n^{(i)}(s)T dg_i(s)\right]^T + g_i(T)S_n^{(i)}(T)\right)_{l=1, \ldots, M, i=1,2}.

Note that $\int_0^T g_i(s) S_n^{(i)}(ds) \overset{P}{\to} \int_0^\infty g_i(s) S_n^{(i)}(ds)$ and $\int_0^T g_i(s) T S_n^{(i)}(ds) \overset{P}{\to} \int_0^\infty g_i(s) T S_n^{(i)}(ds)$ as $T \to \infty$. Moreover, for $\varepsilon > 0$ a conclusion of Markov’s inequality is

$$
\mathbb{P}\left(\left\|\frac{1}{\sqrt{n\Delta_n}} \sum_{j=[T/\Delta_n]+1}^n g_i(\Delta_n j) \left[S_n^{(1)}(j\Delta_n) - S_n^{(1)}((j-1)\Delta_n)\right]\right\| > \varepsilon\right)

\leq \text{const.} \frac{1}{n\Delta_n} \sum_{k=1}^n \mathbb{E}\left\|\xi_{n,k+1}\right\|^2 \left(\sum_{j=[T/\Delta_n]+1}^\infty \mathbb{E}\left\|g_i(\Delta_n j)\right\|^2\mathbb{E}\left\|\xi_{n,k-j}\right\|^2\right)^{1/2}

\leq \text{const.} \Delta_n \sum_{j=[T/\Delta_n]+1}^\infty \left\|g_i(\Delta_n j)\right\|^2 \leq \text{const.} \int_T^\infty \left\|g_i(s)\right\|^2 ds \overset{T \to \infty}{\to} 0.
$$

The same statement holds with $S_n^{(2)}$ replaced by $S_n^{(1)}$, and taking the transposed processes. Hence, the conclusion follows by a convergence together argument (cf. Billingsley (1999), Theorem 3.2) and the continuous mapping theorem if we take the transpose of $\int_0^\infty g_i(s) T S_n^{(i)}(ds) = \left[\int_0^\infty S_n^{(i)}(ds)g_i(s)\right]^T$. \[ \square \]
Proof of Lemma 3.1

(a) Let \((\gamma_k, \Sigma_k, \nu_k)\) be the characteristic triplet of \((L_t)_{t \geq 0}\) and let \(B^m = \{x \in \mathbb{R}^m : \|x\| \leq \varepsilon\}\) be a ball around \(0_m\) in \(\mathbb{R}^m\) with radius \(\varepsilon > 0\). We factorize the Lévy measure \(\nu_k\) into two Lévy measures

\[
\nu_{l_1}(A) := \nu_k(A \setminus B^m) \quad \text{and} \quad \nu_{l_2}(A) := \nu_k(A \cap B^m) \quad \text{for} \ A \in \mathcal{B}(\mathbb{R}^m \setminus \{0_m\})
\]

such that \(\nu_k = \nu_{l_1} + \nu_{l_2}\). Then we can decompose \((L_t)_{t \geq 0}\) into two independent Lévy processes

\[
L_t = L_{l_1}(t) + L_{l_2}(t) \quad \text{for} \ t \geq 0,
\]

(3.21)

where \(L_{l_i}\) has Lévy measure \(\nu_{l_i}\) and expectation \(0_m\) \((i = 1, 2)\), and \(L_{l_1}\) is without Gaussian part. First, we will show that

\[
\lim_{n \to \infty} \Delta_n^{-1} \mathbb{E}(L_{l_1}(\Delta_n)L_{l_1}(\Delta_n)^T \otimes L_{l_1}(\Delta_n)L_{l_1}(\Delta_n)^T) = \int_{\mathbb{R}^m} \mathbf{x}\mathbf{x}^T \otimes \mathbf{x}\mathbf{x}^T \nu_{l_1}(d\mathbf{x}).
\]

(3.22)

Since the Lévy measure of \(L_{l_1}\) is finite and \(L_{l_1}\) is without Gaussian part, \(L_{l_1}\) has the representation as a compound Poisson process with drift

\[
L_{l_1}(t) = \sum_{k=1}^{N(t)} \mathbf{J}_k(c_1 + c_1 \varepsilon), \quad t \geq 0,
\]

(3.23)

where \((\mathbf{J}_k)_{k \in \mathbb{N}}\) is a sequence of iid random vectors independent of the Poisson process \((N(t))_{t \geq 0}\) with intensity \(\lambda_e := \nu_{l_1}(\mathbb{R}^m)\). The distribution of \(\mathbf{J}_k\) is \(\lambda_e^{-1} \nu_{l_1}\). Moreover, \(c_1\) is a vector in \(\mathbb{R}^{m \times m}\). We will use on the one hand, that for \(l \geq 1\),

\[
\mathbb{P}(N(\Delta_n) = l) = e^{-\lambda_e \Delta_n} \frac{(\lambda_e \Delta_n)^l}{l!} \leq \text{const.} \mathbb{P}(N(1) = l),
\]

(3.24)

and on the other hand, that

\[
\lim_{n \to \infty} \mathbb{P}(N(\Delta_n) = l) \Delta_n = \begin{cases} \lambda_e & \text{for } l = 1, \\ 0 & \text{for } l \geq 2. \end{cases}
\]

(3.25)

Then

\[
\mathbb{E}(L_{l_1}(\Delta_n)L_{l_1}(\Delta_n)^T \otimes L_{l_1}(\Delta_n)L_{l_1}(\Delta_n)^T)
= \mathbb{P}(N(\Delta_n) = 1) \mathbb{E}(\mathbf{J}_1(c_1 + c_1 \varepsilon)\mathbf{J}_1(c_1 + c_1 \varepsilon)^T \otimes \mathbf{J}_1(c_1 + c_1 \varepsilon)\mathbf{J}_1(c_1 + c_1 \varepsilon)^T)
+ \sum_{m=2}^{\infty} \mathbb{P}(N(\Delta_n) = m) \mathbb{E}
\left[
\sum_{k=1}^{m} \mathbf{J}_k(c_1 + c_1 \varepsilon) \mathbf{J}_k(c_1 + c_1 \varepsilon)^T \otimes \sum_{k=1}^{m} \mathbf{J}_k(c_1 + c_1 \varepsilon) \mathbf{J}_k(c_1 + c_1 \varepsilon)^T
\right]
=: I_{n,1} + I_{n,2}.
\]

Due to (3.22), (3.25) and dominated convergence we get \(\lim_{n \to \infty} \Delta_n^{-1} I_{n,2} = 0_{m^2 \times m^2}\), and

\[
\lim_{n \to \infty} \Delta_n^{-1} I_{n,1} = \lambda_e \mathbb{E}(\mathbf{J}_1(c_1 + c_1 \varepsilon)^T \otimes \mathbf{J}_1(c_1 + c_1 \varepsilon)^T)
= \int_{\mathbb{R}^m} \mathbf{x}\mathbf{x}^T \otimes \mathbf{x}\mathbf{x}^T \nu_{l_1}(d\mathbf{x}),
\]
so that (3.22) follows. Moreover, by Hölder’s inequality and [Asmussen and Rosiński, 2001, Lemma 3.1] we have

\[
\Delta_n^{-1} \|E(L_1^{(e)}(\Delta_n) L_1^{(e)}(\Delta_n) - E(L_1^{(e)}(\Delta_n)^T \otimes L_1^{(e)}(\Delta_n)^T)\| \\
\leq [\Delta_n^{-1} E\|L_2^{(e)}(\Delta_n)\|^4]^{1/4} [\Delta_n^{-1} E\|L_1^{(e)}(\Delta_n)\|^4]^{3/4} \\
\xrightarrow{n \to \infty} \left[ \int_{\mathbb{R}^n} |x|^4 v_{1,1}^{(e)}(dx) \right]^{1/4} \left[ \int_{\mathbb{R}^n} |x|^4 v_{1,1}^{(e)}(dx) \right]^{3/4} = 0.
\]

On this way we can recursively derive that

\[
\Delta_n^{-1} \|E(L_1^{(e)}(\Delta_n) L_1^{(e)}(\Delta_n) - E(L_1^{(e)}(\Delta_n)^T \otimes L_1^{(e)}(\Delta_n)^T)\| \\
\leq \Delta_n^{-1} \|E(L_1^{(e)}(\Delta_n) L_1^{(e)}(\Delta_n) - E(L_1^{(e)}(\Delta_n)^T \otimes L_1^{(e)}(\Delta_n)^T)\| \\
+ \Delta_n^{-1} \|E(L_1^{(e)}(\Delta_n) L_1^{(e)}(\Delta_n) - E(L_1^{(e)}(\Delta_n)^T \otimes L_1^{(e)}(\Delta_n)^T)\| \\
+ \Delta_n^{-1} \|E(L_1^{(e)}(\Delta_n) L_1^{(e)}(\Delta_n) - E(L_1^{(e)}(\Delta_n)^T \otimes L_1^{(e)}(\Delta_n)^T)\| \\
+ \Delta_n^{-1} \|E(L_1^{(e)}(\Delta_n) L_1^{(e)}(\Delta_n) - E(L_1^{(e)}(\Delta_n)^T \otimes L_1^{(e)}(\Delta_n)^T)\| \\
\xrightarrow{n \to \infty} 0,
\]

and hence, the statement follows.

(b) When we show that

\[
\lim_{n \to \infty} \Delta_n^{-1} \|E(\xi_{n,1}^{(e)} \otimes \xi_{n,1}^{(e)\Delta_n}) - (BL_{\Delta_n})(BL_{\Delta_n})^T\| = 0,
\]

we can conclude the statement from (a). Therefore, we use that as \( n \to \infty, \)

\[
\Delta_n^{-1} \|E(\xi_{n,1}^{(e)}) - (BL_{\Delta_n})\| \to 0.
\]

This we get from the representation of the components of \( \xi_{n,1} - BL_{\Delta_n} \) as

\[
e_i^T \xi_{n,1} - e_i^T BL_{\Delta_n} = \int_0^{\Delta_n} e_i^T (e^{-\lambda t} - I_p) dL = \sum_{k=1}^{m} \int_0^{\Delta_n} e_i^T (e^{-\lambda t} - I_p) B dL_k(s) \quad (i = 1, \ldots, d).
\]

Applying [Cohen and Lindner, 2013, Lemma 3.2] gives

\[
E(e_i^T \xi_{n,1} - e_i^T BL_{\Delta_n})^4 \\
\leq m^4 \sum_{k=1}^{m} E \left( \int_0^{\Delta_n} e_i^T (e^{-\lambda t} - I_p) B dL_k(s) \right)^4 \\
\leq \text{const.} \sum_{k=1}^{m} \left[ \int_0^{\Delta_n} (e_i^T (e^{-\lambda t} - I_p) B)^4 dL_k(s) + \left( \int_0^{\Delta_n} (e_i^T (e^{-\lambda t} - I_p) B)^2 dL_k(s) \right)^2 \right] = o(\Delta_n),
\]

and hence, (3.27) follows. Using Hölder’s inequality, (3.27) and \( E\|BL_{\Delta_n}\|^4 = O(\Delta_n) \) (cf. Asmussen and Rosiński [2001], Lemma 3.1) we obtain

\[
\Delta_n^{-1} \|E(\xi_{n,1}^{(e)} (BL_{\Delta_n})^T \otimes (BL_{\Delta_n}) (BL_{\Delta_n})^T - (BL_{\Delta_n}) (BL_{\Delta_n})^T\| \\
\leq \text{const.} \left( \Delta_n^{-1} E\|\xi_{n,1}^{(e)} - BL_{\Delta_n}\|^4 \right)^{1/4} [\Delta_n^{-1} E\|BL_{\Delta_n}\|^4]^{3/4} = o(1).
\]

Since \( E\|\xi_{n,1}\|^4 = O(\Delta_n) \) (cf. Fasen [2014], Proposition A.1(b)) as well, we obtain recursively the statement (3.26). \( \square \)
Proof of Proposition 3.1.

Let the assumptions of Proposition 4.1 hold. Then as $n \to \infty$,

$$\lim_{n \to \infty} \Delta_n^{-1} E\left( (\xi_n, \xi_{n+1}, \xi_{n+2}, \xi_{n+3}) \right) = B \mathbf{1} \cdot \Lambda_0 \cdot B^T.$$

Then the assumptions of Proposition 3.3 are satisfied and Proposition 3.3 gives the statement.

4 Asymptotic behavior of the sample autocovariance function of MCARMA models

In this section we present the main results of this paper starting with the asymptotic behavior of the sample autocovariance function of a MCARMA process $Y$ as defined in (2.1) driven by the Lévy process $(L_t)_{t \in \mathbb{R}}$. We will assume that $E[|L_1|^2] < \infty$ and $E(L_1) = 0_m$ so that the autocovariance function $\Gamma_Y(h) = E(Y_0 Y_h^T)$ for $h \in \mathbb{R}$ is well-defined. The sample autocovariance function is defined as

$$\hat{\Gamma}_n(h) = \frac{1}{n-\Delta_n} \sum_{k=1}^{n-\Delta_n} (Y_{k\Delta_n} - \bar{Y}_n)(Y_{k\Delta_n+h} - \bar{Y}_n)^T \quad \text{for} \ h \in \{0, \Delta_n, \ldots, (n-1)\Delta_n\},$$

where $\bar{Y}_n = n^{-1} \sum_{k=1}^{n} Y_{k\Delta_n}$ is the sample mean. In our first result we let the sum going to $n$ and neglect the sample mean $\bar{Y}_n$, i.e., we investigate $\hat{\Gamma}_n(h)$ as in (2.4). Afterwards we derive the general result for the sample autocovariance function.

Proposition 4.1. Let $(Y_t)_{t \in \mathbb{R}}$ be a MCARMA process as defined in (2.1) with kernel function $f$, covariance function $(\Gamma_Y(h))_{h \in \mathbb{R}}$ and driving Lévy process $L$ satisfying $E[|L_1|^2] < \infty$, $E(L_1) = 0_m$, $E(L_1 L_1^T) = \Sigma_L = \Sigma_1^{1/2} \Sigma_1^{1/2}$ and having Lévy measure $\nu$. Suppose $(\Delta_n)_{n \in \mathbb{N}}$ is a sequence of positive constants with $\Delta_n \downarrow 0$ and $\lim_{n \to \infty} n \Delta_n = \infty$. Assume that there exists a sequence of positive constants $l_n \to \infty$ with $n/l_n \to \infty$ and $l_n \Delta_n \to \infty$. Define $Y := \int_{\mathbb{R}^m} xx^T \otimes xx^T v(dx) \in \mathbb{R}^{m^2 \times m^2}$, and denote by $(W_t)_{t \geq 0}$ an $m \times m$-valued standard Brownian motion independent of the $m \times m$-valued random matrix $W^T(Y)$ with $\text{vec}(W^T(Y)) \sim \mathcal{N}(0_m, \Sigma_L)$. Let $\mathcal{H}$ be a finite set in $\prod_{n \geq n_0} \{0, \Delta_n, 2\Delta_n, \ldots, (n-1)\Delta_n\}$, $n_0 \in \mathbb{N}$, and $\Gamma_n(h)$ be as in (2.1). Then as $n \to \infty$,

$$\left( \sqrt{n \Delta_n} \left( \hat{\Gamma}_n(h) - \Gamma_Y(h) \right) \right)_{h \in \mathcal{H}} \overset{p}{\to} \left( \int_0^\infty f(s) \Sigma_1^{1/2} dW_s \Sigma_1^{1/2} f(s+h) d(s+h) \right)_{h \in \mathcal{H}}.$$

From this result we get the asymptotic behavior of the sample autocovariance function.

Theorem 4.1. Let the assumptions of Proposition 4.1 hold. Then as $n \to \infty$,

$$\left( \sqrt{n \Delta_n} \left( \hat{\Gamma}_n(h) - \Gamma_Y(h) \right) \right)_{h \in \mathcal{H}} \overset{p}{\to} \left( \int_0^\infty f(s) \Sigma_1^{1/2} dW_s \Sigma_1^{1/2} f(s+h) d(s+h) \right)_{h \in \mathcal{H}}.$$

A consequence is that in the high frequency setting the convergence rate of the sample autocovariance
function is $\sqrt{n\Delta_n}$ which is slower than the classical $\sqrt{n}$ convergence rate for models in discrete time (cf. Theorem 5.1 below).

Moreover note that only for $h \in \bigcap_{k \geq n} \{0, \Delta_n, 2\Delta_n, \ldots, (n-1)\Delta_n\}$ we received a consistent and asymptotically normally distributed estimator. If $n\Delta_n^2 \to 0$, then $\hat{\Gamma}_n(\Delta_n)$ is a consistent and asymptotically normally distributed estimator for $\Gamma(h)$ for any $h > 0$ as well.

We want to investigate now several special cases where the limit process has a simpler structure. First, where the driving Lévy process is an one-dimensional Lévy process and second, where the driving Lévy process of the MCARMA process is a Brownian motion.

**Corollary 4.1.** Let the assumptions of Proposition 4.1 hold.

(a) Let $m = 1$ such that $L = L$, $W = W$ are one-dimensional processes and let $N$ be a standard normal random variable independent of $W$. Then as $n \to \infty$,

$$\sqrt{n\Delta_n} \left( \hat{\Gamma}_n(h) - \Gamma_Y(h) \right) \Rightarrow 0_{\mathbb{R}^2}.$$ 

(b) Let $L$ be a multivariate Brownian motion. Then $Y = 0_{m^2 \times m^2}$ and hence, $W^*(\Gamma) = 0_{m^2 \times m^2}$.

A different representation of Corollary 4.1 is by the vector-representation which gives an explicit description of the limit covariance matrix. However, it is very technical to write it down for different covariances. For this reason we restrict our attention to a fixed covariance.

**Corollary 4.2.** Let the assumptions of Proposition 4.1 hold. Define

$$\Sigma_Y(u) := \int_0^\infty f(s+u) \otimes f(s)ds \quad \text{and} \quad \Sigma^*_Y(u) := \int_0^\infty f(s) \otimes f(s+u)ds \quad \text{for} \ u \in \mathbb{R}.$$ 

Let $P_{m,m}$ be the Kronecker permutation matrix and $h \geq 0$. Then $E(\text{vec}(Y_0^TY_h^T)) = E(\text{vec}(Y_h \otimes Y_0)) = \Sigma_Y(h) \text{vec}(\Sigma_L)$ and as $n \to \infty$,

$$\sqrt{n\Delta_n} \left( \text{vec}(\hat{\Gamma}_n(h) - \Gamma_Y(h)) \right) \Rightarrow 0_{\mathbb{R}^2}, \Sigma_Y(h) \cdot Y \cdot \Sigma_Y(h)^T + \int_0^\infty \Sigma_Y(u+h) \cdot \Sigma_L \otimes \Sigma_L \cdot \Sigma_Y(u+h)^T du$$

$$+ \int_0^\infty \Sigma^*_Y(u-h) \cdot \Sigma_L \otimes \Sigma_L \cdot \Sigma^*_Y(u-h)^T du$$

$$+ \int_0^\infty \Sigma_Y(u+h) \cdot \Sigma^{1/2}_L \otimes \Sigma^{1/2}_L \cdot P_{m,m} \cdot \Sigma^{1/2}_L \otimes \Sigma^{1/2}_L \cdot \Sigma^*_Y(u-h)^T du$$

$$+ \int_0^\infty \Sigma_Y(u-h) \cdot \Sigma^{1/2}_L \otimes \Sigma^{1/2}_L \cdot P_{m,m} \cdot \Sigma^{1/2}_L \otimes \Sigma^{1/2}_L \cdot \Sigma_Y(u+h)^T du.$$ 

The advantage of the representation of the limit distribution as in Theorem 6.1 is that we are able to understand the dependence in the model quite well. For this reason we get several extensions from this including the asymptotic behavior of cross-covariances and cross-correlations between the components. The next corollary shows the behavior of the cross-covariances for the different components of a MCARMA process. It is also straightforward to calculate the cross-correlations.

**Corollary 4.3.** Let the assumptions of Proposition 4.1 hold, and denote by $\gamma(h) = E(Y_0^TY_h^T)$ the covariance function of the $i$-th component and by $\gamma_{ij}(h) = E(Y_0^TY_h^T)$, $h \in \mathbb{R}$, the cross-covariance function between the $i$-th and the $j$-th component of $(Y_t)_{t \geq 0}$. Furthermore,

$$\gamma_{ij}(h) = e^T \Gamma_n(h) e_j = \frac{1}{n} \sum_{k=1}^{n-h/\Delta_n} \left( Y_{k+\Delta_n} - \overline{Y}_n \right) \left( Y_{k+\Delta_n+h} - \overline{Y}_n \right) \quad \text{for} \ h \in \{0, \Delta_n, \ldots, (n-1)\Delta_n\},$$

$$\gamma^{(i)}(h) = e^T \hat{\Gamma}_n(h) e_j = \frac{1}{n} \sum_{k=1}^{n-h/\Delta_n} \left( \overline{Y}_n - \overline{Y}_n \right) \left( \overline{Y} \right) \left( \overline{Y} \right) \quad \text{for} \ h \in \{0, \Delta_n, \ldots, (n-1)\Delta_n\},$$

and hence, $\gamma^{(i)}(h)$ is $\sqrt{n\Delta_n}$ asymptotically normally distributed.
is the sample cross-covariance function between the $i$-th and the $j$-th component, and 
$\hat{Y}_n^{(i)} = e_i^T \hat{Y}_n = \frac{1}{n} \sum_{k=1}^{n} \hat{Y}_{k\Delta}^{(i)}$ is the sample mean of the $i$-th component of $(\hat{Y}_t)_{t \geq 0}$.

(a) Then as $n \to \infty$,
$$\sqrt{n\Delta_n} \left( \frac{\hat{\gamma}_{ij}(h)}{\hat{\gamma}_j(h)} - \frac{\gamma_j(h)}{\gamma(h)} \right) \xrightarrow{d} \mathcal{N} \left( 0, \int_0^\infty \left( \int_0^h \left( e_i^T f(s) \mathbf{x} \cdot (e_j^T f(s+h) \mathbf{x}) d\mathbf{x} \right)^2 \nu(ds) + 2 \int_0^h \gamma_j(s) \gamma_j(s+h) \gamma_j(s-h) ds \right) \right).$$

(b) Assume $\Sigma_1^{-1/2} \mathbf{L}$ has independent and identically distributed components, identically distributed as $\mathcal{L}$. Then as $n \to \infty$,
$$\sqrt{n\Delta_n} \left( \frac{\hat{\gamma}_j(h)}{\gamma_j(h)} \right) \xrightarrow{d} \mathcal{N} \left( 0, \frac{1}{2} \int_{0}^{\infty} \left( \int_0^h \left( e_i^T f(s) \Sigma_1^{1/2} e_i \right) (e_j^T f(s+h) \Sigma_1^{1/2} e_i) ds \right)^2 \nu(ds) + 2 \int_0^h \gamma_j(s) \gamma_j(s+h) \gamma_j(s-h) ds \right).$$

Finally, we want to present Bartlett’s formula for a CARMA process.

**Corollary 4.4.** Let $(Y_t)_{t \in \mathbb{R}}$ be a CARMA process satisfying the assumptions of Proposition 4.1.

(a) The autocovariance function of $(Y_t)_{t \in \mathbb{R}}$ is denoted by $(\gamma(h))_{h \in \mathbb{R}}$ and the sample autocovariance function by $(\hat{\gamma}_n(h))_{h \in [0, \Delta, \ldots, (n-1)\Delta]}$. Then as $n \to \infty$,
$$\left( \sqrt{n\Delta_n} (\hat{\gamma}_n(h) - \gamma(h)) \right)_{h \in \mathbb{R}} \xrightarrow{d} \mathcal{N} \left( 0, (m_{ij})_{i,j \in \mathbb{R}} \right),$$
where $m_{ij} = (\mathbb{E}(\hat{L}_1^j))^{-2} \mathbb{E}(\hat{L}_1^j) - 3 \mathbb{E}(\hat{L}_1^j) \gamma(t) + \int_{-\infty}^\infty \gamma(u+s) \gamma(u+t) \gamma(u+s) \gamma(u-t) du$.

(b) The autocorrelation function of $(Y_t)_{t \in \mathbb{R}}$ is denoted by $(\rho(h))_{h \in \mathbb{R}}$ and $\hat{\rho}_n(h) = \hat{\gamma}_n(h)/\hat{\gamma}_n(0)$ for $h \in \{0, \Delta, \ldots, (n-1)\Delta\}$ denotes the sample autocorrelation function. Then as $n \to \infty$,
$$\left( \sqrt{n\Delta_n} (\hat{\rho}_n(h) - \rho(h)) \right)_{h \in \mathbb{R}} \xrightarrow{d} \mathcal{N} \left( 0, (v_{ij})_{i,j \in \mathbb{R}} \right),$$
where
$$v_{ij} = \int_{-\infty}^\infty \rho(u+s) \rho(u+t) - \rho(u+s) \rho(u+t) + 2 \rho(s) \rho(t) \rho(u)^2$$
$$-2 \rho(s) \rho(u) \rho(u+t) - 2 \rho(t) \rho(u) \rho(u+s) du. \quad (4.1)$$

**Remark 4.1.** Cohen and Lindeberg (2013). Theorem 3.5, derived the asymptotic behavior of the sample autocovariance function of a CARMA process sampled at an equidistant time-grid with distance $\Delta > 0$. We want to compare their and our results. They proved that as $n \to \infty$,
$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} (Y_{k\Delta} - \gamma(0))^2 \xrightarrow{d} \mathcal{N} \left( 0, (\mathbb{E}(\hat{L}_1^2))^{-2} \mathbb{E}(\hat{L}_1^4) - 3(\mathbb{E}(\hat{L}_1^2))^2 \int_0^\Delta f_\Delta(u)^2 du + 2\gamma(0)^2 + 4 \sum_{k=1}^{\infty} \gamma(k\Delta)^2 \right),$$
where $f_\Delta : [0, \Delta] \to \mathbb{R}$ is defined as $f_\Delta(u) = \sum_{k=-\infty}^{\infty} f(u+k\Delta)^2$. If we multiply the variance of the Gaussian limit distribution by $\Delta$ and let $\Delta \to 0$, then
$$\Delta \left[ (\mathbb{E}(\hat{L}_1^2))^{-2} \mathbb{E}(\hat{L}_1^4) - 3(\mathbb{E}(\hat{L}_1^2))^2 \int_0^\Delta f_\Delta(u)^2 du + 2\gamma(0)^2 + 4 \sum_{k=1}^{\infty} \gamma(k\Delta)^2 \right]$$
\[ \Delta_{ij}^0 \left( [\mathbb{E}(L_t^i)]^{-2} \mathbb{E}(L_t^j) - 3 ] \gamma(0)^2 + 4 \int_0^\infty \gamma(s)^2 \, ds \right). \] (4.2)

Note that the second term \( \Delta \cdot 2 \gamma(0)^2 \) converges to 0. The limit result \(4.2\) is in line with Corollary \(4.4\), since as \( n \to \infty \),

\[ \frac{\sqrt{\Delta}}{n} \sum_{k=1}^n (Y_{k\Delta} - \gamma(0)) \xrightarrow{d} N \left( 0, \{ [\mathbb{E}(L_t^i)]^{-2} \mathbb{E}(L_t^j) - 3 ] \gamma(0)^2 + 4 \int_0^\infty \gamma(s)^2 \, ds \right). \]

Hence, for the limit distribution it does not matter if first \( n \to \infty \) and then \( \Delta \to 0 \), or \( \Delta_n \to 0 \) and \( n \to \infty \) at the same time. The analogous phenomenon holds also for the sample autocorrelation function in the high frequency and the discrete-time setting as given in [Cohen and Lindner, 2013, Theorem 3.5] as well.

4.1 Proofs

Proof of Proposition 4.1

Let us define \( f(x) := e^{-\Delta_n x} \mathbb{1}_{[0,\infty)}(x) \) and \( \Gamma_n(h) = \mathbb{E}(Z_0 Z_h^T) \) for \( h \in \mathbb{R} \) with \( Z_n \) as given in (2.2). By the state space representation (2.1) we have the equalities \( f(x) = E f(x), \Gamma_n(h) = \Gamma Z(h) \Gamma^T \) and

\[ \sum_{k=1}^n \left( Y_{k\Delta} Y_{k\Delta+h} - \Gamma Y(h) \right) = E \left( \sum_{k=1}^n \left[ \sum_{r=1}^h Z_{k\Delta} Z_{k\Delta+h} - \Gamma Z(h) \right] \right) \Gamma^T. \]

Hence, it is sufficient to investigate the asymptotic behavior of \( \sum_{k=1}^n \sum_{h=1} \left( Z_{k\Delta} Z_{k\Delta+h} - \Gamma Z(h) \right) \).

The proof has a common ground with the proof of [Fasen, 2014, Theorem 3.6]. A multivariate version of the second order Beveridge-Nelson decomposition presented in [Phillips and Solo, 1992, Equation (28)] gives the representation

\[ Z_{k\Delta} Z_{k\Delta+h}^T = \sum_{j=0}^\infty e^{-\Delta_n \gamma} \mathbb{1}_{[0\Delta]}(\gamma) \mathbb{1}_{[-\Delta_n,0)}(\gamma) e^{-\Delta_n (\gamma+h)} \Gamma (\gamma+h) \]

where

\[ F_{n,k}(h) = \sum_{j=0}^\infty \sum_{x=j+1}^\infty \sum_{s=0}^{\Delta_n \gamma} e^{-\Delta_n \gamma} \mathbb{1}_{[0\Delta_n]}(\gamma) \mathbb{1}_{[-\Delta_n,0)}(\gamma) e^{-\Delta_n (\gamma+h)} \Gamma (\gamma+h), \]

\[ F_{n,k,r}(h) = \sum_{j=\max(0,-r-[-(\gamma+h)])}^\infty \sum_{x=j+1}^\infty \sum_{s=0}^{\Delta_n \gamma} e^{-\Delta_n \gamma} \mathbb{1}_{[0\Delta_n]}(\gamma) \mathbb{1}_{[-\Delta_n,0)}(\gamma) e^{-\Delta_n (\gamma+h)} \Gamma (\gamma+h), \]

Then

\[ \sum_{k=1}^n Z_{k\Delta} Z_{k\Delta+h}^T = \sum_{j=0}^\infty \sum_{k=1}^n \sum_{r=0}^{\Delta_n \gamma} e^{-\Delta_n \gamma} \mathbb{1}_{[0\Delta_n]}(\gamma) \mathbb{1}_{[-\Delta_n,0)}(\gamma) e^{-\Delta_n (\gamma+h)} \Gamma (\gamma+h) \]

\[ + \sum_{k=1}^n \sum_{r=1}^\infty \sum_{s=0}^{\Delta_n \gamma} e^{-\Delta_n \gamma} \mathbb{1}_{[0\Delta_n]}(\gamma) \mathbb{1}_{[-\Delta_n,0)}(\gamma) e^{-\Delta_n (\gamma+h)} \Gamma (\gamma+h), \]

\[ =: J_{n,1}(h) + J_{n,2}(h) + J_{n,3}(h) + J_{n,4}(h). \] (4.3)

The proof is divided in several parts. We will show the following:
Finally, we denote by 

\[ (\mathbf{Fasen}, 2014, \text{Lemma 5.7}) \]

The proof of (ii) and (iv) follows directly from the proof of (\mathbf{Fasen}, 2014, Lemma 5.7).

Proof of (i). We will use the equality 

\[ \Gamma_{\mathbf{Z}}(h) = \sum_{j=0}^{\infty} e^{-\Delta_{n,j}} \{ \mathbb{E}(\varphi_{j,1} \varphi_{n,1}) e^{-\Lambda T} \Delta_{n,j} h \} \] 

An application of Proposition 5.3 yields as \( n \to \infty \),

\[ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} [\varphi_{j,k} - \mathbb{E}(\varphi_{j,1})] \to \text{W}^{*}(\mathbf{Y}) \mathbf{B}^{T}. \]

Finally, we denote by \( \mathbf{g}_{n}^{(h)} \) and \( \mathbf{g}^{(h)} \) maps from \( \mathbb{R}^{p_{d} \times p_{d}} \to \mathbb{R}^{p_{d} \times p_{d}} \) with

\[ \mathbf{g}_{n}^{(h)}(\mathbf{C}) = \Delta_{n} \sum_{j=0}^{\infty} e^{-\Delta_{n,j}} [\mathbf{C} e^{-\Lambda T} \Delta_{n,j} e^{-\Lambda T} \mathbf{C}] \quad \text{and} \quad \mathbf{g}^{(h)}(\mathbf{C}) = \int_{0}^{\infty} e^{-\Lambda T} \mathbf{C} e^{-\Lambda T} (s+h) \, ds. \]

Since \( \mathbf{g}_{n}^{(h)} \) and \( \mathbf{g}^{(h)} \) are continuous with \( \lim_{n \to \infty} \mathbf{g}_{n}^{(h)}(\mathbf{C}_{n}) = \mathbf{g}^{(h)}(\mathbf{C}) \) for any sequence \( \mathbf{C}_{n}, \mathbf{C} \in \mathbb{R}^{p_{d} \times p_{d}} \) with \( \lim_{n \to \infty} \| \mathbf{C}_{n} - \mathbf{C} \| = 0 \), we can apply a generalized version of the continuous mapping theorem (cf. \mathbf{Whitt} (2002), Theorem 3.4.4) to obtain as \( n \to \infty \),

\[ \frac{\sqrt{n}}{\sqrt{n}} \sum_{k=1}^{n} \left[ \varphi_{j,k} \right] \mathbf{g}_{n}^{(h)}(\mathbf{C}) \to \text{W}^{*}(\mathbf{Y}) \mathbf{B}^{T}. \]

Proof of (iii). An application of Proposition 5.3 and a generalized continuous mapping theorem as above gives

\[ \left( \frac{\sqrt{n}}{\sqrt{n}} \sum_{k=1}^{n} \left[ \varphi_{j,k} \right] \mathbf{g}_{n}^{(h)}(\mathbf{C}) \right) \to \left( \Delta_{n} \sum_{j=0}^{\infty} e^{-\Delta_{n,j}} \left\{ \int_{0}^{\infty} \mathbf{F}_{n,k}^{(h)}(s) \mathbf{W}_{n}^{1/2} \mathbf{T} e^{-\Lambda T} (h+s) \, ds \right\} \right) \]

\[ + \int_{0}^{\infty} e^{-\Lambda T} \left( \mathbf{F}_{n,k}^{(h)}(s) \mathbf{W}_{n}^{1/2} \mathbf{T} e^{-\Lambda T} (h+s) \right) \, ds. \]
For the proof we use Corollary 4.2 and denote by $\mathbf{f}$ means in particular that, and applying our Proposition 4.1. (Brockwell and Davis, 1991, Proposition 7.3.4) for MA processes in discrete time by proving

$$\sqrt{n\Delta_n} \left( \hat{\Gamma}_n(h) - \hat{\Gamma}_n(h) \right) \overset{p}{\to} 0_{d \times d},$$

and applying our Proposition 4.1.

Proof of Corollary 4.2

In the multivariate case we get the alternative representation of the limit distribution as

$$\text{vec} \left( \int_0^\infty \int_0^\infty [\mathbf{f}(s) \otimes \Sigma_H \Sigma_L] ds \cdot \mathbf{f}(s+u-h)^T ds + \int_0^\infty \int_0^\infty \mathbf{f}(s+u-h)^T \otimes \Sigma_H \Sigma_L^{1/2} d\text{vec}(\mathbf{W}_u) \right)$$

and the rest of the proof goes as in (Brockwell and Davis, 1991, Proposition 7.3.4) for MA processes in discrete time by proving

$$\sqrt{n\Delta_n} \left( \hat{\Gamma}_n(h) - \hat{\Gamma}_n(h) \right) \overset{p}{\to} 0_{d \times d},$$

and applying our Proposition 4.1.

Proof of Corollary 4.3

(a) For the proof we use Corollary 4.2 and denote by $\mathbf{f} = e_i^T \mathbf{f}$ the $i$-th row of $\mathbf{f}$. The rules for Kronecker products in (1.6) give

$$e_j^T \otimes e_i^T \cdot \left[ \int_0^\infty \left[ \mathbf{Y}_n \cdot \Sigma_L \otimes \Sigma_L \cdot \mathbf{Y}_n^T \right] du \right] \cdot e_j \otimes e_i$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \mathbf{f}_j(s) \Sigma_L \mathbf{f}_j(t)^T \cdot \mathbf{f}_j(u-h+s) \Sigma_L \mathbf{f}_j(u-h+t)^T ds dt du$$

$$= 2 \int_0^\infty \left[ \int_0^\infty \mathbf{f}_j(u-s) \Sigma_L \mathbf{f}_j(u)^T du \right] \mathbf{f}_j(t+s) \Sigma_L \mathbf{f}_j(t)^T dt ds,$$

and on the same way,

$$e_j^T \otimes e_i^T \cdot \left[ \int_0^\infty \left[ \mathbf{Y}_n \cdot \Sigma_L \otimes \Sigma_L \cdot \mathbf{Y}_n^T \right] du \right] \cdot e_j \otimes e_i$$

$$= 2 \int_0^\infty \left[ \int_0^\infty \mathbf{f}_j(u+h+s) \Sigma_L \mathbf{f}_j(u)^T du \right] \mathbf{f}_j(t+s) \Sigma_L \mathbf{f}_j(t)^T dt ds.$$

Putting both equalities together yields

$$e_j^T \otimes e_i^T \cdot \left[ \int_0^\infty [\mathbf{Y}_n \cdot \Sigma_L \otimes \Sigma_L \cdot \mathbf{Y}_n^T + \mathbf{Y}_n \cdot \Sigma_L \otimes \Sigma_L \cdot \mathbf{Y}_n^T] du \right] \cdot e_j \otimes e_i$$

$$= 2 \int_0^\infty \left[ \int_0^\infty \mathbf{f}_j(u+s) \Sigma_L \mathbf{f}_j(u)^T du \right] \mathbf{f}_j(t+s) \Sigma_L \mathbf{f}_j(t)^T dt ds = 2 \int_0^\infty \mathbf{r}(s) \mathbf{r}(s) ds. \quad (4.6)$$
The equality in (1.7) and analogous calculations as above give
\[ e_j^T \otimes e_i^T \cdot \left[ \int_0^\infty \left( \Sigma_Y(u+h) \cdot \Sigma_{L}^{1/2} \otimes \Sigma_{L}^{1/2} \cdot P_{m,m} \cdot \Sigma_{L}^{1/2} \otimes \Sigma_{L}^{1/2} \cdot \Sigma_Y(u-h)^T \right. \right. \]
\[ + \Sigma_Y(u-h) \cdot \Sigma_{L}^{1/2} \otimes \Sigma_{L}^{1/2} \cdot P_{m,m} \cdot \Sigma_{L}^{1/2} \otimes \Sigma_{L}^{1/2} \cdot \Sigma_Y(u+h)^T \left. \right] du \right] \cdot e_j \otimes e_i \]
\[ = 2 \int_0^\infty \gamma_j(s+h) \gamma_i(s-h) \, ds \quad (4.7) \]
as well. Then (a) follows from Corollary 4.2 (4.6) and (4.7).
(b) is a conclusion of (a) and Remark 5.1 b).
\[ \square \]

**Proof of Corollary 4.4.**
(a) can be calculated similarly to Corollary 4.1 (a).
(b) The proof can be done step by step as for MA processes in [Brockwell and Davis, 1991, Theorem 7.2.1] using (a).
\[ \square \]

## 5 Asymptotic behavior of the sample autocovariance function of MA models

In Section 4 we derived the asymptotic behavior of the sample autocovariance function of a MCARMA process. On a similar way we derive the analogous results for the sample autocovariance function of a multivariate MA process in discrete time. The proofs are only slightly different, and are therefore omitted.

The first authors who investigated the asymptotic behavior of the sample autocovariance function for multivariate MA processes in a very general setup are Su and Lund (2012). A difference between their study and our study is that they define the covariance of two random matrices \( U, V \) with \( E(U) = E(V) = 0_{m \times m} \) as \( \text{Cov}_{SU}(U, V) := E(U \otimes V) \) where we use \( \text{Cov}_{F}(U, V) := E(\text{vec}(U) \text{vec}(V)^T) \). The covariance of Su and Lund \( \text{Cov}_{SU}(U, U)^T = E(U^T \otimes U^T) \) is not necessarily symmetric if \( U \) is not a symmetric random matrix in contrast \( \text{Cov}_{F}(U, U)^T = \text{Cov}_{F}(U, U) \).

A multivariate MA process has the representation
\[ Y_k = \sum_{j=0}^{\infty} C_{k-j} \xi_j \quad \text{for } k \in \mathbb{Z}, \quad (5.1) \]
where \((\xi_k)_{k \in \mathbb{Z}}\) is a sequence of iid random vectors in \( \mathbb{R}^n \) and \((C_j)_{j \in \mathbb{N}_0}\) is a sequence of deterministic matrices in \( \mathbb{R}^{m \times m} \). We will assume that \( E(\xi_1) = 0_m, E(\|\xi_1\|^2) < \infty \) and \( \sum_{j=0}^{\infty} E(\|C_j\|^2) < \infty \) so that the autocovariance function \( \Gamma_Y(h) = E(Y_0 Y_h^T) \) for \( h \in \mathbb{Z} \) is well-defined. The sample autocovariance function is defined as
\[ \widehat{\Gamma}_n(h) = \frac{1}{n-h} \sum_{k=1}^{n-h} (Y_k - \bar{Y}_n)(Y_{k+h} - \bar{Y}_n)^T \quad \text{for } h \in \{0, 1, \ldots, n-1\}, \]
where \( \bar{Y}_n = \frac{1}{n} \sum_{k=1}^{n} Y_k \) is the sample mean. It has the following asymptotic behavior in analogy to Theorem 4.1.

**Theorem 5.1.** Let \((Y_k)_{k \in \mathbb{Z}}\) be a multivariate MA process as defined in (5.1) with noise sequence \((\xi_k)_{k \in \mathbb{Z}}\) satisfying \( E(\|\xi_1\|^2) < \infty, E(\xi_1) = 0_m \) and \( \Sigma_{\xi} = E(\xi_1 \xi_1^T) \). Moreover, we assume that \((C_j)_{j \in \mathbb{N}_0}\) satisfies \( \sum_{j=0}^{\infty} E(\|C_j\|^2) < \infty \) and \( C_j = 0_{m \times m} \) for \( j < 0 \). Suppose \( N^*(\Gamma^*) \) is an \( \mathbb{R}^m \times m \)-dimensional normal random matrix with \( \text{vec}(N^*(\Gamma^*)) \sim \mathcal{N}(0_{m^2}, \Sigma) \) where
\[ \Gamma^* = E((\xi_1 \otimes \xi_1) \cdot (\xi_1 \otimes \xi_1)^T) - E(\xi_1 \otimes \xi_1)E(\xi_1 \otimes \xi_1)^T. \]
Furthermore, assume \( N^*(\Gamma^*) \) is independent from the sequence of iid \( \mathbb{R}^m \times m \)-valued random matrices \((N_j)_{j \in \mathbb{N}}\) with independent standard normally distributed components. Let \( \mathcal{H} \subseteq \mathbb{N}_0 \) be a finite set. Then as
Let the assumptions of Theorem 5.1 hold. Define
\[
\Sigma_r := \sum_{j=0}^{\infty} C_{j+r} \otimes C_j \quad \text{and} \quad \Sigma_r^* := \sum_{j=0}^{\infty} C_j \otimes C_{j+r} \quad \text{for } r \in \mathbb{Z}.
\]

Let \( P_{m,m} \) be the Kronecker permutation matrix and \( h \in \mathbb{N}_0 \). Then \( \mathbb{E}(\text{vec}(Y_0 Y_0^T)) = \Sigma_h \text{vec}(\Sigma) \) and as \( n \to \infty \),
\[
\sqrt{n} \text{vec} \left( \hat{\Gamma}_n(h) - \Gamma_Y(h) \right)
\]
\[ \overset{d}{\to} \mathcal{N} \left( 0_{m^2} \cdot \Sigma_h \cdot Y^* \cdot \Sigma_h^* + \sum_{r=1}^{\infty} \left[ \Sigma_{r+h} \cdot \Sigma_q \cdot \Sigma_{r-h}^T + \Sigma_{r-h} \cdot \Sigma_q \cdot \Sigma_{r+h}^T \right] \right)
\]
\[
+ \sum_{r=1}^{\infty} \left[ \Sigma_{r+h} \cdot \Sigma_q^{1/2} \cdot \Sigma_q^{1/2} \cdot P_{m,m} \cdot \Sigma_q^{1/2} \cdot \Sigma_q^{1/2} \cdot \Sigma_q \cdot \Sigma_q \cdot \Sigma_q^T \right]
\]
\[ + \sum_{r=1}^{\infty} \left[ \Sigma_{r+h} \cdot \Sigma_q^{1/2} \cdot \Sigma_q^{1/2} \cdot \Sigma_q \cdot \Sigma_q \cdot \Sigma_q \cdot \Sigma_q \cdot \Sigma_q \cdot \Sigma_q^T \right].
\]

Remark 5.1. The assumption \( \sum_{j=0}^{\infty} j ||C_j||^2 < \infty \) is not a necessary assumption. We require this for our way of proof because they are necessary for the discrete-time versions of \( J_{n,1}, \ldots, J_{n,4} \) given in \((4.3)\) to be well-defined. A guess is that \( \sum_{j=0}^{\infty} ||C_j|| < \infty \) is a sufficient assumption; it is also sufficient in the one-dimensional case. □

The vector-representation of this limit result is the following.

Corollary 5.1. Let the assumptions of Theorem 5.1 hold. Define
\[
\Sigma_r := \sum_{j=0}^{\infty} C_{j+r} \otimes C_j \quad \text{and} \quad \Sigma_r^* := \sum_{j=0}^{\infty} C_j \otimes C_{j+r} \quad \text{for } r \in \mathbb{Z}.
\]

Let \( P_{m,m} \) be the Kronecker permutation matrix and \( h \in \mathbb{N}_0 \). Then \( \mathbb{E}(\text{vec}(Y_0 Y_0^T)) = \Sigma_h \text{vec}(\Sigma) \) and as \( n \to \infty \),
\[
\sqrt{n} \text{vec} \left( \hat{\Gamma}_n(h) - \Gamma_Y(h) \right)
\]
\[ \overset{d}{\to} \mathcal{N} \left( 0_{m^2} \cdot \Sigma_h \cdot Y^* \cdot \Sigma_h^* + \sum_{r=1}^{\infty} \left[ \Sigma_{r+h} \cdot \Sigma_q \cdot \Sigma_{r-h}^T + \Sigma_{r-h} \cdot \Sigma_q \cdot \Sigma_{r+h}^T \right] \right)
\]
\[ + \sum_{r=1}^{\infty} \left[ \Sigma_{r+h} \cdot \Sigma_q^{1/2} \cdot \Sigma_q^{1/2} \cdot P_{m,m} \cdot \Sigma_q^{1/2} \cdot \Sigma_q^{1/2} \cdot \Sigma_q \cdot \Sigma_q \cdot \Sigma_q^T \right].
\]

The limit structure in the discrete-time model (Corollary 5.1) and in the continuous-time model (Corollary 4.2) are the same: sums are only replaced by integrals, and \( Y \) by \( Y^* \).

The cross-covariances between the \( i \)-th and the \( j \)-th component of \( Y \) is presented next.

Corollary 5.2. Let the assumptions of Theorem 5.1 hold, and denote by \( \gamma(h) = \mathbb{E}(Y_0(i) Y_h(j)) \) the autocovariance function of the \( i \)-th component and by \( \gamma_{kj}(h) = \mathbb{E}(Y_0(i) Y_h(j)) \), \( h \in \mathbb{Z} \), the cross-covariance function between the \( i \)-th and the \( j \)-th component of \( (Y_k)_{k \in \mathbb{Z}} \). Furthermore,
\[
\hat{\gamma}_{kj}^{(ij)}(h) := e_i^T \hat{\Gamma}_n(h) e_j = \frac{1}{n} \sum_{k=1}^{n-h} (Y_k(i) - Y_n(i)) (Y_{k+h}(j) - Y_n(j)) \quad \text{for } h \in \{0,1,\ldots,n-1\},
\]

is the sample cross-covariance function between the \( i \)-th and the \( j \)-th component, and \( \hat{\gamma}_{kj}^{(ij)}(h) = e_i^T \hat{\Gamma}_n e_j = \frac{1}{n} \sum_{k=1}^{n-h} Y_k(i) Y_{k+h}(j) \) is the sample mean of the \( i \)-th component of \( (Y_k)_{k \in \mathbb{Z}} \). Then as \( n \to \infty \),
\[
\sqrt{n} \left( \hat{\gamma}_{kj}^{(ij)}(h) - \gamma_{kj}(h) \right)
\]
\[ \overset{d}{\to} \mathcal{N} \left( 0 \mathbb{E} \left( \sum_{r=0}^{\infty} (e_i^T C_r \xi) \cdot (e_j^T C_{r+h} \xi) \right) - 3 \gamma_{ij}(h)^2 + \sum_{r=-\infty}^{\infty} \gamma(r) \gamma_j(r+h) \gamma_{ij}(r-h) \right). 
\]

Most results in the literature, with exception of Su and Lund (2013), restricted their attention to cross-covariances for either Gaussian processes or independent processes where the fourth moment part can be neglected. The result presented here is an extension.

Finally, we present the well-known Bartlett’s formula (see Theorem 7.2.1 and Proposition 7.3.4 in Brockwell and Davis (1991), and Theorem 6.3.6 and Corollary 6.3.6.1 in Fuller (1996)). It is a direct consequence of Theorem 5.1.
Corollary 5.3. Let \((Y_k)_{k \in \mathbb{Z}}\) be an one-dimensional MA process satisfying the assumptions of Theorem 4.4.

(a) The autocovariance function of \((Y_k)_{k \in \mathbb{Z}}\) is denoted by \((\gamma(h))_{h \in \mathbb{Z}}\) and \((\hat{\gamma}_n(h))_{h \in \{0, \ldots, n-1\}}\) denotes the sample autocovariance function. Then as \(n \to \infty\),
\[
\left(\sqrt{n}(\hat{\gamma}_n(h) - \gamma(h))\right)_{h \in \mathbb{H}} \xrightarrow{d} \mathcal{N}(0, (m_{s,t})_{s,t \in \mathbb{H}}),
\]
where
\[
m_{s,t} = \left((\mathbb{E}(\xi_1^2))^2 - 2\mathbb{E}(\xi_1^4) - 3\gamma(s)\gamma(t) + \sum_{r=-\infty}^{\infty} \gamma(r+s)\gamma(r+t) + \gamma(r+s)\gamma(r-t)\right).
\]
(b) The autocorrelation function of \((Y_t)_{t \in \mathbb{R}}\) is denoted by \((\rho(h))_{h \in \mathbb{Z}}\) and \((\hat{\rho}_n(h))_{h \in \{0, \ldots, n-1\}}\) denotes the sample autocorrelation function. Then as \(n \to \infty\),
\[
\left(\sqrt{n}(\hat{\rho}_n(h) - \rho(h))\right)_{h \in \mathbb{H}} \xrightarrow{d} \mathcal{N}(0, (v_{s,t})_{s,t \in \mathbb{H}}),
\]
where \((v_{s,t})_{s,t \in \mathbb{H}}\) is as in (4.1).

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References


