TAIL PROBABILITIES OF RANDOM LINEAR FUNCTIONS OF REGULARLY VARYING RANDOM VECTORS

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We provide a new extension of Breiman’s Theorem on computing tail probabilities of a product of random variables to a multivariate setting. In particular, we give a complete characterization of regular variation on cones in \([0, \infty)^d\) under random linear transformations. This allows us to compute probabilities of a variety of tail events, which classical multivariate regularly varying models would report to be asymptotically negligible. We illustrate our findings with applications to risk assessment in financial systems and reinsurance markets under a bipartite network structure.

1. Introduction. In this article we study the probability of tail events for random linear functions of regularly varying random vectors. Suppose \(Z\) is a non-negative random vector with multivariate regularly varying tail distribution on \(\mathbb{E}_d^{(1)} := [0, \infty)^d \setminus \{0\}\) with index \(-\alpha_1 \leq 0\), denoted \(\text{MRV}(\alpha_1, \mathbb{E}_d^{(1)})\), a precise definition of this notion is given in Section 2. Furthermore, let \(A\) be a \(q \times d\) random matrix independent of \(Z\). For \(X = AZ\), our goal is to find \(P(X \in tC)\) for large values of \(t\) and a wide variety of sets \(C \subset [0, \infty)^q\).

A classical result on the tail behavior of a product of random variables, now known as Breiman’s Theorem, states that given independent non-negative random variables \(Z\) and \(A\), where \(Z\) has a univariate regularly varying tail distribution with index \(-\alpha \leq 0\) and \(E[A^{\alpha + \delta}] < \infty\) for some \(\delta > 0\), the tail distribution of \(X = AZ\) is also regularly varying with index \(-\alpha\). More precisely,

\[
P(X > x) \sim E[A^\alpha]P(Z > x), \quad \text{as } x \to \infty.
\]  

(1.1)

This was stated first in Breiman [3] for \(\alpha \in [0, 1]\) and established for all \(\alpha \geq 0\) in Cline and Samorodnitsky [6]. The inherent applicability of this result to stochastic recurrence equations and portfolio tail risk computations has led to a few generalizations in the past decades. A generalization of Breiman’s Theorem by relaxing the assumption of independence of random variables \(A\) and \(Z\) to asymptotic independence was provided in Maulik, Resnick and Rootzén

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A vector-valued generalization of (1.1) was obtained in Basrak, Davis and Mikosch [1, Proposition A.1] where a $d$-dimensional non-negative random vector $Z \in \mathcal{MRV}(\alpha_1, \mathbb{E}_d^{(1)})$ is independent of a $q \times d$ random matrix $A$ with $\mathbb{E}\|A\|^{\alpha_1+\delta} < \infty$. The result states, that in such a case $X = AZ \in \mathcal{MRV}(\alpha_1, \mathbb{E}_d^{(1)})$ where $\mathbb{E}_q^{(1)} = \{0, \infty\}^q \setminus \{0\}$. A generalization of this result with respect to the dependence and joint regular variation assumptions on $(A, Z)$ was given in Fougères and Mercadier [12]. On the other hand, Janssen and Drees [15, Theorem 2.3] generalizes Proposition A.1 in Basrak, Davis and Mikosch [1] so that one may compute probabilities of tail sets $C$ contained in $\mathbb{E}_q^{(1)} = (0, \infty)^q$ when $q = d$ and $A$ is of full rank (and certain other conditions). For $X = AZ \in \mathcal{MRV}(\alpha_1, \mathbb{E}_q^{(1)})$, they show that $X = AZ \in \mathcal{MRV}(\alpha_q, \mathbb{E}_q^{(1)})$, and here $\alpha_q \geq \alpha_1$.

Consider the following example to fix ideas in this setting. Let $Z$ be comprised of iid Pareto random variables with $P(Z_i > x) = x^{-\alpha}, x > 1$ where $\alpha > 0$, and let $A$ be a $d \times d$ random matrix independent of $Z$ satisfying the conditions for both [1, Proposition A.1] and [15, Theorem 2.3]. It is easy to check that $Z \in \mathcal{MRV}(\alpha_1 = \alpha, \mathbb{E}_q^{(1)})$ and $Z \in \mathcal{MRV}(\alpha_d = d\alpha, \mathbb{E}_d^{(1)})$; furthermore implying that $X = AZ \in \mathcal{MRV}(\alpha, \mathbb{E}_q^{(1)})$ and $X = AZ \in \mathcal{MRV}(d\alpha, \mathbb{E}_d^{(1)})$.

Hence for sets of the form $[0, x]^c$ and $(x, \infty)$ with $x > 0$, we are able to compute for $t \to \infty$:

$$P(X \in t[0, x]^c) \sim t^{-\alpha} E[\mu_1(AZ \in [0, x]^c)];$$

(1.2)

$$P(X \in t(x, \infty)) \sim t^{-d\alpha} E[\mu_d(AZ \in (x, \infty)],$$

(1.3)

for some measures $\mu_1(\cdot)$ and $\mu_d(\cdot)$ to be elaborated on later. Moreover, the quantities on the right hand side of both (1.2) and (1.3) are non-trivial and finite; hence our probability estimates are valid. Thus (1.2) allows us to compute probabilities of events described as “at least one of the components of $X$ is large”, whereas (1.3) allows us to compute probabilities of events described as “all components of $X$ are large”. Natural questions to inquire of here would be, what if we want to compute such probabilities when the matrix $A$ is not invertible, or perhaps $q \neq d$. We may also wish to find the probability that “at least three of the components of $X$ are large” or “exactly two of the components of $X$ are large”. We can check that, although a probability computation akin to (1.2) is possible in such a case, it will often render the measure $\mu_1$ and hence the right hand side of (1.2) to be zero. On the other hand (1.3) will fail to answer such a question if either $q \neq d$ or the particular set of concern does not have all components to be large. To the best of our knowledge, (1.2) and (1.3) are the only results that compute probabilities of extreme sets for random linear functions of regularly varying vectors. In our work, we provide a generalization of Breiman’s Theorem which allows us to compute such probabilities for more general extreme sets. For example, in this particular setting of $Z$ being iid Pareto, our results show that

$$P(X \in tC) \sim t^{-i\alpha} \mathbb{P}(C), \quad t \to \infty,$$

(1.4)

where $\mathbb{P}(\cdot)$ is an appropriate measure with $i \in \{1, \ldots, d\}$ depending on the form of the set $C$ as well as the matrix $A$. Moreover, we can show that $AZ \in \mathcal{MRV}(i\alpha, \mathbb{P}, \mathbb{E}^*)$ where $\mathbb{E}^*$ is an appropriate subspace of $\mathbb{E}_q^{(1)}$ depending on the set $C$ and $\mathbb{P}$ is a measure on $\mathbb{E}^*$.

Further related literature: A few other publications have also exhibited interesting applications and generalizations of the result of Breiman, albeit in different contexts. In Jessen and
Mikosch [16], the authors provide partial converses to Breiman’s result; assuming $A$ and $Z$ to be non-negative independent random variables, if $AZ$ has a regularly varying tail distribution, they find conditions when $Z$ will also have a regularly varying tail distribution. Recently, Tillier and Wintenberger [26] have extended Breiman’s multivariate result to vectors of random length, determined for instance by a Poisson random variable. In a more general setting, Chakraborty and Hazra [5], extend Breiman’s result for multiplicative Boolean convolution of regularly varying measures. Finally, the monograph Buraczewski, Damek and Mikosch [4] provides many applications of Breiman’s result and its generalizations in the area of stochastic modeling with power-law tail.

Our interest in computation of probabilities of the form (1.4) is motivated by a wide range of applications in mind. Regularly varying distributions have been used to model power-law tail behavior in stochastic models in applications including hydrology, finance, insurance, telecommunications, social networks and much more. A regularly varying random vector like $Z \in [0, \infty)^d$ can be used to represent returns from multiple stocks (in finance) or losses pertaining to different insurance companies (in an insurance context). In such a case a $q \times d$ random matrix $A$ represents portfolios of a group of stockholders or business entities, or, exposures of reinsurers to the insurance companies, respectively. Thus a common quantity of interest to compute here is $P(AZ \in tC)$ for tail sets $C$ representing a variety of worst case scenarios relating to multiple portfolio, or bankruptcy or loss for multiple insurers.

Our paper is organized as follows. We provide a summary of notations used in the paper in Section 1.1 to finish up the introduction. In Section 2, we discuss multivariate regular variation with $\mathcal{M}$-convergence in different subspaces of $[0, \infty)^d$ which provides a set up for the main result of the paper. Our main result extending Breiman’s result is developed in Section 3. In Section 4, we provide applications of the model in the context of bipartite networks, where $q$ agents can be exposed to the risk of $d$ objects where $Z \in [0, \infty)^d$ are the risks of the objects. The exposures of the agents is represented by $X = AZ$ and illustrates the behavior of tail risk of the agents for possible structures of the weighted adjacency matrix $A \in [0, \infty]^{q \times d}$. We conclude with indications to future directions of research in Section 5.

1.1. Notations. Various notations and concepts used in this paper are summarized in this section. Bold letters are used to denote vectors, with capital letters used for random vectors and small letters for non-random vectors, e.g., $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$. Vector operations are always understood component-wise, e.g., for vectors $x$ and $y$, $x \leq y$ means $x_i \leq y_i$ for all $i$.

For a constant $c \in \mathbb{R}$ and a set $A \subset \mathbb{R}^d$, we denote by $cA := \{ cx : x \in A \}$. Further notations are tabulated below. References are provided wherever applicable.

- $\mathcal{RV}_\beta$: Regularly varying functions with index $\beta \in \mathbb{R}$; that is, functions $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying $\lim_{t \to \infty} f(tx)/f(t) = x^\beta$, for $x > 0$. See Definition 2.1 and [2, 9, 24] for further details.
- $\mathbb{R}_d^d$: $[0, \infty)^d$ for dimension $d \geq 1$.
- $v^{(1)}, \ldots, v^{(d)}$: Order statistics of $v = (v_1, \ldots, v_d) \in \mathbb{R}_+^d$ such that $v^{(1)} \geq v^{(2)} \geq \ldots \geq v^{(d)}$.
- $\mathcal{CA}_d^{(i)}$: $\{ v \in \mathbb{R}_+^d : v^{(i)} = 0 \}$ for $i = 1, \ldots, d - 1$. Also define $\mathcal{CA}_d^{(d)} = \{0\}$. 
The necessary definitions and results formulated with respect to multivariate regular variation on a sequence of subspaces of \( \mathbb{R}^d \) extending results from Basrak, Davis and Mikosch [1], Janssen and Drees [15]. The particular choice of subsets where we seek hidden regular variation are natural, thus we often omit one or more of the arguments. See Definition 2.2 for details.

\[ \tau^{(k)}_{\theta}(x) = d(x, CA \theta^{(k-1)}) = x^{(k)}, \text{ the distance between } x \in \mathbb{R}^d_+ \text{ and } CA \theta^{(k-1)} \]

where \( d(x, y) = \|x - y\|_2 \); see Section 3 for details.

\[ \tau^{i,k}_{d,q}(A) = \sup_{x \in \mathbb{R}^d_+} \frac{\tau^{(k)}_{\theta}(Az)}{x^{(i)}(z)} \text{ for } A \in \mathbb{R}^{q \times d}; \text{ see Section 3 for details.} \]

2. Multivariate regular variation and convergence concepts. All random objects in this paper are defined on a probability space \((\Omega, \mathcal{F}, P)\). We use the notion of \(M\)-convergence of measures to define multivariate regular variation on Euclidean spaces and subsets thereof; see Das, Mitra and Resnick [7], Lindskog, Resnick and Roy [19] for details. In particular, we investigate hidden regular variation of a random vector \( X \), which is given as \( X = AZ \), where \( Z \in \mathbb{R}^d_+ = [0, \infty)^d \) is multivariate regularly varying with index \(-\alpha \leq 0\) and \( A \) is a \( q \times d \) random matrix independent of \( Z \) such that \( \mathbb{E}[\|A\|^{\alpha + \delta}] < \infty \) for some \( \delta > 0 \) and an operator norm \( \| : \| \) for matrices.

Our goal is to obtain a complete picture concerning linear functions \( X \) which possess multivariate regular variation on a sequence of subspaces of \( \mathbb{R}^d_+ \) (also called hidden regular variation), thus extending results from Basrak, Davis and Mikosch [1], Janssen and Drees [15]. The particular choice of subsets where we seek hidden regular variation are natural, depending on the type of extreme sets for which we seek to find probabilities; see Mitra and Resnick [22] for examples. The necessary definitions and results formulated with respect to \( M \)-convergence are discussed below.

**Definition 2.1 (Regular variation).** A positive measurable function \( f \) defined on \((0, \infty)\) is regularly varying at infinity with index \( \beta \in \mathbb{R} \), if \( \lim_{t \to \infty} f(tx)/f(t) = x^\beta \) holds for all \( x \in \mathbb{R} \). We write \( f \in \mathcal{R} \mathcal{V}_\beta \). A real-valued random variable \( V \) with distribution function \( F \) is regularly varying (at infinity) if \( F := 1 - F \in \mathcal{R} \mathcal{V}_{-\alpha} \) for some \(-\alpha \leq 0\). We then write \( V \in \mathcal{R} \mathcal{V}_{-\alpha} \).

Consider the space \( \mathbb{R}^d_+ \) endowed with a metric \( d(x, y) \) satisfying for some \( c > 0 \)

\[ d(cx, cy) = cd(x, y), \quad (x, y) \in \mathbb{R}^d_+ \times \mathbb{R}^d_. \quad (2.1) \]

Any metric \( d \) defined by a norm as \( d(x, y) = \|x - y\| \) will always satisfy (2.1). Recall that a cone \( C \subseteq \mathbb{R}^d_+ \) is a set which is closed under scalar multiplication: if \( x \in C \) then \( cx \in C \) for \( c > 0 \). A closed cone of course, is a cone which is a closed set in \( \mathbb{R}^d_+ \). Now we define multivariate regular variation using convergence of measures on a closed cone \( C \subseteq \mathbb{R}^d_+ \) with...
a closed cone $C_0 \subseteq R^d$ deleted. Moreover, we say that a subset $\Lambda \subseteq R^d \setminus C_0$ is bounded away from $C_0$ if $d(\Lambda, C_0) = \inf\{d(x, y) : x \in \Lambda, y \in C_0\} > 0$. In this paper, we use the $\ell^2$-norm (or Euclidean norm) as our choice of metric $d$, since the distance of a point $y \in R^d_+$ to specific closed sets can be represented as an order statistics of the co-ordinates of $y$; see (3.4).

The convergence concept we use for defining regular variation on cones is $M$-convergence, which is slightly different from vague convergence which has been traditionally used in multivariate regular variation. Reasons for the preference of $M$-convergence are presented in Das and Resnick [8, Remark 1.1.]; see also Das, Mitra and Resnick [7], Lindskog, Resnick and Roy [19]. The class of Borel measures on $R^d \setminus C_0$ that assign finite measure to all Borel sets $B \subseteq R^d \setminus C_0$, which are bounded away from $C_0$, is denoted by $M(R^d \setminus C_0)$. In the space $E^{(1)}_d = R^d_+ \setminus \{0\}$ the notions of vague convergence and $M$-convergence are identical.

DEFINITION 2.2 (Multivariate regular variation; Definition 3.2 of [19]). Let $C_0 \subseteq R^d \subseteq R^d_+$ be closed cones containing $0$. A random vector $V = (V_1, \ldots, V_d) \in R^d$ is regularly varying on $R^d \setminus C_0$ if there exists a function $b(\cdot) \in RV_{1/\alpha}$ for $\alpha \geq 0$, called the scaling function, and a non-null (Borel) measure $\mu(\cdot) \in M(R^d \setminus C_0)$ called the limit or tail measure such that

$$t \mathbf{P}(V/b(t) \in \cdot) \to \mu(\cdot), \quad t \to \infty,$$

in $M(R^d \setminus C_0)$. We write $V \in MRV(\alpha, b(\cdot), \mu, R^d \setminus C_0)$ or, if the scaling function is contextually irrelevant and $R^d \setminus C_0 = [0, \infty)^d \setminus \{0\} =: E^{(1)}_d$, we simply write $V \in MRV(\alpha, \mu, E^{(1)}_d)$ or $V \in MRV(\alpha, \mu)$.

A possible choice of $b(\cdot)$ is given by using $t \mathbf{P}(\max\{V_1, \ldots, V_d\} > b(t)) \to 1$ as $t \to \infty$. Since $b(\cdot) \in RV_{1/\alpha}$, the limit measure $\mu(\cdot)$ has a scaling property:

$$\mu(c \cdot) = c^{-\alpha} \mu(\cdot), \quad c > 0. \quad (2.3)$$

It is often convenient to restate Definition 2.2 using generalized polar coordinates, cf. Das and Resnick [8, Section 1.4.1]. Denote the unit ball and unit sphere in $R^d \setminus C_0$ by

$$R_{C_0} = \{x \in R^d \setminus C_0 : d(x, C_0) \geq 1\}, \quad \partial R_{C_0} = \{x \in R^d \setminus C_0 : d(x, C_0) = 1\}. \quad (2.4)$$

Define the polar co-ordinate transform relative to the deleted closed cone $C_0$ by $GPOLAR : R^d \setminus C_0 \to (0, \infty) \times \partial R_{C_0}$ defined as

$$GPOLAR(x) = (d(x, C_0), \frac{x}{d(x, C_0)}). \quad (2.5)$$

The following result holds; see Mitra and Resnick [22, Proposition 3.1] and Lindskog, Resnick and Roy [19, Corollary 4.4].

PROPOSITION 2.3. A random vector $V \in R^d$ satisfies $V \in MRV(\alpha, b(\cdot), \mu, R^d \setminus C_0)$ with $\mu(R_{C_0}) = 1$ if and only if

$$t \mathbf{P}(GPOLAR(V)/b(t) \in \cdot) \to \nu_\alpha \times S_{C_0}(\cdot), \quad (2.6)$$

in $M((0, \infty) \times \partial R_{C_0})$. Moreover, for $\alpha > 0$,

$$\nu_\alpha(z, \infty) = z^{-\alpha}, \quad z > 0, \quad (2.7)$$

and, for appropriate choice of $b(\cdot)$, $S_{C_0}(\cdot)$ is a probability measure on the unit sphere $\partial R_{C_0}$.
DEFINITION 2.4 (Hidden regular variation). Let $C_1 \subset C_2 \subset C \subset \mathbb{R}^d_+$ be closed cones containing $0$. Let $F_i = C \setminus C_i, i = 1, 2$. Suppose the random vector $V = (V_1, \ldots, V_d) \in C$ is such that $V \in \mathcal{MRV}(\alpha_1, \beta_1, \nu_1, F_1)$. Moreover, $\nu_1(F_2) = 0$ and there exists $b_2(t) \in \mathcal{RV}_{V_1/\alpha_2}$ with $\alpha_2 \geq \alpha_1$ and $b_1(t)/b_2(t) \to \infty$ as $t \to \infty$ such that $V \in \mathcal{MRV}(\alpha_2, \beta_2, \nu_2, F_2)$. Then we say $V$ has regular variation on $F_1$ with hidden regular variation on $F_2$.

2.1. Hidden regular variation on a sequence of subspaces. We define regular variation on a specific sequence of subspaces of $\mathbb{R}^d_+$ following Mitra and Resnick [22]. For $V \in \mathbb{R}^d_+$, write $v = (v_1, \ldots, v_d)$. Moreover, the order statistics for any vector $v \in \mathbb{R}^d_+$ is defined as

$$v^{(1)} \geq v^{(2)} \geq \ldots \geq v^{(d)}.$$  

(2.8)

where $v^{(i)}$ denotes the $i$th largest component of $v$. First we define closed sets which we think of as union of co-ordinate hyper-planes of various orders in $\mathbb{R}^d_+$. Define $CA_d^{(0)} := \{0\}$ and for $1 \leq i \leq d - 1$ define

$$CA_d^{(i)} = \bigcup_{1 \leq j_1 < \ldots < j_{d-1}+1 \leq d} \{v \in \mathbb{R}^d_+ : v_{j_1} = 0, \ldots, v_{j_{d-1}+1} = 0\} = \{v \in \mathbb{R}^d_+ : v^{(i)} = 0\}.$$  

Here $CA_d^{(i)}$ represents the union of all $i$-dimensional co-ordinate hyperplanes in $\mathbb{R}^d_+$. Now define the following sequence of subcones of $\mathbb{R}^d_+$:

$$E_d^{(1)} := \mathbb{R}^d_+ \setminus CA_d^{(0)} = \mathbb{R}^d_+ \setminus \{0\} = \{v \in \mathbb{R}^d_+ : v^{(1)} > 0\},$$  

(2.9)

$$E_d^{(i)} := \mathbb{R}^d_+ \setminus CA_d^{(i-1)} = \{v \in \mathbb{R}^d_+ : v^{(i)} > 0\}, \quad 2 \leq i \leq d.$$  

(2.10)

Hence, $E_d^{(1)}$ is the non-negative orthant with $\{0\} = CA_d^{(0)}$ removed, $E_d^{(2)}$ is the non-negative orthant with all one-dimensional co-ordinate axes removed, $E_d^{(3)}$ is the non-negative orthant with all two-dimensional co-ordinate hyperplanes removed, and so on. Clearly, we have

$$E_d^{(1)} \supset E_d^{(2)} \supset \ldots \supset E_d^{(d)}.$$  

(2.11)

We also denote the unit balls and unit spheres in $E_d^{(i)}$ for $1 \leq i \leq d$ by

$$N_d^{(i)} = \{x \in E_d^{(i)} : d(x, CA_d^{(i-1)}) \geq 1\}, \quad \partial N_d^{(i)} = \{x \in E_d^{(i)} : d(x, CA_d^{(i-1)}) = 1\}.$$  

(2.12)

A recipe for finding hidden regular variation in the above sequence of cones can be devised as follows. To start with, suppose $V \in \mathcal{MRV}(\alpha_1, b_1(\cdot), \mu_1, E_d^{(1)})$.

1. If $\mu_1(E_d^{(d)}) > 0$, we seek no further regular variation on cones of $\mathbb{R}^d_+$.
2. If $\mu_1(E_d^{(d)}) = 0$, we can find $2 \leq i \leq d$ such that $\mu_1(E_d^{(i-1)}) > 0$, yet $\mu_1(E_d^{(i)}) = 0$. Hence $\mu_1$ concentrates on $CA_d^{(i-1)}$. So we seek regular variation in $E_d^{(i)} = \mathbb{R}^d_+ \setminus CA_d^{(i-1)}$. Suppose there exists $b_i(t) \uparrow \infty$ with $\lim_{t \to \infty} b_i(t)/b_{i-1}(t) = \infty$ and $\mu_i \neq 0$ on $E_d^{(i)}$ such that $V$ is regularly varying on $E_d^{(i)}$. Then there exists $\alpha_i \geq \alpha_1$ such that $V \in \mathcal{MRV}(\alpha_i, b_i(\cdot), \mu_i, E_d^{(i)})$. Moreover, $b_i(\cdot) \in \mathcal{RV}_{V_1/\alpha_i}$ and $\mu_i(c(\cdot)) = c^{-\alpha_i} \mu_i(\cdot)$ for $c > 0$. Hence $V$ has hidden regular variation on $E_d^{(i)}$ with parameter $-\alpha_i$.
3. In the next step, if $\mu_i(E_d^{(d)}) > 0$, we stop looking for regular variation; otherwise we keep seeking hidden regular variation through $E_d^{(i+1)}, \ldots, E_d^{(d)}$ sequentially.
Example 2.5. Suppose $V_1, V_2, \ldots, V_d$ are iid Pareto($\alpha$) random variables with $\alpha > 0$. This means $V_i$ has tail probability $P(V_i > t) = t^{-\alpha}$, $t \geq 1$. Then for $d \geq 2$,
\[
\frac{P(V \in [c, \infty]^d)}{P(\|V\| > t)} = \frac{P(V_1 > tc, \ldots, V_d > tc)}{P(V_1 + \cdots + V_d > t)} \sim c^{-\alpha}t^{-\alpha} \to 0, \quad t \to \infty.
\]

However, we find regular variation on all subspaces $E_d^{(i)}$ for $1 \leq i \leq d$ with $\alpha_i = i\alpha$ and $b_i(t) = t^{1/(\alpha_i)}$; see Example 5.1 in [21] and Example 2.2 in [22]. The limit measure $\nu_i$ on $E_d^{(i)}$ is such that for any $z = (z_1, \ldots, z_d) \in E_d^{(i)}$,
\[
\nu_i\left(\{y \in E_d^{(i)} : y_{j_1} > z_{j_1}, \ldots, y_{j_i} > z_{j_i} \text{ for some } 1 \leq j_1 < \cdots < j_i \leq d\}\right) = \sum_{1 \leq j_1 < \cdots < j_i \leq d} (z_{j_1}z_{j_2}\cdots z_{j_i})^{-\alpha}.
\]

\[\square\]

Remark 1. Although multivariate regular variation can be defined for a very general class of cones in $\mathbb{R}_+^d$ (see [7, 19, 22] for examples); for the purposes of this paper, restricting to the sub-cones $E_d^{(1)}, \ldots, E_d^{(d)}$ defined in (2.9) and (2.10) suffice. For an example of regular variation with infinite sequence of indices on an infinite sequence of cones contained in the space $\mathbb{R}_+^2$; see Das, Mitra and Resnick [7, Example 5.3].

Remark 2. If $V^{(i)} \in \mathcal{MRV}(\alpha_i, b_i(\cdot), \mu_i, E_d^{(i)})$ for some $1 \leq i \leq d$ then a choice of the function $b_i(t)$ is such that:
\[
\lim_{t \to \infty} b_i(t)/F_{V^{(i)}}(1 - 1/t) = 1. \tag{2.13}
\]

where $V^{(i)}$ has distribution $F_{V^{(i)}}$. For the rest of the paper, unless specified otherwise, we call such a choice the canonical choice of $b_i$ when $V^{(i)} \in \mathcal{MRV}(\alpha_i, b_i(\cdot), \mu_i, E_d^{(i)})$. Clearly, there are many other alternative choices of $b_i$, which would lead to limit measures $\nu_i$, which are unique up to a constant.

3. Breiman’s theorem and regular variation on Euclidean subspaces. In this section we provide a complete characterization of the vector-valued generalization addressed in Basrak, Davis and Mikosch [1, Proposition A.1] for the space $E_d^{(1)}$ and its subsequent extension to $E_d^{(q)}$ for $q = d$ provided in Janssen and Drees [15, Theorem 2.3]. We investigate the vector $X = AZ$, where $A \in \mathbb{R}^{q \times d}$ is a random matrix and $Z \in \mathbb{R}_+^d$ is independent of $A$ satisfying $Z \in \mathcal{MRV}(\alpha)$ on a variety of subspaces of $\mathbb{R}_+^d$ and provide asymptotic rates of convergence and hence estimates of tail probabilities for $P(X \in tC)$ for sets $C \subset E_d^{(k)}$ where $1 \leq k \leq q$. For the sake of convenience, first we present the two available results addressing this issue.

Most results quoted from previous papers appeared with asymptotic properties and definitions in terms of vague convergence, we restate them here with respect to $\bar{M}$-convergence.

Theorem 3.1 (Basrak, Davis and Mikosch [1, Proposition A.1]). Let $Z \in \mathbb{R}_+^d$ such that $Z \in \mathcal{MRV}(\alpha_1, \mu_1)$ with $\alpha_1 \geq 0$ and $A \in \mathbb{R}^{q \times d}$ be a random matrix independent of $Z$ with
0 < E[\|A\|^{\alpha_1+\delta}] < \infty \text{ for some } \delta > 0. \text{ Then}

\[ \frac{P(t^{-1}AZ \in \cdot)}{P(\|Z\| > t)} \to E \left[ \mu_1(\{z \in E_d^{(1)} : Az \in \cdot\}) \right] =: \overline{\mu}_1(\cdot), \quad t \to \infty, \tag{3.1} \]

in \( M(E_d^{(1)}) \). In particular, we have \( AZ \in \mathcal{M}\mathcal{R}\mathcal{V}(\alpha_1, \overline{\mu}_1, E_d^{(1)}) \).

**Remark 3.** A couple of remarks are in order here.

(i) For \( \|Z\| \) to become large, it suffices that only one component of \( Z \) becomes large. Hence \( P(\|Z\| > t) \sim c P(\max_{1 \leq i \leq d} Z_i > t) \) for some constant \( c > 0 \).

(ii) Although the right hand side of (3.1) may not be identically zero, the structure of \( A \) may lead to zero appearing on the right hand side of (3.1) for certain sets (cf. Example 2.5), especially in the presence of asymptotic independence.

(iii) The observation in (1.2) is an easy consequence of this theorem.

Note that getting a zero on the right hand side of (3.1) is quite uninformative and a partial solution to providing a non-zero limit for \( P(t^{-1}AZ \in \cdot) \) under such a situation is provided in Janssen and Drees [15] when \( q = d \). The convergence occurs in the space \( E_d^{(d)} = (0, \infty)^d \), which means that we look for the rate of convergence of sets, where all components of \( X = AZ \) are large, translated into the event \( \{X^{(d)} > t\} \).

The formal setting in [15] is as follows. Define \( \tau : \mathbb{R}_+^d \to \mathbb{R}_+ \) to be the distance of \( x \) from the space \( C_{A(d-1)} := [0, \infty)^d \setminus (0, \infty)^d \) in the Euclidean norm, given by \( \tau(z) := d(z, C_{A(d-1)}) = \min_{1 \leq i \leq d} (z_i) \). For a matrix \( A \in \mathbb{R}^{d \times d} \), we define the analog

\[ \tau(A) := \sup_{z \in E_d^{(d)} : \tau(z) = 1} \tau(Az) = \sup_{z \in \partial E_d^{(d)}} \tau(Az). \tag{3.2} \]

**Theorem 3.2** (Janssen and Drees [15, Theorem 2.3]). Let \( Z \in \mathbb{R}_+^d \) be such that \( \|Z\| \to \infty \) and \( \mathcal{A} \subseteq \mathbb{R}^{d \times d} \) be a random matrix independent of \( Z \). Assume \( \tau(A) > 0 \) almost surely and \( E[\tau(A)^{\alpha_1+\delta}] < \infty \text{ for some } \delta > 0 \). Then

\[ \frac{P(t^{-1}AZ \in \cdot)}{P(\tau(Z) > t)} \to E \left[ \mu_d(\{z \in E_d^{(d)} : Az \in \cdot\}) \right] =: \overline{\mu}_d(\cdot), \quad t \to \infty, \tag{3.3} \]

in \( M(E_d^{(d)}) \). In particular, we have \( AZ \in \mathcal{M}\mathcal{R}\mathcal{V}(\alpha_d, \overline{\mu}_d, E_d^{(d)}) \).

**Remark 4.** A couple of remarks are necessary to explain the result obtained.

(i) Note that \( P(\tau(Z) > t) = P(\min_{1 \leq i \leq d} Z_i > t) \) which provides the rate of convergence of \( P(t^{-1}AZ \in \cdot) \) to zero.

(ii) The observation in (1.3) is an easy consequence of Theorem 3.2.

(iii) Theorem 3.2 was designed for a specific situation of stochastic volatility models; it is quite restrictive in its assumptions and fails to capture a variety of instances, where the right hand side of (3.1) is zero. In particular, for square matrices \( A \) with non-negative entries, Theorem 3.2 requires that \( A \) is invertible and, moreover, that its inverse has non-negative entries (see Janssen and Drees [15, Lemma 2.2]). This entails that \( A \) is a row permutation of a diagonal matrix with positive diagonal entries; cf. Ding and Rhee [11].
3.1. Extension of Breiman’s theorem to Euclidean subspaces. In light of the previous results, we provide a multivariate extension to Breiman’s theorem which entails non-trivial convergence for a multitude of forms of $A$. Let $Z \in \mathbb{R}^d_+$ and $A \in \mathbb{R}^{q \times d}_+$ and $X = AZ \in \mathbb{R}^q_+$. We define the analog sequence of subcones of $\mathbb{R}^d_+$ as in (2.9)-(2.10) and proceed as follows. For every $1 \leq k \leq q$, let $\tau_q^{(k)} : \mathbb{R}^q_+ \to \mathbb{R}_+$ denote the distance of a point $x \in \mathbb{R}^q_+$ from $\mathcal{A}_q^{(k-1)}$ which is given by

$$\tau_q^{(k)}(x) = d(x, \mathcal{A}_q^{(k-1)}) = x^{(k)}.$$  

(3.4)

For a matrix $A \in \mathbb{R}^{q \times d}_+$ we define in analogy to (3.2) the function $\tau_{q,d}^{(k,i)} : \mathbb{R}^q_+ \to \mathbb{R}_+$ given by

$$\tau_{q,d}^{(k,i)}(A) = \sup_{z \in \mathbb{E}_q^{(i)}} \frac{\tau_q^{(k)}(Az)}{\tau_d^{(i)}(z)} = \sup_{z \in \mathbb{E}_q^{(i)}} \frac{(Az)^{(k)}}{z^{(i)}} = \sup_{z \in \partial \mathbb{E}^{(i)}} \tau_q^{(k)}(Az).$$  

(3.5)

Note that $\tau_{q,d}^{(q,d)}(A) = \tau(A)$ from (3.2) if $q = d$.

Although the functions $\tau_q^{(k)}, \tau_d^{(k,i)}$ are not seminorms on the induced vector space (see [13, Section 5.1]), they have some nice properties as listed below. We call a row of $A$ trivial, if it is a zero vector.

**Lemma 3.3** (Properties). For every matrix $A \in \mathbb{R}^{q \times d}_+$ and $z \in \mathbb{R}^d_+$ the following holds for $i \in \{1, \ldots, d\}$ and $k \in \{1, \ldots, q\}$:

(a) $\tau_q^{(k)}(Az) \leq \tau_{q,d}^{(k,i)}(A)\tau_d^{(i)}(z)$.

(b) $\tau_{q,d}^{(k,i)}(A) \leq \tau_{q,d}^{(k-1,i)}(A)$.

(c) $\tau_{q,d}^{(k,i)}(A) \leq \tau_{q,d}^{(k,i+1)}(A)$.

(d) $\tau_{q,d}^{(q,1)}(A) > 0$ if and only if all rows of $A$ are non-trivial.

(e) $\tau_{q,d}^{(k,1)}(A) \leq \tau_{q,d}^{(1,1)}(A) < \infty$.

**Proof.** (a) By definition we have

$$\tau_q^{(k)}(Az) = \tau_q^{(k)}\left(\frac{A}{\tau_d^{(i)}}z\right)\tau_d^{(i)}(z) \leq \tau_{q,d}^{(k,i)}(A)\tau_d^{(i)}(z).$$  

(3.6)

(b) and (c) immediately follow from the definition.

(d) If $A = (a_{ij})_{i,j}$ has non-trivial rows, denoting $e = (1, \ldots, 1)^T$, we have

$$\tau_{q,d}^{(q,1)}(A) \geq \frac{\tau_q^{(q)}(Ae)}{\tau_d^{(1)}(e)} = \min_{1 \leq j \leq q} \sum_{i=1}^d a_{ij} > 0,$$

the final domination being a consequence of each row of $A$ having at least one positive entry.

On the other hand, suppose that $\tau_{q,d}^{(q,1)}(A) > 0$ and $A$ has a trivial row. W.l.o.g., assume this is the first row. Then for any $z \in \mathbb{E}_d^{(1)}$, we have

$$\tau_q^{(q)}(Az) = \min_{1 \leq i \leq q} \sum_{j=1}^d a_{ij}z_j = 0.$$
This implies
\[ \tau_{q,d}^{(1)}(A) = \sup_{z \in \partial \mathbb{N}^{(1)}_d} \frac{\tau_q(Az)}{\tau_d^{(1)}(z)} = 0, \]
which is a contradiction. Hence \( A \) cannot have a trivial row.

(e) The first inequality follows from (b). Moreover
\[ \tau_{q,d}^{(1,1)}(A) = \sup_{z \in \partial \mathbb{N}^{(1)}_d} (Az) = \sup_{z \in \mathbb{E}^{(1)}_{d,i} : z^{(1)} = 1} (Az) \leq \max_{1 \leq i \leq q, 1 \leq j \leq d} a_{ij} < \infty. \]
\[
\Box
\]

For \( A \in \mathbb{R}^{q \times d}_+ \) and \( C \subseteq \mathbb{R}^q_+ \), the pre-image of \( C \) is given by \( A^{-1}(C) = \{ z \in \mathbb{R}^d_+ : Az \in C \} \).

The following lemma characterizes the mapping of the subspaces of \( \mathbb{R}^+_d \) under the linear map \( A \) and is key to the results to follow.

**Lemma 3.4.** Let \( A \in \mathbb{R}^{q \times d}_+ \) be a deterministic matrix with all rows non-trivial. Then for fixed \( i \in \{1, \ldots, d\} \) and \( k \in \{1, \ldots, q\} \) the following are equivalent:

(a) \( A^{-1}(\mathbb{E}^{(k)}_q) \subseteq \mathbb{E}^{(i)}_d \).
(b) \( 0 < \tau_{q,d}^{(k,i)}(A) < \infty \).

**Proof.** (a) \( \Rightarrow \) (b): Let \( A^{-1}(\mathbb{E}^{(k)}_q) \subseteq \mathbb{E}^{(i)}_d \). First suppose that \( \tau_{q,d}^{(k,i)}(A) = 0 \). Hence by definition, from (3.5) we have that \( \tau_q(Az) = (Az)^{(k)} = 0 \) for every \( z \in \mathbb{E}^{(i)}_d \). Thus,
\[ A^{-1}(\mathbb{E}^{(k)}_q) \cap \mathbb{E}^{(i)}_d = \emptyset \]
contradicting the premise.

Now suppose that \( \tau_{q,d}^{(k,i)}(A) = \infty \). Let \( M = \tau_q^{(1)}(Ae) \) where \( e = (1, 1, \ldots, 1)^T \). Then there exists \( z \in \partial \mathbb{N}^{(i)}_d = \{ z \in \mathbb{R}^d_+ : z^{(i)} = 1 \} \) such that \( \tau_q^{(k)}(Az) \geq M + d \). Fix such a \( z \) and without loss of generality assume that \( z_1 \geq z_2 \geq \ldots \geq z_d \) (otherwise we may arrange columns of \( A \) accordingly). Hence, \( z^{(i)} = z_i = 1 \). Define \( z^* \in \mathbb{R}^d_+ \) by converting the last \( d - i \) components of \( z \) to 1. Hence,
\[ z^* = (z_1, \ldots, z_{i-1}, 1, \ldots, 1). \]

Since component-wise \( z^* \geq z \) and the components of \( z^* \) and \( z \) are ordered, we have \( \tau_q^{(k)}(Az^*) \geq \tau_q^{(k)}(Az) \geq M + d \). Now, define
\[ y = z^* - e = (z_1 - 1, \ldots, z_{i-1}, 0, \ldots, 0). \]
Clearly \( y \in \mathbb{R}^d_+ \) as well as \( y \notin \mathbb{E}^{(i)}_d \) since \( y^{(i)} = y_i = 0 \). Note that \( \tau_q^{(k)}(Az^*) \geq M + d \) means at least \( k \)-elements of \( Az^* \) are larger than \( M + d \), whereas \( \tau_q^{(k)}(Ae) \leq \tau_q^{(1)}(Ae) = M \) by definition. Hence, all elements of \( Ae \) are at most \( M \). Since \( Ay = Az^* - Ae \), at least \( k \) elements of \( Ay \) are greater or equal to \( d \). Therefore, \( \tau_q^{(k)}(Ay) \geq d > 0 \). Thus, \( Ay \in \mathbb{E}^{(k)}_q \) which is a contradiction.
The following example illustrates the equivalence shown in Lemma 3.4. Let
\[ A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \]
Let \( z = (z_1, z_2, z_3, z_4) \). Then
\[ x = Az = (z_1 + z_2 + z_3, z_1 + z_2 + z_4, z_1 + z_3 + z_4, z_2 + z_3 + z_4) \top. \]

For \( k = q = 4 \) we find

\[ \tau_{4,4}^{(1,1)}(A) = \sup_{z \in \mathbb{E}_4^{(1)}} \frac{x^{(4)}}{z^{(1)}} = 3 < \infty, \quad \tau_{4,4}^{(1,2)}(A) = \sup_{z \in \mathbb{E}_4^{(2)}} \frac{x^{(4)}}{z^{(2)}} = 3 < \infty, \]

\[ \tau_{4,4}^{(4,3)}(A) = \sup_{z \in \mathbb{E}_4^{(3)}} \frac{x^{(4)}}{z^{(3)}} = \infty. \]

The supremum value of 3 in the first two cases is attained at \( z = (z, z, z, z) \) for \( z > 0 \). The final equality is attained by using \( z^* = (z^4, z^3, z^2, z) \) for \( z > 0 \), where \( z^* \in \mathbb{E}_4^{(3)} \). Hence according to Lemma 3.4 we have
\[ A^{-1}(\mathbb{E}_4^{(4)}) \subseteq \mathbb{E}_4^{(2)} \] (and by inclusion also \( \mathbb{E}_4^{(1)} \)).

This means that the pre-image \( A^{-1}(\mathbb{E}_4^{(4)}) \) contains vectors \( z \in \mathbb{R}_4^+ \), whose largest two components are positive, and the other two components can be either zero or positive.

This example can be compared to [15, Lemma 2.2] where only \( \tau_{4,4}^{(4,4)}(A) \) is considered, which for this example by Lemma 3.3(c) is infinite. The only choice for \( A \) where \( \tau_{4,4}^{(4,4)}(A) < \infty \) are permutations of diagonal matrices with positive diagonal entries; see Remark 4 (iii) for further explanation.

3.2. Main Result. The key result extending Theorems 3.1 and 3.2 incorporating general linear transformations \( A \in \mathbb{R}_+^{q \times d} \) and a wide variety of tail sets is provided in this section. If \( Z \in \mathcal{MRV}(\alpha, \mu) \) with asymptotically independent components, meaning \( \mu(\mathbb{R}_+^d \setminus CA_d^{(d-1)}) = \mu(\{ z \in \mathbb{R}_+^d : z^{(d)} > 0 \}) = 0 \), we may seek and find hidden regular variation in spaces \( \mathbb{E}_d^{(i)} \) for \( 1 \leq i \leq d - 1 \) as seen in Section 2.1. Theorem 3.6 provides the appropriate non-null limit and
its rate in the presence of such hidden regular variation for a random linear transformation of \( \mathbf{Z} \). The result presented requires the presence of multivariate (hidden) regular variation on each cone \( \mathbb{R}^d_i \), \( 1 \leq i \leq d \). Note that rate functions \( b_i(\cdot) \) need not be different in each case, meaning for some \( i \in \{2, \ldots \} \) we have \( b_{i-1}(t)/b_i(t) \to \infty \) as \( t \to \infty \) for \( 2 \leq i \leq d \); in some other cases we have \( b_{i-1}(t) \sim c_{i-1} b_i(t) \) implying \( \mu_{i-1} = c_{i-1}^{-1} \mu_i \), for some constant \( c_{i-1} > 0 \), and \( \alpha_{i-1} = \alpha_i \). In Example 2.5, we have \( \mathbf{V} \in \mathcal{M} \mathcal{R} \mathcal{V}(i, b_i, \mu_i, \mathbb{E}_i^{(d)}) \) for \( i = 1, \ldots, d \) with \( b_{i-1}(t)/b_i(t) \to \infty \) as \( t \to \infty \) for \( 2 \leq i \leq d \).

Define for \( 1 \leq i \leq d \),

\[
\mathcal{C} \mathcal{A}_d^{(i)} \setminus \mathcal{C} \mathcal{A}_d^{(i-1)} = \{ \mathbf{v} \in \mathbb{R}_+^d : \text{exactly } i \text{-coordinates of } \mathbf{v} \text{ are positive} \}
\]

\[
= \bigcup_{j=1}^{d} \mathcal{C} \mathcal{A}_d^{(i)}(j),
\]

(3.8)

where \( \mathcal{C} \mathcal{A}_d^{(i)}(j) \) denotes the \( j \)-th \( i \)-dimensional coordinate hyperplane in \( \mathbb{R}_+^d \) with \( i \) positive and \( d - i \) zero co-ordinates in some ordering of the hyperplanes.

**Theorem 3.6.** Recall the probability space \((\Omega, \mathcal{A}, \mathbf{P})\). Let \( \mathbf{Z} \in \mathbb{R}_+^d \) be a random vector such that \( \mathbf{Z} \in \mathcal{M} \mathcal{R} \mathcal{V}(\alpha_i, b_i(\cdot), \mu_i, \mathbb{E}_i^{(d)}) \), with canonical choice of \( b_i(\cdot), 1 \leq i \leq d \) as in (2.13), and let \( \mathbf{A} \in \mathbb{R}_+^{q \times d} \) be a random matrix with almost surely non-trivial rows independent of \( \mathbf{Z} \). For fixed \( k \in \{1, \ldots, q\} \) and \( \mathbf{A}_\omega := \mathbf{A}(\omega), \omega \in \Omega \) define

\[
i_{k}(\mathbf{A}_\omega) = \arg \max \{ i \in \{1, \ldots, d\} : \tau_{q,d}^{(k,i)}(\mathbf{A}_\omega) < \infty \},
\]

which creates a partition of \( \Omega \) given by

\[
\Omega_i^{(k)} := \{ \omega \in \Omega : i_{k}(\mathbf{A}_\omega) = i \}, \quad i = 1, \ldots, d.
\]

Denote by \( \mathbf{P}_i^{(k)} := \mathbf{P}|_{\Omega_i^{(k)}} \) and \( \mathbf{E}_i^{(k)} := \mathbf{E}|_{\Omega_i^{(k)}} \) the restrictions of \( \mathbf{P} \) and \( \mathbf{E} \) to \( \Omega_i^{(k)} \), respectively.

Suppose that for \( 1 \leq i \leq d \) the following conditions are satisfied:

(i) \( \mathbf{P}(\Omega_i^{(k)}) > 0 \),
(ii) for some \( \delta = \delta(i,k) > 0 \) we have

\[
\mathbf{E}_i^{(k)} \left[ \tau_{q,d}^{(k,i)}(\mathbf{A})^{\alpha_i+\delta} \right] := \int_{\Omega_i^{(k)}} \tau_{q,d}^{(k,i)}(\mathbf{A})^{\alpha_i+\delta} \ d\mathbf{P}_i^{(k)} < \infty,
\]

(iii) and, \( \mu_i(\mathcal{C} \mathcal{A}_d^{(i)}(j)) > 0 \) for all \( j \in \{1, \ldots, d\} \).

Then, for \( 1 \leq i \leq d \),

\[
\frac{\mathbf{P}_i^{(k)}(t^{-1} \mathbf{A} \mathbf{Z} \in \cdot)}{\mathbf{P}_i^{(k)}(\tau_{d}^{(i)}(\mathbf{Z}) > t)} \to \mathbf{E}_i^{(k)} \left[ \mu_i(\{ \mathbf{z} \in \mathbb{E}_i^{(d)} : \mathbf{A} \mathbf{z} \in \cdot \}) \right] =: \bar{\mu}_{i,k}(\cdot), \quad t \to \infty,
\]

(3.9)

in \( \mathcal{M}(\mathbb{E}_i^{(k)}) \). Hence on \( \Omega_i^{(k)} \) we have \( \mathbf{A} \mathbf{Z} \in \mathcal{M} \mathcal{R} \mathcal{V}(\alpha_i, \bar{\mu}_{i,k}, \mathbb{E}_i^{(k)}) \).
PROOF. First we fix a $k \in \{1, \ldots, q\}$. Also fix $i \in \{1, \ldots, d\}$ which satisfies conditions (i)-(iii) of Theorem 3.6. Hence $P(\Omega_i^{(k)} > 0)$ and $E_i^{(k)}[\tau_{q,d}^{(k,i)}(A)^{\alpha_i + \delta}] < \infty$ for some $\delta > 0$. For any Borel set $C \subset E_q^{(k)}$ bounded away from $C\Lambda_q^{(k-1)}$, there exists a constant $\delta_C$ such that $\tau_d^{(i)}(z) = z^{(i)} > \delta_C$ for all $z \in C$. Using Lemma 3.3(a), we have for all $t > 0$, $M > 0$

$$P_i^{(k)}(AZ \in tC, \tau_{q,d}^{(k,i)}(A) > M) \leq P_i^{(k)}(\tau_q^{(k)}(AZ) > t\delta_C, \tau_{q,d}^{(k,i)}(A) > M) \leq P_i^{(k)}(\tau_{q,d}^{(k,i)}(A)\tau_d^{(i)}(Z) > t\delta_C, \tau_{q,d}^{(k,i)}(A) > M).$$

Since $\tau_d^{(i)}(Z) = Z^{(i)} \in RV_{-\alpha_i}$, and $A$ and $Z$ are assumed to be independent, the univariate version of Breiman’s lemma in combination with $E_i^{(k)}[\tau_{q,d}^{(k,i)}(A)^{\alpha_i + \delta}] < \infty$ yields

$$\limsup_{t \to \infty} \frac{P_i^{(k)}(AZ \in tC, \tau_{q,d}^{(k,i)}(A) > M)}{P_i^{(k)}(\tau_d^{(i)}(Z) > t)} \leq \limsup_{t \to \infty} \frac{P_i^{(k)}(1_{\{\tau_{q,d}^{(k,i)}(A) > M\}} \tau_{q,d}^{(k,i)}(A)\tau_d^{(i)}(Z) > t\delta_C)}{P_i^{(k)}(\tau_d^{(i)}(Z) > t)} = \delta_C^{-\alpha_i} E_i^{(k)}[\tau_{q,d}^{(k,i)}(A)^{\alpha_i} 1_{\{\tau_{q,d}^{(k,i)}(A) > M\}}].$$

Note that $A^{-1}(C) := \{z \in \mathbb{R}_+^d : Az \in C\}$ is again a.s. bounded away from $C\Lambda_d^{(k-1)}$, since for $x \in C$, $\omega \in \Omega_i^{(k)}$, and $zx \in A_{\omega}^{-1}(C) \subseteq \mathbb{R}_+^d$ we have by Lemma 3.3(a),

$$\tau_d^{(i)}(zx) \geq \tau_q^{(k)}(A_{\omega}zx) = \frac{\tau_q^{(k)}(x)}{\tau_{q,d}^{(k,i)}(A_{\omega})} > \frac{\delta_C}{\tau_{q,d}^{(k,i)}(A_{\omega})} > 0$$

and, thus, $P_i^{(k)}(A^{-1}(C) \subseteq E_q^{(k)}) = 1$. Hence, abbreviating $a := A_{\omega}$ and conditioning on $A$, by independence of $Z$ and $A$,

$$\lim_{t \to \infty} \frac{P_i^{(k)}(AZ \in tC, \tau_{q,d}^{(k,i)}(A) \leq M)}{P_i^{(k)}(\tau_d^{(i)}(Z) > t)} = \lim_{t \to \infty} \int_{\{\tau_{q,d}^{(k,i)}(a) \leq M\}} \frac{P_i^{(k)}(Z \in ta^{-1}(C))}{P_i^{(k)}(\tau_d^{(i)}(Z) > t)} dP_i^{(k)}(a)$$

$$= \int_{\{\tau_{q,d}^{(k,i)}(a) \leq M\}} \mu_i(a^{-1}(C)) dP_i^{(k)}(a)$$

$$= E_i^{(k)}[\mu_i \left( \left\{ z \in E_d^{(i)} : A1_{\{\tau_{q,d}^{(k,i)}(A) \leq M\}} z \in C \right\} \right)]$$

$$= E_i^{(k)} \left[ \mu_i \left( A^{-1}(C) 1_{\{\tau_{q,d}^{(k,i)}(A) \leq M\}} \right) \right],$$

where we used for the second equality that $E_i^{(k)}[\mu_i(\partial A^{-1}(C))] = 0$ in combination with Pratt’s lemma [23], since for $\tau_{q,d}^{(k,i)}(A_{\omega}) \leq M$ we have for the integrand

$$\frac{P_i^{(k)}(Z \in tA_{\omega}^{-1}(C))}{P_i^{(k)}(\tau_d^{(i)}(Z) > t)} \leq \frac{P_i^{(k)}(\tau_{q,d}^{(k,i)}(A_{\omega})\tau_d^{(i)}(Z) > t\delta_C)}{P_i^{(k)}(\tau_d^{(i)}(Z) > t)}$$

$$\leq \frac{P_i^{(k)}(M\tau_d^{(i)}(Z) > t\delta_C)}{P_i^{(k)}(\tau_d^{(i)}(Z) > t)} \to M^{\alpha_i} \delta_C^{-\alpha_i} (t \to \infty).$$
We need to show that \( E_i^{(k)}[\mu_i(A^{-1}(C))] < \infty \). Define \( B_d^{(i)}(\delta) := \{ z \in \mathbb{R}^d_+ : \tau_d^{(i)}(z) \leq \delta \} \). By homogeneity of \( \mu_i \) we have
\[
E_i^{(k)}[\mu_i(A^{-1}(C))] \leq E_i^{(k)}[\mu_i(B_d^{(i)}(\delta_C/\tau_{q,d}^{(k,i)}(A))^c)] = \mu_i\left(\left(B_d^{(i)}(\delta_C)\right)^c\right) E_i^{(k)}[\tau_{q,d}^{(k,i)}(A)^{\omega_i}] < \infty.
\]
To finish the proof it remains to show that \( E_i^{(k)}[\mu_i(A^{-1}(E_q^{(k)}))] > 0 \).
Let \( \omega \in \Omega_q^{(k)} \). First assume that \( 1 \leq i < d \). We know from Lemma 3.4 that \( A_{\omega}^{-1}(E_q^{(k)}) \subseteq E_d^{(i)} \). By definition, \( CA_{d,i}^{(i)} \cap CA_{d,i}^{(i-1)} \subseteq E_d^{(i)} \). We claim that
\[
(CA_{d,i}^{(i)} \cap CA_{d,i}^{(i-1)}) \cap A_{\omega}^{-1}(E_q^{(k)}) \neq \emptyset.
\]
If not, then we have \( A_{\omega}^{-1}(E_q^{(k)}) \subseteq E_d^{(i)} \setminus (CA_{d,i}^{(i)} \cap CA_{d,i}^{(i-1)}) = \mathbb{R}^d_+ \setminus CA_{d,i}^{(i)} = E_d^{(i+1)} \). Therefore by Lemma 3.4, \( \tau_{q,d}^{(k,i+1)}(A_{\omega}) < \infty \) for all \( \omega \in \Omega_q^{(k)} \) and, hence, \( d^{(k)}(\tau_{q,d}^{(k,i+1)}(A) < \infty) = 1 \). But this is a contradiction to the definition of \( \Omega_q^{(k)} \).
So let \( z \in (CA_{d,i}^{(i)} \cap CA_{d,i}^{(i-1)}) \cap A_{\omega}^{-1}(E_q^{(k)}) \). Then by (3.8), we have \( z \in \widetilde{CA}_{d,i}^{(i)}(j^*) \) for some \( 1 \leq j^* \leq \binom{d}{i} \). Let \( I_z := \{ j \in \{1, \ldots, d\} : z_j > 0 \} \). Clearly,
\[
\widetilde{CA}_{d,i}^{(i)}(j^*) = \{ z \in \mathbb{R}_+^d : z_j > 0, \ j \in I_z, \ z_j = 0, \ j \in \{1, \ldots, d\} \setminus I_z \}.
\]
Hence for every \( z^* \in \widetilde{CA}_{d,i}^{(i)}(j^*) \) we have that some component of \( A_{\omega}z^* \) is positive if and only if the corresponding component of \( A_{\omega}z \) is positive, since \( A_{\omega} \) has only non-negative entries. Thus, \( A_{\omega}z^* \in E_q^{(k)} \); i.e., \( z^* \in \widetilde{CA}_{d,i}^{(i)}(E_q^{(k)}) \). Hence, we get that
\[
\widetilde{CA}_{d,i}^{(i)}(j^*) \subseteq A_{\omega}^{-1}(E_q^{(k)}) \subseteq E_d^{(i)}.
\]
Since \( \mu_i \) has positive mass on each of the \( \binom{d}{i} \) hyperplanes \( \widetilde{CA}_{d,i}^{(i)}(j) \), this results in
\[
\mu_i(A_{\omega}^{-1}(E_q^{(k)})) \geq \mu_i(\widetilde{CA}_{d,i}^{(i)}(j^*)) \geq \min_j \mu_i(\widetilde{CA}_{d,i}^{(i)}(j)) > 0,
\]
and
\[
E_i^{(k)}[\mu_i(A^{-1}(E_q^{(k)}))] \geq \min_j \mu_i(\widetilde{CA}_{d,i}^{(i)}(j)) > 0,
\]
which proves the claim for \( 1 \leq i < d \).
Finally let \( i = d \). Then for every \( z \in A_{\omega}^{-1}(E_q^{(k)}) \subseteq E_{q,d}^{(d)} \) all components are positive. Moreover, \( A_{\omega}z \in E_q^{(k)} \). Then for every \( z^* \in E_{d}^{(d)} \) we get that \( A_{\omega}z^* \) has only positive components if and only if \( A_{\omega}z \) has positive components, i.e., \( A_{\omega}z^* \in E_q^{(k)} \). This results in \( E_d^{(d)} = A_{\omega}^{-1}(E_q^{(k)}) \) and
\[
E_i^{(k)}[\mu_d(A^{-1}(E_q^{(k)}))] = \mu_d(E_{d}^{(d)}) > 0.
\]

**Remark 5.** The condition that \( \mu_i(\widetilde{CA}_{d,i}^{(i)}(j)) > 0 \) for all \( j \in \{1, \ldots, \binom{d}{i}\} \) could be relaxed to \( \mu_i(\widetilde{CA}_{d,i}^{(i)}(j)) > 0 \) for at least one \( j \in \{1, \ldots, \binom{d}{i}\} \), but showing that the limit measure is non-zero turns out to be a cumbersome exercise and needs to be done with proper care. In many examples, the measures \( \mu_i \) turn out to be exchangeable with respect to their co-ordinates and the assumption being true for all \( j \in \{1, \ldots, \binom{d}{i}\} \) is not uncommon; one such example is given in Example 2.5 where \( Z_i \) are iid Pareto(\( \alpha \)) for \( \alpha > 0 \) and we have \( \nu_i(\widetilde{CA}_{d,i}^{(i)}(j)) > 0 \) for all \( j = 1, \ldots, \binom{d}{i} \).
Remark 6. Note that in (3.9), we have \( P_i^{(k)}(\tau_d^{(i)}(Z) > t) = P(\tau_d^{(i)}(Z) > t) \) since the argument does not depend on the partition of \( \Omega \), and \( A \) and \( Z \) are independent. Theorem 3.6 provides regular variation limit measure for sets in \( \mathbb{E}_q^{(k)} \) restricted to \( \Omega_i^{(k)} \), whenever its three conditions are satisfied and, as a consequence, we have the following Proposition 3.7.

Proposition 3.7. Let the notation and assumptions of Theorem 3.6 hold. Let \( C \subset \mathbb{E}_q^{(k)} \) be a Borel set bounded away from \( \mathbb{A}_{q}^{(k-1)} \).

(a) Then

\[
P(AZ \in tC) = \sum_{i=1}^{d} \left[ P(Z^{(i)} > t) E_i^{(k)}[\mu_i(A^{-1}(C))] P(\Omega_i^{(k)}) + o(1/(b_i^{+}(t))) \right], \quad t \to \infty.
\]

(b) Suppose that

\[
i_i^* := \arg\min\{i \in \{1, \ldots, d\} : P(\Omega_i^{(k)}) > 0\}. \tag{3.10}
\]

Then

\[
P(t^{-1}AZ \in C) \to E_i^{(k)}[\mu_i^{*}(A^{-1}(C))] P(\Omega_i^{(k)}) =: \mu_{i_i^*, k}(C), \quad t \to \infty, \tag{3.11}
\]

in \( \mathcal{M}(\mathbb{E}_q^{(k)}) \). Hence, we have \( AZ \in \mathcal{MRV}(\alpha_i^{*}, \mu_{i_i^*, k}, \mathbb{E}_q^{(k)}) \).

Proof. (a) Note that \( \{\omega \in \Omega : AZ \in tC\} = \bigcup_{i=1}^{d} \{\omega \in \Omega_i^{(k)} : AZ \in tC\} \) giving a partition of the event. Hence, \( P(AZ \in tC) = \sum_{i=1}^{d} P_i^{(k)}(AZ \in tC)P(\Omega_i^{(k)}) \). Now the result follows now from (3.9) and observing that

\[
P_i^{(k)}(\tau_d^{(i)}(Z) > t) = P(\tau_d^{(i)}(Z) > t) = P(Z^{(i)} > t) \sim c_i/b_i^{+}(t), \quad t \to \infty,
\]

for some constant \( c_i > 0 \).

(b) Since \( \Omega_i^{(k)}, 1 \leq i \leq d \) forms a partition of \( \Omega \), at least one of them has positive probability and hence \( P(\Omega_i^{(k)}) > 0 \). Note that using part (i), we have for any Borel set \( C \subset \mathbb{E}_q^{(k)} \) bounded away from \( \mathbb{A}_{q}^{(k-1)} \), and as \( t \to \infty \),

\[
P(t^{-1}AZ \in C) \to (1 + o(1)) \sum_{i=1}^{d} \frac{P(Z^{(i)} > t)}{P(Z_i^{(i)}) > t} E_i^{(k)}[\mu_i(A^{-1}(C))] P(\Omega_i^{(k)})
\]

\[
= (1 + o(1)) \sum_{i=1}^{d} \frac{P(Z^{(i)} > t)}{P(Z_i^{(i)}) > t} E_i^{(k)}[\mu_i(A^{-1}(C))] P(\Omega_i^{(k)})
\]

\[
= (1 + o(1))E_i^{(k)}[\mu_i^{*}(A^{-1}(C))] P(\Omega_i^{(k)})
\]

\[
+ (1 + o(1)) \sum_{i=i_i^*+1}^{d} \frac{P(Z^{(i)} > t)}{P(Z_i^{(i)}) > t} E_i^{(k)}[\mu_i(A^{-1}(C))] P(\Omega_i^{(k)})
\]

\[
\to E_i^{(k)}[\mu_i^{*}(A^{-1}(C))] P(\Omega_i^{(k)}), \quad t \to \infty,
\]
since for any $i^*_{k} < i \leq d$ we have $b_{i^*_{k}}(t)/b_{i}(t) \to 0$, and hence
\[
\frac{\mathbf{P}(Z^{(i)} > t)}{\mathbf{P}(Z^{(i^*_{k})} > t)} \sim \frac{b_{i^*_{k}}^{-}(t)}{b_{i}^{-}(t)} \to 0, \quad t \to \infty.
\]

**Remark 7.** It is easy to see that Proposition 3.7 (ii) provides the correct rate of decay of the probability of the event. Part (i) becomes useful in case we can quantify $o(1/b_{i}^{-}(t))$. For a deterministic matrix $A$, we have $\mathbf{P}(\Omega_{i^*_{k}}) = 1$, and parts (i) and (ii) are the same.

**Remark 8.** If $Z$ has asymptotically independent components, and each component has distribution tail $\mathbf{P}(Z_{j} > x) \sim c_{j}x^{-\alpha}$ for some $\alpha > 0$, then we get a generalization of Theorem 3.2 of Kley, Klüppelberg and Reinert [17]. We investigate such structures further in the next section.

**Example 3.8.** This example illustrates the transformation of sets and where the limit measure lies in a 3-dimensional setting. Let $Z = (Z_{1}, Z_{2}, Z_{3})$ have iid Pareto($\alpha$) marginal distributions, meaning $\mathbf{P}(Z_{i} > x) = x^{-\alpha}, x > 1$ for some $\alpha > 0$. Then $Z \in \mathcal{MRY}(i\alpha, \mu, \mathbb{E}^{(i)})$ for $i = 1, 2, 3$ where
\[
\begin{align*}
\mu_{1}\left(\bigcup_{i=1}^{3}\{z \in \mathbb{R}_{+}^{3} : x_{i} > z_{i}\}\right) &= z_{1}^{-\alpha} + z_{2}^{-\alpha} + z_{3}^{-\alpha},
\mu_{2}\left(\bigcup_{1 \leq i \neq j \leq 3}\{z \in \mathbb{R}_{+}^{3} : x_{i} > z_{i}, x_{j} > z_{j}\}\right) &= (z_{1}z_{2})^{-\alpha} + (z_{2}z_{3})^{-\alpha} + (z_{3}z_{1})^{-\alpha},
\mu_{3}\left((z_{1}, \infty) \times (z_{2}, \infty) \times (z_{3}, \infty)\right) &= (z_{1}z_{2}z_{3})^{-\alpha}.
\end{align*}
\]
Consider the matrix
\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \]

Then under the map \( A : z \mapsto x \), the region \( C = (1, \infty)^3 \subset E_3 \) is transformed to
\[ A^{-1}(C) = \{ x \in \mathbb{R}_+^3 : x_1 > t, x_2 > t \} \cup \{ x \in \mathbb{R}_+^3 : x_2 > t, x_3 > t \} \cup \{ x \in \mathbb{R}_+^3 : x_3 > t, x_1 > t \}. \]

It is easy to check that \( i_3^*(A) = 2 \). Hence,
\[ P(X \in tC) \sim P(Z(2) > t)\mu_2(A^{-1}(C)) \sim 3t^{-2\alpha}, \quad t \to \infty. \]

In the left plot of Figure 1 the region \( C \) is shaded in blue. In the right plot, the blue shaded region is \( A^{-1}(C) \). The light red region are the walls where the measure \( \mu_2 \) lies. The rest of the blue shaded region has zero mass under \( \mu_2 \).

4. Applications to bipartite networks. Risk-sharing in complex systems is often modeled using a graphical network model, one such example is the bipartite network structure for modeling claims in insurance markets or financial investment risk as proposed in Kley, Klüppelberg and Reinert [17], Kley, Klüppelberg and Reinert [18]. In the same spirit, we consider a vertex set of agents \( \mathcal{A} = \{1, \ldots, q\} \) and a vertex set of objects (insurance claims or investment risks) \( \mathcal{O} = \{1, \ldots, d\} \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{bipartite_network.png}
\caption{A bipartite network with \( q = 3 \) agents and \( d = 4 \) objects.}
\end{figure}

Each agent \( k \in \mathcal{A} \) chooses a number of objects \( i \in \mathcal{O} \) to connect with; Figure 2 provides an example of such a network. This choice can be random according to some probability distribution. A basic model assumes \( k \) and \( i \) connects with probability
\[ P(k \sim i) = p_{ki} \in [0, 1], \quad k \in \{1, \ldots, q\}, i \in \{1, \ldots, d\}, \]
independently for each pair. Let \( Z_i \) denote the risk attributed to the \( i \)-th object and \( Z = (Z_1, \ldots, Z_d)^T \) forms the risk vector. Also assume that the graph creation process is independent of \( Z \). The proportion of loss of object \( i \) affecting agent \( k \) is denoted by
\[ f_k(Z_i) = 1(k \sim i)W_{ki}Z_i \quad (4.1) \]
Our goal is to find the probability of tail risks in terms of $X$, where $X$ can be represented as $X = (X_1, \ldots, X_q) \top$, where $X_k = \sum_{i=1}^d f_k(Z_i)$ can be represented as

$$X = AZ.$$  

Our goal is to find the probability of tail risks in terms of $X$.

**Example 4.1.** Assume two possible investments $Z_1, Z_2$, which are independent and $P(Z_i > x) \sim c_i x^{-\alpha}$ as $x \to \infty$ for $i = 1, 2$. There are three investors, who choose by tossing a coin independently of each other. This results in the model

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} B_1 \ 1-B_1 \\ B_2 \ 1-B_2 \\ B_3 \ 1-B_3 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = AZ,$$

where $B_1, B_2, B_3$ are independent Bernoulli variables with $P(B_k = 1) = p_k, k = 1, 2, 3$ and $0 < p_k < 1$. Then each of the 8 realised matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, A_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, A_7 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, A_8 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix},$$

occur with positive probability; for example $P(A_3) = p_1(1-p_2)p_3$. In the market, we want to assess the risk for all investments being above $t > 0$. Hence given $t > 0$ we compute

$$P(X_k > t, k = 1, 2, 3) = P(X \in t(1, \infty)).$$

Note that the components of $X$ are always dependent. We compute $i_k^*$ as defined in Proposition 3.7 based on $r_{k, l}^*(A_\omega)$ for $k = 1, 2, 3$ and all $A_\omega$.

For $A_1$ we note that $x = A_1 z = (z_1, z_1, z_1)$ and $x = A_8 z = (z_2, z_2, z_2)$, giving complete dependence. Then we have for $\omega = 1, 8$,

$$r_{3,2}^{(3,1)}(A_\omega) = r_{3,2}^{(2,1)}(A_\omega) = r_{3,2}^{(1,1)}(A_\omega) = \sup_{z \in \{1\}^2} \frac{z_1 1_{\{\omega=1\}} + z_2 1_{\{\omega=8\}}}{z_1 \lor z_2} = 1,$$

$$r_{3,2}^{(3,2)}(A_\omega) = r_{3,2}^{(2,2)}(A_\omega) = r_{3,2}^{(1,2)}(A_\omega) = \sup_{z \in \{2\}^2} \frac{z_1 1_{\{\omega=1\}} + z_2 1_{\{\omega=8\}}}{z_1 \land z_2} = \infty.$$

Hence, for these matrices $i_k(A_\omega) = 1$ for $\omega = 1, 8$ and $k = 1, 2, 3$. 

where $W_{ki} > 0$ denotes the proportional effect of the $i$-th object on the $k$-th agent. Now define the $q \times d$ adjacency matrix $A$ by

$$A_{ki} = 1(k \sim i)W_{ki}. \quad (4.2)$$

Hence the total exposure of the agents given by $X = (X_1, \ldots, X_q)$, where $X_k = \sum_{i=1}^d f_k(Z_i)$ can be represented as

$$X = AZ. \quad (4.3)$$
For all other matrices the vector $x$ has two equal components, either $z_1$ or $z_2$ and a third one, which is different. For $A_2, A_3, A_4$ we have $z_1$ occurring twice and for $A_5, A_6, A_7$ we have $z_2$ occurring twice. In either of the cases we can check that

$$
\tau_{3,2}^{(3,1)}(A_\omega) = \tau_{3,2}^{(2,1)}(A_\omega) = \tau_{3,2}^{(1,1)}(A_\omega) = \sup_{z \in E^{(1)}_2} \frac{z_1 \lor z_2}{z_1 \lor z_2} = 1,
$$

$$
\tau_{3,2}^{(3,2)}(A_\omega) = \sup_{z \in E^{(2)}_2} \frac{z_1 \land z_2}{z_1 \land z_2} = 1, \quad \tau_{3,2}^{(2,2)}(A_\omega) = \tau_{3,2}^{(1,2)}(A_\omega) = \sup_{z \in E^{(2)}_2} \frac{z_1 \lor z_2}{z_1 \land z_2} = \infty.
$$

Hence, $i_3(A_\omega) = 2$, $i_2(A_\omega) = 1$, and $i_1(A_\omega) = 1$.

We apply Theorem 3.6. First note that among the 8 matrices and for $k = 3$ and the six matrices $A_2, \ldots, A_7$ we have $i_3^* = 2$ and $A^{-1}_\omega(E^{(3)}_3) \subseteq E^{(2)}_2$, and for the matrices $A_1$ and $A_8$ have $i_3^* = 1$ and hence $A^{-1}_\omega(E^{(3)}_3) \subseteq E^{(1)}_2$. From (3.9) we know that for $t \to \infty$,

$$
P_1^{(3)}(AZ \in t(1, \infty)) \sim P(Z^{(1)} > t) \sum_{\omega \in \{1,8\}} \frac{q_\omega}{q_1 + q_8} \mu_1(\{z \in E^{(1)}_2 : A_\omega z \in (1, \infty)\}),
$$

$$
P_2^{(3)}(AZ \in t(1, \infty)) \sim P(Z^{(2)} > t) \sum_{\omega \in \{2,3,4,5,6,7\}} \frac{q_\omega}{1 - q_1 - q_8} \mu_2(\{z \in E^{(2)}_2 : A_\omega z \in (1, \infty)\}),
$$

where $q_\omega = P(A_\omega)$. We compute the constants on the right-hand sides as:

$$
\mu_1(\{z \in E^{(1)}_2 : A_1 z \in (1, \infty)\}) = \mu_1(\{z_1 > 1\}) = \frac{c_1}{c_1 + c_2},
$$

$$
\mu_1(\{z \in E^{(1)}_2 : A_8 z \in (1, \infty)\}) = \mu_1(\{z_1 > 1\}) = \frac{c_2}{c_1 + c_2},
$$

$$
\mu_2(\{z \in E^{(2)}_2 : A_\omega z \in (1, \infty)\}) = \mu_2(\{z \in E^{(2)}_2 : z \in (1, \infty)\}) = \mu_2(\{z_2 > 1\}) \mu_2(\{z_2 > 1\}) = 1, \quad \omega \in \{2,3,4,5,6,7\}.
$$

Hence we can compute $P(X \in t(1, \infty))$ by Proposition 3.7 (i):

$$
P(X \in t(1, \infty)) \sim (q_1 + q_8)P_1^{(3)}(AZ \in t(1, \infty)) + (1 - q_1 - q_8)P_2^{(3)}(AZ \in t(1, \infty)) = (q_1 c_1 + q_8 c_2) \alpha^{-\alpha} + (1 - q_1 - q_8) c_1 c_2 \alpha^{-2\alpha}.
$$

\[\square\]

Remark 9. (i) This is a bipartite graph model with 3 agents and 2 objects, and each agent selects independently and independently of each object 1 and object 2 with probability $p_k$ for $k = 1, 2, 3$. Extending such a result to an arbitrary bipartite graph model with $d$ objects and $q$ agents, arbitrary $p$ and adjacency matrix $A$ as in (4.2) will provide insight into such models. Clearly, the number of free parameters in the model implies that getting an analytical closed form value for probabilities of tail events in such cases is non-trivial. Nevertheless, under certain modest assumptions we are able to provide a solution.

(ii) Relevant models have weighted adjacency matrices with entries

$$
A_{ki} = \frac{1(k \sim i)}{\deg(i)} \quad \text{and} \quad A_{ki} = \frac{1(k \sim i)}{\deg(k)}, \quad k = 1, \ldots, q, \ i = 1, \ldots, d,
$$

where $\deg(i)$ is the object degree and $\deg(k)$ is the agent degree. The first weighted adjacency matrix distributes a risk, e.g. an insurance claim, to those agents which connect to a specific
object in equal proportions. The second one distributes a risk, e.g. an investment risk of an investor in equal proportions to the different investments opportunities they select. Below we restrict our investigations to the unweighted adjacency matrix, as the weights do not change the dependence structure, they only yield weighted quantities in the calculations.

4.1. Agents connected to a fixed number of objects. Suppose we assume $W_{ki} = 1$ for all $k, i$ in (4.1) and each agent is able to connect to the risk objects according to a probability distribution. We also assume that agent choices are independent of each other. Hence we may assume

$$P(A_{ki} = 1 \text{ for } i \in J, \text{ and } A_{ki} = 0 \text{ for } i \in J^c) = p_J$$

for $J \subset O = \{1, \ldots, d\}$ and $\sum_{J \subset O} p_J = 1$.

Our first result provides tail probabilities for the agent exposures for a model where each agent connects to exactly one risk and the risk objects are independent of each other.

**Proposition 4.2.** Suppose $Z_1, \ldots, Z_d$ are independent random variables with $P(Z_i > z) = c_i z^{-\alpha}, z > c_i^{1/\alpha}$ for $c_i > 0$ for all $i = 1, \ldots, d$. Let $A \in \mathbb{R}^{q \times d}_+$ be such that

$$A_{ki} = 1 \quad (k \sim i),$$

where for all $k \in A = \{1, \ldots, q\}$ independently,

$$P(A_{ki} = 1 \text{ and } A_{kj} = 0 \text{ for } j \neq i) = \frac{1}{d}.$$

Let $X = AZ$.

(a) Then as $t \to \infty$,

$$P(X \in t(1, \infty)) = \sum_{j=1}^{d} c_j \left(\frac{1}{d}\right)^q t^{-\alpha} + o(t^{-\alpha}). \quad (4.4)$$

(b) Moreover, for $1 \leq i \leq \min(d, q)$, we have as $t \to \infty$,

$$P^{(q)}(X \in t(1, \infty)) = \left(\sum_{1 \leq j_1 < \cdots < j_i \leq d} \prod_{l=1}^{i} c_{j_l}\right) \frac{1}{(d)} t^{-i\alpha} + o(t^{-i\alpha}). \quad (4.5)$$

(c) Subsequently, as $t \to \infty$, we can write

$$P(X \in t(1, \infty)) \sim \sum_{i=1}^{\min(q, d)} \left[ \sum_{1 \leq j_1 < \cdots < j_i \leq d} \prod_{l=1}^{i} c_{j_l} \left\{ \left(\frac{i}{d}\right)^q - i \left(\frac{i-1}{d}\right)^q \right\} \right] t^{-i\alpha}. \quad (4.6)$$

**Proof.** (a) Since the $Z_i$ are independent with power law tails of the same order, following Example 2.5, we can check that $Z \in \mathcal{MRV}(i\alpha, \mu_i, E^{(i)}_d)$ with $b_i(t) = C(i)t^{1/(i\alpha)}$ for $i = 1, \ldots, d$. For any $z = (z_1, \ldots, z_d) \in E^{(i)}_d$ we compute,

$$\mu_i(\{y \in E^{(i)}_d : y_{j_1} > z_{j_1}, \ldots, y_{j_i} > z_{j_i} \text{ for some } 1 \leq j_1 < \cdots < j_i \leq d\})$$

$$= (C(i))^{-i\alpha} \sum_{1 \leq j_1 < \cdots < j_i \leq d} c_{j_1} c_{j_2} \cdots c_{j_i} (z_{j_1} z_{j_2} \cdots z_{j_i})^{-\alpha}$$
\[
(C(i))^{-i\alpha} \sum_{1 \leq j_1 < \cdots < j_i \leq d} \prod_{l=1}^{i} c_{j_l} z_{j_l}^{-\alpha},
\]
(4.7)

where \( C(i) = \left( \sum_{1 \leq j_1 < \cdots < j_i \leq d} \prod_{l=1}^{i} c_{j_l} \right)^{1/(i\alpha)} \). We can also check that for \( 1 \leq i \leq \min(d, q) \),

\[
P(Z^{(i)} > t) \sim C(i)^{i\alpha} t^{-i\alpha}, \quad t \to \infty.
\]
(4.8)

Also, referring to Remark 5 and (4.7), we have \( \mu_i(C_k_{B_d}(j)) > 0 \) for all \( j = 1, \ldots, (d)_i \), satisfying condition (iii) in Theorem 3.6.

We are interested in computing the probability of \( \{ X \in t(1, \infty) \} \subset E_q \). Now since each row of \( A \) has exactly one entry 1 and rest elements zero with probability \( 1/d \), we have for \( i = 1, \ldots, \min(d, q) \),

\[
\Omega_i^{(q)} = \{ \omega \in \Omega : \text{exactly } i \text{ columns } A_\omega \text{ have 1 appearing at least once} \}
\]

and therefore

\[
P(\Omega_i^{(q)}) = \binom{d}{i} \left\{ \left( \frac{i}{d} \right)^{q} - i \left( \frac{i-1}{d} \right)^{q} \right\}.
\]
(4.9)

Thus condition (i) of Theorem 3.6 is satisfied for \( i = 1, \ldots, \min(d, q) \). Moreover, the structure of \( A \) guarantees that \( E_q \left[ \tau_{q,d}^{k,i}(A)^{i\alpha} \right] < \infty \) satisfying condition (ii) of Theorem 3.6. Clearly we have \( i^*_q = 1 \). Using Proposition 3.7(ii) along with (4.7), (4.8) and (4.9), we have as \( t \to \infty \),

\[
P(X \in t(1, \infty)) \sim E_q^{(1)}(z \in R^d_+: z_i > 1) \left( \frac{1}{d} \right)^{q} (C(1))^{i\alpha} t^{-i\alpha}
\]

which shows (4.4).

(b) Now, when we restrict to the space \( \Omega_i^{(q)} \), we are in the class of matrices \( A \) which have at least \( i \) columns appearing at least once. A similar argument as in (a) shows that (4.5) holds, using Theorem 3.6.

(c) Finally using Proposition 3.7 (i) along with (4.5) we get (4.6).

\( \square \)

**Remark 10.** If each agent connects to more than one object, formulas akin to (4.5) and (4.6) turn out to be rather intricate. Nevertheless under this model we can still find the first order approximation for \( P(X \in t(1, \infty)) \) which is \( O(t^{-\alpha}) \).
Proposition 4.3. Suppose $Z_1, \ldots, Z_d$ are independent random variables with $P(Z_i > z) = c_i z^{-\alpha}, z > T$ for $c_i > 0$ and a fixed $T > 0$ for all $i = 1, \ldots, d$. Let $A \in \mathbb{R}_+^{q \times d}$ be such that $A_{ki} = 1 (k \sim i)$, where for all $k \in A = \{1, \ldots, q\}$

$$P(A_{ki} = 1 \text{ for } i \in J, \text{ and } A_{ki} = 0 \text{ for } i \in J^c) = \frac{1}{\binom{d}{r}},$$

for all $J \subset O = \{1, \ldots, d\}$ such that $|J| = r$. Let $X = AZ$. Then as $t \to \infty$,

$$P(X \in t(1, \infty)) = \sum_{l=1}^{r} \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_l \leq d} \left( \sum_{s=1}^{l} c_{i_s} \right) \left[ \binom{d-l}{r-l} - (d-l) \binom{d-l-1}{r-l-1} \right] \right) t^{-\alpha} + o(t^{-\alpha}).$$

Proof. This can be shown in a similar manner as Proposition 4.2, and is omitted here. $\Box$

In the above examples, all extreme events are of the form $t(1, \infty)$; meaning the portfolio of all agents are above a threshold $t$. In case we want to solve the problem for a specific set of agents, in each of the cases it reduces to having less rows for the adjacency matrix $A$. In the following section we find probabilities of more general extreme sets.

4.2. Deterministic connections. In the previous section, the risk events for which we found probabilities were of the type $t(1, \infty)$ which are subsets of $E_q^{(q)}$. The asymptotic results in Section 3 allow us to compute also probabilities for sets which belong to $E_q^{(i)}$ for some $1 \leq i \leq q$. Sometimes the form of randomness in matrix $A$ makes it too cumbersome to provide general analytical formulas; hence we concentrate on a deterministic weighted adjacency matrix here. One may consider this to be an analog of (4.5), where we condition the matrix to realize a particular form.

Example 4.4. Suppose $Z_1, Z_2, Z_3$ are independent random variables with $P(Z_i > x) = c_i x^{-\alpha}, x > c_i^{1/\alpha}$, for $\alpha > 0, c_i > 0, i = 1, 2, 3$. Let $Z = (Z_1, Z_2, Z_3)$ and $X = AZ$ where

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \\ a_{41} & a_{42} & 0 \\ 0 & a_{52} & a_{53} \end{bmatrix}.$$ (4.10)

Assume $a_{11}, a_{22}, a_{33}, a_{41}, a_{42}, a_{52}, a_{53} > 0$. Also for the convenience of computing measures of sets we assume $a_{41}a_{11}^{-1} + a_{42}a_{22}^{-1} > 1$ and $a_{52}a_{22}^{-1} + a_{53}a_{33}^{-1} > 1$.

We can think of $X$ as portfolios of five agents each of whom connects to a subset of three objects whose risks are given by $Z$. Now we can find probability estimates of the following events for $k = 1, \ldots, 5$:

$$P(\text{risk of at least } k \text{ of the portfolios } > t) =: P(X \in tD_k).$$ (4.11)
First of all, as $t \to \infty$,

$$
P(Z^{(1)} > t) = (c_1 + c_2 + c_3)t^{-\alpha} + o(t^{-\alpha}),
$$

$$
P(Z^{(2)} > t) = (c_1c_2 + c_2c_3 + c_3c_1)t^{-2\alpha} + o(t^{-2\alpha}),
$$

$$
P(Z^{(3)} > t) = (c_1c_2c_3)t^{-3\alpha}.
$$

Here, we have $Z \in \mathcal{MRV}(\alpha, b_i, \mu, \mathcal{E}_3^{(i)})$ with canonical $b_i$ as in (2.13) given by

$$
b_1(t) = (c_1 + c_2 + c_3)^{1/\alpha}t^{1/\alpha}, \quad b_2(t) = (c_1c_2 + c_2c_3 + c_3c_1)^{1/2\alpha}t^{1/2\alpha},
$$

$$
b_3(t) = (c_1c_2c_3)^{1/3\alpha}t^{1/3\alpha},
$$

Figure 3. Top two plots: probabilities of the five events in Example 4.4 for $\alpha = 1$ and $20 \leq t \leq 100$; bottom two plots: probabilities of the five events in Example 4.4 for $\alpha = 2$ and $20 \leq t \leq 100$. 
and
\[
\begin{align*}
\mu_1(\bigcup_{i=1}^{3} \{ z \in \mathbb{R}^3_+ : x_i > z_i \}) & = (c_1 + c_2 + c_3)^{-1} \sum_{i=1}^{3} c_i z_i^{-\alpha}, \\
\mu_2\left( \bigcup_{1 \leq i \neq j \leq 3} \{ z \in \mathbb{R}^3_+ : x_i > z_i, x_j > z_j \} \right) & = (c_1c_2 + c_2c_3 + c_3c_1)^{-1} \sum_{1 \leq i \neq j \leq 3} c_i c_j (z_i z_j)^{-\alpha}, \\
\mu_3((z_1, \infty) \times (z_2, \infty) \times (z_3, \infty)) & = (z_1 z_2 z_3)^{-\alpha}. 
\end{align*}
\]

(4.14)

Note that $D_k \subset \mathbb{B}_5^{(k)}$ for $k = 1, \ldots, 5$. We can check from the form of $A$ that
\[i_1^1(A) = 1, i_2^2(A) = 1, i_3^3(A) = 1, i_4^4(A) = 2, i_5^5(A) = 3.\]

Hence, using Proposition 3.7, along with (4.12) and (4.14), we have as $t \to \infty$,
\[
P(X \in t D_1) \sim P(Z^{(1)} > t) \mu_1(A^{-1}(D_1)) \\
\sim [c_1(\max(a_{11}^\alpha, a_{11}^\alpha)) + c_2(\max(a_{22}^\alpha, a_{42}^\alpha, a_{52}^\alpha)) + c_3(\max(a_{33}^\alpha, a_{53}^\alpha))]^{-\alpha}.
\]

Similarly, we can show that as $t \to \infty$,
\[
P(X \in t D_2) \sim P(Z^{(1)} > t) \mu_1(A^{-1}(D_2)) \\
\sim [c_1(\min(a_{11}^\alpha, a_{41}^\alpha)) + c_2(\min(a_{22}^\alpha, a_{42}^\alpha, a_{52}^\alpha)) + c_3(\min(a_{33}^\alpha, a_{53}^\alpha))]^{-\alpha}.
\]

The forms for $P(X \in t D_4)$ and $P(X \in t D_5)$ become more complicated if we do not assume $a_{11}, a_{22}, a_{33}, a_{41}, a_{42}, a_{52}, a_{53} > 0$.

As an illustration, we fix $c_1 = 1, c_2 = 2, c_3 = 3$. Moreover, let $a_{11} = a_{22} = a_{33} = 1$, $a_{41} = 2, a_{42} = 2$, and $a_{52} = a_{53} = 3$. The five probabilities obtained above are plotted for the case $\alpha = 1, 2$ in Figure 3.

Finally we provide an example where the underlying risk objects are not independent; the overall structure is still closely related to Example 4.4 in order for us to be able to compare.

**Example 4.5.** Assume that $Z = (Z_1, Z_2, Z_3)$ has a probability distribution given by
\[
P(Z_1 \leq z_1, Z_2 \leq z_2, Z_3 \leq z_3) = \prod_{i=1}^{3} (1 - c_i z_i^{-\alpha})(1 + \theta(c_1c_2c_3)^{\rho}(z_1 z_2 z_3)^{-\rho \alpha}),
\]

for $z_i \geq c_i^{1/\alpha}$ where $c_i > 0, i = 1, 2, 3, \alpha > 0, \rho > 1, 0 \leq \theta \leq 1$. Example 4.4 illustrates the case $\theta = 0$ where $Z_i$s are independent Pareto. Here we restrict to the case $\rho = 1, \theta = 1$. Such dependence in terms of copulas have been discussed in [25].
Let $X = AZ$ where $A$ is defined in (4.10). Assume $a_{11}, a_{22}, a_{33}, a_{41}, a_{42}, a_{52}, a_{53} > 0$, and $a_{41}a_{11}^{-1} + a_{42}a_{22}^{-1} > 1, a_{52}a_{22}^{-1} + a_{53}a_{33}^{-1} > 1$ in order to keep computations simple. We want to find probabilities $P(X \in tD_k), k = 1, \ldots, 5$ as in (4.11). Now, as $t \to \infty$, we have

$$P(Z^{(1)} > t) = (c_1 + c_2 + c_3)t^{-\alpha} + o(t^{-\alpha}),$$

$$P(Z^{(2)} > t) = (c_1c_2 + c_2c_3 + c_3c_1)t^{-2\alpha} + o(t^{-2\alpha}),$$

$$P(Z^{(3)} > t) = c_1c_2c_3(c_1 + c_2 + c_3)t^{-4\alpha} + o(t^{-4\alpha}).$$  \hspace{1cm} (4.15)

Notice that the only change from (4.12) is in the term $P(Z^{(3)} > t)$. Hence we have $Z \in \mathcal{MRV}(i\alpha, b_i, \mu_i, E^{(i)}_3)$ with $b_i$ as in (4.13) and $\mu_i$ as in (4.14) for $i = 1, 2$. On the other hand, with $b_3(t) = (c_1c_2c_3(c_1 + c_2 + c_3))^{1/4\alpha}t^{1/4\alpha}$, we have $Z \in \mathcal{MRV}(4\alpha, b_3, \mu_3, E^{(3)}_3)$ where

$$\mu_3((z_1, \infty) \times (z_2, \infty) \times (z_3, \infty)) = (c_1 + c_2 + c_3)^{-1}\sum_{i=1}^{3} c_iz_i^{-\alpha}(z_1z_2z_3)^{-\alpha}. \hspace{1cm} (4.16)$$

As in Example 4.4, we have $i_1^*(A) = 1, i_2^*(A) = 1, i_3^*(A) = 1, i_4^*(A) = 2, i_5^*(A) = 3$. Using Proposition 3.7, along with (4.12) and (4.14), we have the same limits for $P(X \in tD_k)$ for $k = 1, \ldots, 4$. The only change occurs for the case $k = 5$, where we have for $t \to \infty$,

$$P(X \in tD_5) \sim P(Z^{(3)} > t)\mu_3(A^{-1}(D_5)) \sim [c_1c_2c_3a_1^{\alpha}a_2^{\alpha}a_3^{\alpha}(c_1a_1^{\alpha} + c_2a_2^{\alpha} + c_3a_3^{\alpha})]t^{-4\alpha}.$$

\hfill \Box

In our examples, the underlying distribution of $Z$ has either independent marginals or at least has a tractable form; the forms of the adjacency matrix $A$ are relatively simple, since they provide more interpretable illustrations. Most of the sets $C \subset E_q^{(i)}$ that we look for have linear boundaries and hence form a polytope, whose pre-image also turns out to be a polytope in $\mathbb{R}^d_+$. Computing the limit measures $\mu_i(A^{-1}(C))$ in such cases results in finding the appropriate boundaries of the polytope which can become quite complicated. Nevertheless, for moderate values of $d, q$, numerical solutions can be obtained even when the distributional forms of $Z$ and $A$ are more complicated.

5. Conclusion. This work is motivated by the need to find probabilities of a variety of extreme events under a linear transformation of regularly varying random vectors. By an extension of Breiman’s Theorem we have shown that probabilities of many such events can be calculated, if we have information on regular variation property of the underlying random vector on subspaces of the Euclidean space. We envisage wide application of such results in areas of risk management. There are clear implications of this result for computing conditional value at risk, as well as a variety of conditional risk measures. We also believe that an alternative characterization of the rate of decay of tail probabilities can be provided via connectivity of the row components (in the bipartite network model, the agents); this work is under current investigation.
REFERENCES


