Minkowski Tensors of Anisotropic Spatial Structure

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Abstract—We describe a novel approach to morphology and anisotropy analysis of complex spatial structure using tensor-valued Minkowski functionals, or Minkowski tensors. Minkowski tensors are generalizations of the well-known scalar Minkowski functionals and are explicitly sensitive to anisotropic aspects of morphology, relevant for example for elastic moduli or permeabilities of porous materials. We provide explicit linear-time algorithms to compute these measures for three-dimensional shapes given by triangulations of their bounding surface, including triangulations obtained from experimental gray-scale image data by isosurface extraction. Eigenvalue ratios of Minkowski tensors provide a robust and versatile definition of intrinsic anisotropy. This analysis is applied to biopolymer networks under shear and to cellular complexes of dense bead packs. We validate our numerical method by computing Minkowski tensors of triply-periodic minimal surfaces (often used as structural models for self-assembled amphiphilic phases) for which analytic expressions of these tensors can be derived from the Weierstrass parametrization.

Index Terms—Mathematical morphology, anisotropy characterization, porous and cellular structures, Minkowski functionals, bicontinuous space partitions, minimal surfaces

The morphology of complex spatial microstructures is often classified qualitatively into types such as cellular, porous, network-like, fibrous, percolating, periodic, lamellar, hexagonal, disordered, fractal, etc. Various quantitative measures of morphology have been defined often applicable to one specific type only, e.g., moments of the distributions of angles of tangent vectors with a fixed specified direction as anisotropy characterization of a network structure. Apart from the concept of correlation functions, few measures are defined sensibly and robustly for all types. In this article, we describe the class of Minkowski tensors that apply generically to any type of bi-phasic structure. The Minkowski tensors are defined as integrals of powers of normal and position vectors and surface curvatures (or curvature measures) over the solid body. Because of their tensorial nature they are explicitly sensitive to anisotropic and orientational aspects of spatial structure, and robust measures of intrinsic anisotropy and alignment can be derived (A shape measure is intrinsic if, for a homogeneous body, its value is independent of the size and shape of the observation window). Figure 1 shows examples of systems where subtle anisotropy of the spatial structure influences the physical properties and to which the analysis of this article is applicable.

Scalar Minkowski functionals are not explicitly sensitive to features of the morphology that relate to orientation or directional anisotropy, as motion-invariance is one of their defining properties. Therefore they do not provide explicit quantification of anisotropy and can not capture the orientation-dependence of physical processes, such as elastic properties or permeability of anisotropic porous or microstructured materials or in copolymers [7], to the dewetting dynamics of thin films [8], and to Turing patterns [9]. They have also been shown to be the only morphological quantities on which the thermodynamic properties of simple fluids near curved solid interfaces depend [10], [11]. The mathematical theory of Minkowski functionals and their generalizations has been comprehensively developed in the context of integral geometry [12]–[14].

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systems with external fields. This motivates their generalization to tensorial quantities. The Minkowski tensors have already been shown to be the relevant morphological descriptors for a density functional theory of fluids of non-spherical particles [15] and of DNA conformations [16], and of a simple model for transport of molecular motors [17]. They have also been used, in 2D, as morphology descriptors of arrangements of neuronal cells [18], galaxies [19], and Turing patterns [20]. The mathematical discipline of integral geometry has proven statements regarding robustness and completeness equivalent to the scalar case [21]–[24]. However an algorithm for the computation of the Minkowski tensors applicable to experimental 3D data – a prerequisite for the systematic morphological analysis of experimental systems – has thus far been lacking.

A primary application of Minkowski tensors is the quantitative analysis of the degree of intrinsic anisotropy of materials with complex spatial structure. Scalar measures of anisotropy are easily derived as eigenvalue ratios of these tensors. Alternative methods for the characterization of anisotropy and alignment exist. Fourier transforms are a common way to characterize anisotropy, and have been applied e.g. for trabecular bone [25], for electrodeposited patterns [26], for fiber systems [27], and for structured polyethylene mats [28]. Related methods based on correlation functions are also known [29], [30]. As well as anisotropy, Fourier methods that analyze the amplitude of the Fourier transform of a gray-scale image in polar coordinates can also quantify alignment, e.g. of copolymer films in electric fields [1]. A difficulty with Fourier methods is that great care must be taken with respect to the finite window of observation. Measures of anisotropy derived from the normal vector distribution of a given shape, similar in spirit to the Minkowski tensors, have been used to describe the shape anisotropy of simulated 3D foam cells [31]. An anisotropy measure applicable to porous media is derived from the directional variations of average chord lengths. For a binary composite, i.e. consisting of a solid and a void phase, a chord is a segment of an infinite straight line that is fully contained in one of the two phases. Analyses of chord lengths and the derived mean intercept length ellipsoid are used for the investigation of the microstructure of bone [32]–[37], see also ref. [38] for a comparison of anisotropy measures based on mean-intercept length, star-volume and star-length distributions. Deformations of cellular or granular material have recently been quantified using the so-called texture tensor, defined as the sum \( \sum \mathbf{l} \otimes \mathbf{l} \) over a subset of link vectors in the structure [39], [40]. The texture tensor can be used to characterize anisotropy, e.g. for Antarctic ice crystals [41] and liquid foam cells [42]. Further anisotropy measures are based on the Steiner compact [43], wavelet analysis [44], the orientation of volumes [45] or star-volumes [46]. Two-dimensional measures of the anisotropy measures discussed in this article have previously been used for the analysis of the shape of neuronal cells [18] and galaxies [19], and are discussed in detail in [20]. (Parts of the analysis of this article represent the thesis work in ref. [47]).

The paper is organized as follows: Section I provides an overview of the theory of Minkowski tensors and functionals, including a discussion of the definition based on measure theory. Section II describes algorithms to compute Minkowski tensors for triangulated bodies and voxelized data sets. Section III describes anisotropy measures derived from the Minkowski tensors and illustrates their application to two experimental data sets and the mathematical structure of triply-periodic minimal surfaces. The appendix provides analytic expressions for the Minkowski tensors of minimal surfaces derived from Weierstrass parametrizations.

I. DEFINITION AND FUNDAMENTAL PROPERTIES OF MINKOWSKI TENSORS

The definition of scalar Minkowski functionals and their generalizations, used in the mathematical disciplines of integral and convex geometry, is based on measure theory, see section I-D. An equivalent approach, based on curvature-weighted integrals, has been more popular in the physical sciences and forms the basis of the algorithm described in this article. While some of the derivation is detailed for the sake of mathematical rigor and comprehensiveness, the underlying principle is simple and elegant, and generalizes the concepts of tensors of inertia and pole figures.

For a compact set \( K \) with nonempty interior, also called a body, embedded in Euclidean space \( \mathbb{E}^d \), with a sufficiently smooth bounding surface \( \partial K \), the scalar Minkowski functionals are defined as

\[
W_0(K) = \int_K dV, \quad \text{and} \quad W_\nu(K) = \frac{1}{3} \int_{\partial K} G_\nu d\mathcal{O}
\]

in space-dimension \( d = 3 \) and with \( \nu = 1, 2, 3 \). The scalar functions \( G_\nu \), are \( G_1 = 1 \), the mean curvature \( G_2 = (\kappa_1 + \kappa_2)/2 \) and the point-wise Gaussian curvature \( G_3 = \kappa_1 \cdot \kappa_2 \) of the bounding surface \( \partial K \); \( dV \) is the infinitesimal volume and \( d\mathcal{O} \) the scalar infinitesimal area element. This definition naturally applies to both convex and non-convex bodies of arbitrary topology with a sufficiently smooth bounding surface. Partly because of the integral nature of \( W_\nu \), these definitions generalize to bodies with bounding surfaces that have curvature discontinuities, such as polyhedra discussed in section II. The normalizing prefactor is chosen such that for a sphere \( B_R \) at the origin with radius \( R \) the scalar Minkowski functionals are \( W_\nu(B_R) = \kappa_\nu R^3\nu \); \( \kappa_3 = 4\pi/3 \) is the volume of the 3-dimensional unit sphere.

Minkowski tensors of rank two are defined using tensor products of position vectors \( \mathbf{r} \) and normal vectors \( \mathbf{n} \) of \( \partial K \). With the
symmetrized tensor product
\[(a \odot b)_{ij} := (a_ib_j + a_jb_i)/2,\]
\[(2)\]
of two vectors \(a\) and \(b\), and \(a^2 := a \odot a = a \odot a\) and \(ab := a \odot b\) the Minkowski tensors of rank two are defined as
\[W_0^r,0(K) := \int r^2 \, \mathrm{d}V,\]
\[(3)\]
\[W_{r,s}^\nu(K) := \frac{1}{3} \int G_\nu r^p \, n^s \, \mathrm{d}O.\]
\[(4)\]
with \(\nu = 1, 2, 3\) and \((r, s) = (2, 0), (1, 1)\) or \((0, 2)\). For ease of notation, we set \(W_0^r,0 := 0\) for \(s > 0\) and \(W_{r,s}^\nu := 0\) if \(\nu < 0\) or \(\nu > 3\). For a three-dimensional body, this definition yields ten Minkowski tensors (not counting the ones that vanish by definition for all bodies). This set is linearly independent in the vector space of isometry-covariant additive functionals. The prefactors in eqs. (3) and (4) cannot be chosen arbitrarily but are fixed by demanding that \(W_{\nu}(B_R) = \kappa^3 R^{-3-\nu}\) and that the translation covariance formula, eq. (5), holds. Other normalizations are possible.

Minkowski tensors of rank one, called Minkowski vectors, are defined by \(W_1^1,0 := \int_K r \, \mathrm{d}V\) and by \(W_1^1,\nu := \frac{1}{\partial K} \int \partial K \, r \, \mathrm{dO}\) for \(\nu = 1, 2, 3\). The prefactors are chosen such that, for a sphere centered at \(c\) the so-called curvature centroids \(W_1^{1,0}/W_1^{1,1}\) are equal to \(c\). Formally, vectors proportional to \(\int_{\partial K} n \, \mathrm{dO}\) for \(\nu = 1, 2, 3\) are also defined, however they vanish for any body (with a closed bounding surface).

### A. Fundamental properties

Minkowski tensors are isometry covariant, that is their behavior under translations and rotations is given by
\[W_{\nu}^{r,s}(K \oplus t) = \sum_{p=0}^r \binom{r}{p} t^p W_{\nu}^{r-p,s}(K),\]
\[(5)\]
\[W_{\nu}^{r,s}(\tilde{U}K) = \tilde{U}_{r+s} W_{\nu}^{r,s},\]
\[(6)\]
where \(\tilde{U}\) is an orthogonal transformation (rotation or mirror reflection). The body \(K\) translated by a vector \(t\) is denoted \(K \oplus t\). The rotation \(\tilde{U}_{r+s}\) is the rotation operator for a tensor of rank \(r + s\) corresponding to \(\tilde{U}\).

For \(r = 0\), equation (5) gives \(W_{\nu}^{0,s}(K) = W_{\nu}^{r,s}(K \oplus t)\). A tensor that fulfills this equation for all \(K\) is called translation invariant. For the sake of brevity, we will use the term translation covariant to denote specifically the genuinely translation covariant tensors (excluding the translation invariant ones). In dimension \(d = 3\), also \(W_1^{1,1}, 2^{1,1} \) and \(3^{1,1}\) are translation invariant due to the envelope theorems of Müller [23]; see also eq. (22).

For the purpose of intrinsic shape description this distinction is very important. Table I specifies the covariance behavior of the Minkowski functionals.

### Minkowski tensors can be defined for finite unions of convex bodies, by using their additivity property
\[W_{\nu}^{r,s}(K \cup K') = W_{\nu}^{r,s}(K) + W_{\nu}^{r,s}(K') - W_{\nu}^{r,s}(K \cap K').\]
\[(7)\]
This extension of eqs. (1), (3) and (4) will require the definition of Minkowski tensors also for bodies which do not have smooth boundaries, since intersections of bodies with smooth bounding surfaces do not in general have smooth bounding surfaces. In section I-D we will introduce Minkowski tensors for general convex bodies so that the right-hand side of eq. (7) is always well-defined.

The scaling behavior under rescaling of all coordinates by a factor \(\lambda \geq 0\), mathematically speaking the degree of homogeneity of \(W_{\nu}^{r,s}\), is given by
\[W_{\nu}^{r,s}(\lambda K) = \lambda^{3+r-s} W_{\nu}^{r,s}(K).\]
\[(8)\]
Not all tensor valuations of rank two are linearly independent, i.e. contain independent shape information. For tensors of rank two the linear dependencies
\[E_3 W_{\nu} = \nu W_{\nu}^{0,2} + (3 - \nu) W_{\nu}^{1,1}\]
\[(9)\]
are valid for any body \(K\) in \(\mathbb{E}^3\) (with closed bounding surface) and \(\nu = 1, 2, 3\); in particular, it follows that \(E_3 W_3 = 3 W_3^{0,2}\). Moreover, we have
\[E_3 W_0 = 3 W_0^{1,1}.\]
\[(10)\]
These relations are special cases of [22, eq. (1.1)] or [49, eq. (1.5)].

### TABLE I

<table>
<thead>
<tr>
<th>Homogeneity [unit]</th>
<th>rank 0</th>
<th>rank 1</th>
<th>rank 2</th>
<th>translation behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda^3 [m^3])</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>genuinely t. cov.</td>
</tr>
<tr>
<td>(\lambda^3 [m^3])</td>
<td>–</td>
<td>(W_1^{0,0})</td>
<td>(W_1^{1,0})</td>
<td>genuinely t. cov.</td>
</tr>
<tr>
<td>(\lambda^3 [m^3])</td>
<td>(W_1)</td>
<td>(W_2^{0,0})</td>
<td>(W_2^{1,0})</td>
<td>genuinely t. cov.</td>
</tr>
<tr>
<td>(\lambda^2 [m^2])</td>
<td>–</td>
<td>–</td>
<td>(W_1)</td>
<td>t. invariant</td>
</tr>
<tr>
<td>(\lambda^1 [m^2])</td>
<td>(W_2)</td>
<td>(W_2^{0,2})</td>
<td>(W_2^{1,2})</td>
<td>t. invariant</td>
</tr>
<tr>
<td>(\lambda^0 [1])</td>
<td>(W_3)</td>
<td>(W_3)</td>
<td>(W_3)</td>
<td>t. invariant</td>
</tr>
</tbody>
</table>

The vector space of continuous, isometry covariant tensor valuations of rank two in \(\mathbb{E}^3\) has dimension 10. A particular basis of this vector space consists of the six tensor valuations \(W_0^{2,0}\), \(W_1^{1,0}, W_2^{1,0}, W_3^{1,0}\), \(W_1^{0,2}\) and \(W_2^{0,2}\), which contain important independent shape information, and of the four tensor valuations \(E_3 W_{\nu}\), \(\nu = 0, \ldots, 4\). A summary is provided in Table I. In the
mathematical literature, this set is referred to as basic tensor valuations, whereas the tensors in eqs. (3) and (4) are referred to as Minkowski tensors; compare the remark in ref. [22, p. 150, line 23].

Aleksic’s theorem [21] makes a strong statement about the completeness of the Minkowski tensors for the purpose of shape description. For the special case of tensors of rank two, it states that any isometry covariant, additive, continuous functional $\phi$ on general convex bodies in $\mathbb{E}^3$, taking values in the space of symmetric tensors of rank two over $\mathbb{E}^3$, is a linear combination of the basic tensor valuations, that is

$$\varphi(K) = \sum_{\nu=0}^{3} \phi_{\nu}(K) E_3 + \sum_{\nu=1}^{2} \varphi_{0,2,0,2}^{\nu}(K),$$

with coefficients $\varphi_{\nu}^{\nu,s} \in \mathbb{R}$ that do not depend on the convex body $K$. The unit tensor $E_3$ raises the rank of the scalar Minkowski functionals to two. Starting from (11) and using the linear dependencies among the basic tensor valuations, $\varphi$ can be expressed in terms of linearly independent basic tensor valuations which form a basis of the corresponding vector space.

Since $\varphi$ is continuous and additive on convex bodies, it can be extended as an additive functional to finite unions of convex bodies. For this additive extension, eq. (11) remains valid since the right-hand side is a linear combination of additive functionals. It should be emphasized, however, that although all these functionals are continuous on the space of convex bodies, they are not continuous on the space of finite unions of convex bodies, see the example in [20, Fig. 3].

### B. Specific examples

For some simple shapes the Minkowski tensors can be calculated analytically by using explicit surface parametrizations and expressions for surface normals and principal curvatures. Specifically for a sphere of radius $R$ centered at the origin, one obtains

$$W_0 = \frac{4\pi}{3} R^3, \quad W_0^{2,0} = \frac{4\pi}{15} R^5 E_3$$

and, for $\nu = 1, 2, 3$ and $r + s = 2$,

$$W_\nu = \frac{4\pi}{3} R^{4-\nu}, \quad W_\nu^{\nu,\nu,\nu} = \frac{4\pi}{9} R^{3-\nu+r} E_3$$

with the unit tensor $E_3$.

For a convex polytope $P$, we write $\mathcal{F}(P)$ for the set of $i$-dimensional faces of $P$, $i = 0, 1, 2$, that is, $\mathcal{F}(P)$ is the set of vertices, $\mathcal{F}(P)$ is the set of edges, and $\mathcal{F}(P)$ is the set of faces. If $F \in \mathcal{F}(P)$, then we denote by $n(F, P)$ the set of exterior unit normal vectors of $P$ at $F$, which is a $(2-i)$-dimensional subset of the unit sphere. Then we obtain, as a special case of general formulas in section 1-D,

$$W_\nu^{\nu,s}(P) = \frac{1}{3} \sum_{F \in \mathcal{F}_{3-\nu}(P)} \int_{\mathcal{F}(P)} \int_{\mathcal{F}(P)} u^s \, du$$

with $\nu = 1, 2, 3$. The integration symbols $dx$ and $du$ refer to the natural volume measures on the face $F$ and its spherical image $n(P, F)$.

For a rectangular box of size $a_x \times a_y \times a_z$ aligned with the coordinate axes and centered at the origin eq. (14) yields $W_0 = a_x a_y a_z$, $W_1 = \frac{4}{3}(a_x a_y + a_y a_z + a_z a_x) = W_2 = \frac{4}{3}(a_x + a_y + a_z)$, $W_3 = \frac{4\pi}{3}$, $W_3^{1,0} = 0$ and all Minkowski tensors of rank two are diagonal matrices with the following entries

$$(W_0^{2,0})_{ii} = \frac{1}{12} a_i^3 a_j a_k, \quad (W_3^{2,0})_{ii} = \frac{\pi}{3} a_i^2,$$

$$(W_1^{2,0})_{ii} = \frac{1}{6} \left( a_i^3 (a_j + b_k) + a_j^2 a_k a_i \right), \quad (W_3^{2,2})_{ii} = \frac{2a_i a_k}{3},$$

$$(W_2^{2,0})_{ii} = \frac{\pi}{36} \left( a_i^3 + 3a_i^2 (a_j + a_k) \right), \quad (W_2^{0,2})_{ii} = \frac{\pi}{6} (a_j + a_k)\)$$

where $\{i, j, k\} = \{x, y, z\}$ and cyclic permutations thereof.

For an ellipsoid given by $(x/l_x)^2 + (y/l_y)^2 + (z/l_z)^2 = 1$ the surface integrals all result in elliptic integrals and cannot be expressed in closed form. However, the scalar Minkowski functional $W_0 = \frac{4\pi}{3} l_x l_y l_z$ and the tensor $W_0^{2,0}$ is diagonal with

$$(W_0^{2,0})_{ii} = \frac{\pi}{120} l_i^3 l_j l_k$$

where $\{i, j, k\} = \{x, y, z\}$ and cyclic permutations thereof. The integration of all other tensors is easily numerically evaluated.

### C. Minkowski tensors of convex polyhedra

It is instructive to illustrate the Minkowski tensors for convex polyhedral cells and to point out similarities to the tensor of inertia. For a polyhedral cell (a polytope), the tensors $W_0^{2,0}$ characterize the distribution of mass if the cell is solid ($W_0^{2,0}$), a hollow cell ($W_2^{2,0}$), a wire frame ($W_2^{2,0}$) and a cell consisting of points at the vertices only ($W_0^{2,0}$); in the last two cases, however, this distribution of mass is weighted with certain exterior angles. The tensor of inertia $I$, defined by $I_{ij} = \int_K \left(-r_i r_j + \delta_{ij} r_k^2 \right) dV$, is a measure of the weight distribution of a body $K$, relevant for the relationship between a rotation and the resulting moment. As $I$ is not translation-covariant, it is not a linear combination of the Minkowski tensors. However the simple relationship $I(K) = -W_0^{2,0}(K) + \text{tr} \left( W_0^{2,0}(K) \right) E_3$ holds
Fig. 4. Local parallel set where only points are considered for which the normal direction is in a prescribed subset of the unit sphere.

simplicity and by the immediate relationship to the algorithmic implementation. Here we provide some details of the more fundamental measure theoretic definition.

In order to introduce Minkowski tensors, we first consider local versions of the scalar Minkowski functionals. These are the support measures, which are also called generalized curvature measures. If these measures are available, then the Minkowski tensors for convex (or more general) bodies are obtained by integrating tensor functions with respect to these measures. Here we describe the approach for convex sets. The idea underlying the introduction of support measures for convex sets is to generalize the notion of a parallel set (cf. Fig. 2 for \( d = 2 \)) of a convex body \( K \) in \( d \)-dimensional Euclidean space \( \mathbb{E}^d \) by a suitable local construction. The volume of the local parallel sets to be considered turns out to be a polynomial in a distance parameter \( \epsilon \) with coefficients that depend on the given convex body \( K \) and on a given subset \( \eta \) of \( \mathbb{E}^d \times S^{d-1} \) which determines the local parallel set of \( K \) at distance \( \epsilon \), where \( S^{d-1} \) is the Euclidean unit sphere.

A definition of the local parallel set that also applies to bodies \( K \) without smooth boundary is given in the following: We define \( p(K,x) \) as the unique point in \( K \) which is nearest to a given point \( x \in \mathbb{E}^d \). This defines a continuous map \( p(K,\cdot) : \mathbb{E}^d \to K, x \mapsto p(K,x) \). Then \( d(K,x) := ||x-p(K,x)|| \) is the distance from \( x \) to \( K \) and \( u(K,x) := (x-p(K,x))/d(K,x) \), for \( x \in \mathbb{E}^d \setminus K \), is an exterior unit normal of \( K \) at the boundary point \( p(K,x) \in \partial K \). For given \( \eta > 0 \) and \( \eta \subset \mathbb{E}^d \times S^{d-1} \), the local parallel set of \( K \) defined by

\[ M_{\epsilon}(K,\eta) := \{ x \in \mathbb{E}^d \setminus K : d(K,x) \leq \epsilon, (p(K,x),u(K,x)) \in \eta \} \]

(16)

contains all points \( x \in \mathbb{E}^d \) with \( 0 < d(K,x) \leq \epsilon \) such that the pair \( (p(K,x),u(K,x)) \in \eta \). The latter condition restricts \( x \) to a subset of the global outer parallel set \( (K \oplus B_{\epsilon}) \setminus K \), where \( K \oplus B_{\epsilon} \) is the vector sum (point-wise vector addition) of \( K \) and a ball of radius \( \epsilon \) (cf. (23) and Figure 4).

If \( \eta = \beta \times \omega \) with \( \beta \subset \mathbb{E}^d \) and \( \omega \subset S^{d-1} \), then this restriction means that we require \( p(K,x) \in \beta \) and \( u(K,x) \in \omega \). The volume of this local parallel set is \( V_d(M_{\epsilon}(K,\eta)) \). A fundamental result in integral geometry, known as the local Steiner formula \([14],[51]\), states that the map \( \epsilon \mapsto V_d(M_{\epsilon}(K,\eta)) \), \( \epsilon > 0 \), is a polynomial of degree \( d \), that is

\[ V_d(M_{\epsilon}(K,\eta)) = \sum_{j=0}^{d-1} \epsilon^{d-j} \kappa_{d-j} \Lambda_j(K,\eta), \]

(17)

where \( \kappa_n := \pi^{n/2}/\Gamma(\frac{n}{2}+1) \) is the volume of an \( n \)-dimensional unit ball and \( \Lambda_j(K,\eta), j = 0, \ldots, d-1 \), are certain real values of the intermediate radius \( l_0 \) as function of \( r = l_1/l_1 \). Each symbol in Fig. 3 is a homogeneously filled solid.

For a polytope \( K \), the tensor \( W_{2,0}(K) \) reduces to a line integral over the edges of the polytope (as the mean curvature vanishes on the flat facets, see also section II) and is hence related to a mass distribution if \( K \) is given by a wire frame with wires along the edges. However, imposed by the requirement of additivity, the wire diameter cannot be uniform but must be proportional to the mean curvature along the edge (i.e. the dihedral angle). Similarly, the tensor \( W_{2,0}(K) \) reduces to a sum of point contributions, as the Gaussian curvature \( G_3 \) of \( K \) vanishes except at the vertices of the given polyhedral set \( K \). Hence \( W_{2,0}(K) \) corresponds to the mass distribution of \( K \) on the flat facets, see also section II) and is hence related to a mass distribution if \( K \) is given by a wire frame with wires along the edges. However, imposed by the requirement of additivity, the wire diameter cannot be uniform but must be proportional to the mean curvature along the edge (i.e. the dihedral angle). Similarly, the tensor \( W_{2,0}(K) \) reduces to a sum of point contributions, as the Gaussian curvature \( G_3 \) of \( K \) vanishes except at the vertices of the given polyhedral set \( K \). Hence \( W_{2,0}(K) \) corresponds to the mass distribution of \( K \) on
coefficients that depend on $K$ and $\eta$, but not on $\epsilon$. In fact, for eq. (17) to be true for all $\epsilon > 0$, it is crucial that $K$ is convex. Eq (17) is easily confirmed (and evaluated) for a convex polytope $P$. In this case, the set $\mathbb{E}^d \setminus P$ can be decomposed in an elementary way into wedges over the faces $F$ of $P$ as indicated by Fig. 2, and the contributions to $V_\delta(M_{\epsilon}(K, \eta))$ coming from these wedges can be calculated by a simple integration (known as Fubini’s theorem or Cavalieri’s principle [52], [53]). This shows that for $\eta = \beta \times \omega$, with $\beta \in \mathbb{E}^d$ and $\omega \subset S^{d-1}$, eq. (17) holds with

$$\Lambda_j(P, \beta \times \omega) = \frac{1}{\omega_{d-j}} \sum_{F \in E(P)} \int_{F \cap \beta} dx \int_{n(P, F) \cap \omega} du,$$

where $\omega_j = jk_e$ is the surface measure of the $(j-1)$-dimensional unit sphere. Since an arbitrary convex body $K$ can be approximated by polytopes, eq. (17) can be derived by a continuity argument. Moreover, since $\Lambda_j(K, \eta)$ can be expressed as a linear combination of $V_\delta(M_{\epsilon}(K, \eta))$, for $k = 1, \ldots, d$, properties of the local parallel volume are also available for $\Lambda_j(K, \eta)$. In particular, $K \mapsto \Lambda_j(K, \eta)$ is a valuation (additive) for fixed $\eta$, and $\eta \mapsto \Lambda_j(K, \eta)$ is a non-negative measure for fixed $K$. The latter means that if $\eta_i \in \mathbb{E}^d \times S^{d-1}$, $i \in \mathbb{N}$, is a sequence of mutually disjoint (measurable) sets, then

$$\Lambda_j \left( \bigcup_{i=1}^{\infty} \eta_i \right) = \sum_{i=1}^{\infty} \Lambda_j(K, \eta_i).$$

This property, which is called $\sigma$-additivity of $\Lambda_j(K, \cdot)$, has to be distinguished from the finite additivity of $\Lambda_j(K, \eta)$ with respect to the convex body $K$ (and fixed $\eta$).

In particular, $\Lambda_j(K, \cdot)$ can be used to integrate functions over $\mathbb{E}^d \times S^{d-1}$. It is plausible that $\Lambda_j(K, \cdot)$ is concentrated on the normal bundle $N(K)$ of $K$ which is the set of all $(x, u) \in \partial K \times S^{d-1}$ such that $u$ is an exterior unit normal of $K$ at $x$. The measures $\Lambda_j(K, \cdot)$, $j = 0, \ldots, d - 1$, are called support measures and are determined as coefficient measures of the Steiner formula (17). They are local versions of the classical Minkowski functionals, since $\Lambda_j(K, \mathbb{E}^d \times S^{d-1}) = V_j(K)$.

If $K$ is sufficiently smooth, then

$$\Lambda_j(K, \cdot) = \frac{\partial^j}{\partial x_{d-j}} \int_{\partial K} 1 \{ (x, \sigma_K(x)) \in \cdot \} H_{d-1-j}(x) \, dx, \quad (18)$$

where $\sigma_K(x)$ is the exterior unit normal of $K$ at $x$ and $H_1(x)$ is the $i$-th (normalized) elementary symmetric function of the principal curvatures of $K$ at $x$. In $\mathbb{E}^3$, we have $H_0 = G_1 = 1$, $H_1 = G_2 = (k_1 + k_2)/2$ and $H_2 = G_3 = k_1 \cdot k_2$. For general dimensions and $j = d - 1$, eq. (18) holds for general convex bodies $K$.

Having introduced the support measures as local versions of the scalar Minkowski functionals, it is now surprisingly easy to define the Minkowski tensors for a general convex body $K$ by

$$\Phi_{j, \epsilon}^s(K) := \frac{1}{\epsilon^{d-1}} \int_{\partial K} \omega_{d-j} \Phi_{j}^s(K, d(x, u)), \quad (19)$$

hence we obtain $\Phi_{j, \epsilon}^s(K)$ by integrating the tensorial function $x^r u^s$ with respect to the measure $\Lambda_j(K, \cdot)$ over $N(K) \subset \mathbb{E}^d \times S^{d-1}$. If $K$ is a polytope, this yields equation (14) (up to a different normalization), and we obtain eq. (4) in case $K$ is smooth, i.e.

$$\Phi_{j, \epsilon}^s(K) = \frac{(d-1)}{r!s!} \int_{\partial K} x^r u^s H_{d-1-j}(x) \, dx,$$

keeping in mind the different normalization.

The notation $\Phi_{j, \epsilon}^s$ or $\Phi_{j, \epsilon}^s$ for the Minkowski tensors in eq. (19) is preferred in the mathematical literature and differs from the notation $W_{j, \epsilon}^s$ in eqs. (3-4) only by a different indexing scheme and a different normalization. In $\mathbb{E}^3$, i.e. for $d = 3$, the functionals $\Phi_{j, \epsilon}^s$ and $W_{j, \epsilon}^s$ are related by

$$3 \left( \frac{2}{\nu - 1} \right) W_{j, \epsilon}^s(K) = r! s! \omega_{d-j-s} \Phi_{j, \epsilon}^s(K), \quad (20)$$

for $\nu = 1, \ldots, 3$, and

$$W_{0, \epsilon}^0(K) = r! \Phi_{3, \epsilon}^0(K). \quad (21)$$

The additivity and continuity properties of the support measures now immediately yield the corresponding properties of the Minkowski tensors. This approach also shows that if it is possible to define support measures for a class of sets, then eq. (19) yields a corresponding tensor valuation. Since the theory of support measures is well-developed [14], [50], [51], the measure theoretic approach outlined above has some advantages. As a simple illustration, let us explain why $W_{d-j, \epsilon}^s$ is translation invariant for $j = 0, \ldots, d - 1$. Observe that by translation covariance of the support measures

$$\int x u \Lambda_j(K + t, d(x, u)) = \int (x + t) u \Lambda_j(K, d(x, u)) \quad \quad (22)$$

It is a basic property of the measures $\Lambda_j(K, \cdot)$ that they are centered at the origin in the sense that $\int u \Lambda_j(K, d(x, u)) = 0$, which already yields the assertion. A natural and useful extension that is suggested by general measure theory is to introduce local tensor valuations by restricting the integration on the right-hand side of (19) to suitable subsets of $\mathbb{E}^d \times S^{d-1}$.

II. Bodies Bounded by Triangulated Surfaces

We describe the exact computation of all independent scalar, vectorial and Minkowski tensors for arbitrary bodies bounded by piece-wise linear (i.e. triangulated) surfaces. Henceforth such bodies, convex or non-convex, are called polytopes. The formulae are derived for convex bodies with triangulated bounding surfaces by considering parallel $\epsilon$-bodies (that have continuous normal fields and finite curvatures whenever they exist) in the limit of $\epsilon \to 0$. By application of the additivity relation these formulae are then shown to be valid also for bodies that are not convex. Table II summarizes these formulae.

Consider a body $K$ in $\mathbb{E}^3$ with piece-wise linear bounding surface $\partial K$. Without loss of generality the linear facets may be assumed to be triangles. The set of all triangular patches of $\partial K$ is $F_2$, the set of oriented edges $F_1$ and the set of vertices $F_0$. Every edge that is shared between two triangles $T$ and $T'$ is a double-edge consisting of two oriented edges $e$ (being part of $T$) and $e'$ (part of $T'$), constituting an unambiguous assignment of each edge to a triangle. The remaining ambiguity in the edge

2It is an important consequence of the additivity relation that the Minkowski tensors (in contrast to e.g. the texture tensor) do not change if flat polygonal facets are broken up into triangles. This is evidently also true for the algorithmic implementation described here.
Fig. 6. Subdivision of a non-convex body into convex sub-bodies, $K = K_1 \cup K_2 \cup \ldots \cup K_4 \cup K_7$. Note that for the computation of Minkowski tensors the segments $K_2$ and $K_4$ need to be taken into account, even though their volume (3D) or area (2D) measure is $W_0(K_2) = W_0(K_4) = 0$.

orientation is lifted by requiring the triangle normals to point out of the body $K$. For a triangle $T$ with vertices $v_1$, $v_2$, $v_3$ and edges $e_{12} = v_2 - v_1$ and $e_{13} = v_3 - v_1$, the triangle normal is $n(T) = e_{12} \times e_{13} / |e_{12} \times e_{13}|$.

The parallel body construction is illustrated by Fig. 2 (b). For an arbitrary body $K$ the parallel body $K_\epsilon$ with thickness $\epsilon > 0$ is defined as

$$K_\epsilon := K \cup B_\epsilon := \{ x + t x | x \in K, t \in \mathbb{R}^3, |t| \leq \epsilon \}. \quad (23)$$

The bounding surface of $K_\epsilon$ is $\partial K_\epsilon$. For a convex polytope $K$ the bounding surface $\partial K$ has a continuous normal field. The curvatures are patch-wise constant: $G_2 = G_3 = 0$ on the planar patches, $G_2 = (2\epsilon)^{-1}$ and $G_3 = 0$ on the cylindrical caps corresponding to polygon edges, and $G_2 = 1/\epsilon$ and $G_3 = 1/\epsilon^2$ on the conical caps corresponding to polytope vertices. Hence, for convex polytopes, the Minkowski tensors are defined as the surface integrals of eq. (4) evaluated on $\partial K_\epsilon$ in the limit $\epsilon \to 0$.

The result thus obtained is consistent with (19) and (14).

A. Volume $W_0$ and surface area $W_1$

The calculation of the volume of a body $K$ can be transformed into a surface integral by Gauss’ law. With $\nabla r = 3$ one obtains

$$W_0(K) = \int_K dV = \frac{1}{3} \int_K \nabla r \cdot dV = \frac{1}{3} \int_{\partial K} r \cdot dA \quad (24)$$

where $dA$ denotes the oriented infinitesimal area element. The surface integral is a sum over triangles and is easily evaluated yielding the formulae in Tab. II. This result is independent on whether $K$ is convex or not. However, it cannot be applied if $K$ is a translational unit cell of a periodic body, if its bounding surface $\partial K$ is not a closed surface, cf. Fig. 8.

The surface area $W_1(K)$ of $\partial K$ is simply the sum of triangle areas.

B. Integral mean curvature $W_2$

Expressing $W_2(K)$ as the limit of vanishing parallel distance $\epsilon$ of $W_2(K_\epsilon)$ of the parallel body, $W_2(K) = \lim_{\epsilon \to 0} W_2(K_\epsilon)$, the contributions of facets vanish because the mean curvature of a flat face is zero. The contribution of the spherical caps corresponding to vertices vanishes because the integral over spherical caps can be parametrized in spherical coordinates by

$$\int_{\varphi_1}^{\varphi_2} d\varphi \int_{\theta_1}^{\theta_2} \sin^2 \theta / (2\epsilon)$$

which vanishes as $\epsilon \to 0$. The remaining contribution of the edges is given in polar coordinates as

$$W_2(K) = \frac{1}{3} \lim_{\epsilon \to 0} \sum_{e \in F_1} |e| \int_0^{\alpha_e/2} d\alpha \int_0^\alpha \epsilon \frac{\alpha}{\epsilon} = \sum_{e \in F_1} |e| \frac{\alpha_e}{6} \quad (25)$$

where $|e|$ is the length of edge $e$ and $\alpha_e \in [0, \pi]$ is the dihedral angle along edge $e$, i.e. the angle between the surface normals of the two facets adjacent to $e$, see also Fig. 5. Note that $F_1$ is the set of oriented edges, i.e. the edge shared by two triangles is represented by two distinct oriented edges. Therefore the integral extends over only half the dihedral angle.

Eq. (25) remains valid even if $K$ is not convex, as is shown by exploiting additivity. The body $K$ can always be decomposed into a set of convex bodies by cutting it along the symmetric bisector planes of all concave edges, see Fig. II-B. For a concave edge $E$, the symmetric bisector plane is the plane that is spanned by $E$ and the average of the facet normals of the two facets adjacent to $E$. By adding the contributions of all resulting convex bodies using the additivity relationship eq. (7), as outlined in the caption of Fig. 5 one obtains the validity of eq. (25) for non-convex triangulated bodies. The sign of the dihedral angle $\alpha \in [-\pi, \pi]$ determines if the edge is convex or concave.

C. Integral Gaussian Curvature $W_3$ (Euler index $\chi$)

As the point-wise Gaussian curvature $G_3$ on cylinders and flat facets vanishes, only vertices of the triangulation (and their corresponding spherical caps on the parallel body) contribute to $W_3$. For both convex and non-convex polyhedral bodies $K$ the point-wise Gaussian curvature $G_3$ and the integrated Gaussian curvature $W_3$ can be calculated by the well-known simple sum in eq. (27) over angle deficits at surface vertices, derived below and also given in [54], [55]. The non-convex case is treated by exploiting additivity.

The Gaussian curvature contribution of the vertices $c \in F_0$ is derived by the Gauss-Bonnet-formula

$$\int_R G dO = 2\pi - \sum \beta_i - \int_{\partial R} k_g ds \quad (26)$$

where $R$ denotes the spherical cap on the parallel surface $\partial K_\epsilon$ of a polytope corresponding to a vertex $c$ (or more generally,
Definition of geometric properties of a triangulated surface $F_2$. For a triangle $T$, $n_T$ is the normal vector of $T$, $A_T$ its area, and $c_{T,1}$ to $c_{T,3}$ its corners. An edge $e$ is defined by its vertices $c_{e,1}$ and $c_{e,2}$. The angle $\beta_{T,e}$ is the angle between the two edges of the triangle $T$ at the vertex $e$.

Middle) Cross-sectional view along a directed edge $e$. The normal vectors $n_T$ and $n_{T'}$ of the triangle $T$ (that contains $e$) and $T'$ (that is adjacent to $T$ along edge $e$) span the angle $\alpha_e \in [\pi, \pi]$. More precisely, a rotation by $\alpha_e$ around $e$ in counter-clockwise direction maps the triangle normal $n_T$ of the triangle containing $e$ onto the $n_{T'}$. A concave edge has a negative angle $\alpha_e$. The figure also shows the definition of the local coordinate system used for the computation of $W_{0,2}$. The basis vectors $n_e'$, $n_e''$ and $\hat{e}$ are defined as $\hat{e} = e/|e|$, $n_e'' = (n_{e,1} + n_{e,2})/|n_{e,1} + n_{e,2}|$ and $n_e' = \hat{e} \times n_e''$.

(Right) Subdivision of a body $K$ along a concave edge $e$.

**Scalar Measures**

- $W_0 = \int_K dV$, $\frac{1}{2} \sum_{T \in F_2} (C_T, n_T) A_T$
- $W_1 = \frac{1}{3} \int_{\partial K} dO$, $\frac{1}{3} \sum_{T \in F_2} A_T$
- $W_2 = \frac{1}{3} \int_{\partial K} G_2 dO$, $\frac{1}{3} \sum_{e \in F_1} |e| \alpha_e$
- $W_3 = \frac{1}{3} \int_{\partial K} G_3 dO$, $\frac{1}{3} \sum_{e \in F_0} (2\pi - \sum_{T \in F_2(e)} \beta_{T,e}) c_i$

**Vectorial Measures**

- $(W_{0,1}^{1,0})_i = \int_K r_i dV$, $\sum_{T \in F_2} 2A_T I_{T,i,k} n_{T,k}$, see sec. II-D
- $(W_{1,1}^{1,0})_i = \frac{1}{3} \int_{\partial K} r_i dO$, $\frac{1}{3} \sum_{T \in F_2} C_{T,i} A_T$
- $(W_{2,1}^{1,0})_i = \frac{1}{3} \int_{\partial K} G_2 r_i dO$, $\frac{1}{18} \sum_{e \in F_1} |e| (c_{e,1,i} + c_{e,2,i}) \alpha_e$
- $(W_{3,1}^{1,0})_i = \frac{1}{3} \int_{\partial K} G_3 r_i dO$, $\frac{1}{3} \sum_{e \in F_0} (2\pi - \sum_{T \in F_2(e)} \beta_{T,e}) c_i$

**Tensornal Measures**

- $(W_{0,2}^{2,0})_{ij} = \int_K r_i r_j dV$, $\sum_{T \in F_2} 2A_T J_{T,i,j,k} n_{T,k}$, see sec. II-F
- $(W_{1,2}^{2,0})_{ij} = \frac{1}{3} \int_{\partial K} r_i r_j dO$, $\frac{1}{18} \sum_{T \in F_2} \sum_{m=1}^{3} \sum_{l=1}^{3} (c_{T,l,i} c_{T,m,j} + c_{T,m,i} c_{T,l,j}) A_T$
- $(W_{2,2}^{2,0})_{ij} = \frac{1}{3} \int_{\partial K} G_2 r_i r_j dO$, $\frac{1}{18} \sum_{e \in F_1} |e| (c_{e,1,i} c_{e,2,j} + c_{e,2,i} c_{e,1,j})$
- $(W_{3,2}^{2,0})_{ij} = \frac{1}{3} \int_{\partial K} G_3 r_i r_j dO$, $\frac{1}{3} \sum_{e \in F_0} (2\pi - \sum_{T \in F_2(e)} \beta_{T,e}) c_i c_j$
- $(W_{1,0}^{0,2})_{ij} = \frac{1}{3} \int_{\partial K} n_i n_j dO$, $\frac{1}{3} \sum_{T \in F_2} A_T n_{T,i} n_{T,j}$
- $(W_{2,0}^{0,2})_{ij} = \frac{1}{3} \int_{\partial K} G_2 n_i n_j dO$, $\frac{1}{18} \sum_{e \in F_1} |e| (\alpha_e + \sin \alpha_e) n_e' \cdot n_e' + (\alpha_e - \sin \alpha_e) n_e'' \cdot n_e''$

**Table II**

Minkowski tensors in 3 dimensions of a body $K$ with the triangulated surface $\partial K$. The mean and Gaussian curvature are $G_2$ and $G_3$, respectively. The set of facets of the triangulation of $\partial K$ is $F_2$, the set of edges is $F_1$ and the set of vertices $F_0$. The subset of triangles that contain the vertex $e$ is denoted by $F_2(e)$. The nomenclature for triangulated surfaces is defined in Fig. 5 on the left side. $A_T$ is the area, $C_T := \sum_{i=0}^{3} e_i/3$ the center of mass and $I_T$ is given in Tab. III, see also [47].
any simply-connected surface patch of a smooth manifold), \( \partial R \) its oriented boundary curve, and \( k_\gamma \) the geodesic curvature along \( \partial R \). At sharp corners of \( \partial R \) the discontinuity of the tangent vectors is characterized by jumping angles \( \beta_i \), see Fig. (7) (a) [56]. The geodesic curvature \( k_\gamma \) vanishes almost everywhere along \( \partial R(\epsilon) \), because \( \partial R(\epsilon) \) are great circle arcs on the spherical cap and the adjacent cylindrical patch and are thus geodesics; the integral \( \int k_\gamma \, ds \) vanishes.

Furthermore Fig. 7 (a) shows that the tangents of the circular segments of \( \partial R(\epsilon) \) at the corners are perpendicular to the straight lines \( g_{ij} \) between the cylindrical segments and the flat facets; these lines \( g_{ij} \) are parallel to the edges \( e_{ij} \). The angle between adjacent straight lines \( g_{ij} \) at the vertices are \( \beta_i \). The jumping angle \( \beta_i \) of \( R \) at the triangle \( T_i \) is the same as the face angle \( \beta_i \) of the triangle \( T_i \) of \( K \). As a consequence, \( \int_{\partial R(\epsilon)} G_3 \, d\Omega \) is constant for all \( \epsilon \). Equation (26) therefore yields a definition and an explicit formula for the Gaussian curvature \( G_3(P, c) \) of a polytope \( K \) at a vertex \( c \)

\[
G_3(P, c) = 2\pi - \sum_{T \in F_2(c)} \beta_{T, e},
\]

and, by summation, for the corresponding Minkowski functional \( W_3 \), cf. Tab. II. \( F_2(c) \) is the subset of triangles that contain the vertex \( c \).

At a concave vertex \( c \), a polytope can always be decomposed into three separate bodies that have convex vertices in lieu of \( c \). Because of additivity eq. (7) is also valid at concave vertices, see Fig. 7 (b).

**D. Center of mass \( W_0^{1,0} \)**

The Minkowski vector \( W_0^{1,0} \) corresponds to the center of mass of \( K \) multiplied by its volume, if \( K \) is homogeneously filled with material of constant density. This tensor may be computed by transforming the volume integral into a surface integral using Gauss’ theorem

\[
\left( W_0^{1,0}(K) \right)_i = \int_K \mathbf{r}_i \, dV = \int_{\partial K} \langle \mathbf{f}_i, \mathbf{n} \rangle \, d\Omega
\]

with \( \mathbf{f}_i \) satisfying \( \text{div} \mathbf{f}_i = r_i \). For the particular choice of \( \mathbf{f}_i \) given in Table III, this can be explicitly written as

\[
(W_0^{1,0}(K))_i = \sum_{T \in F_2} \int_{T} \mathbf{r}_i \mathbf{n}_{T,k} \, d\Omega
\]

with \( k \) as listed in Table III. (\( k \) is not a summation index.) The \( I_{i,k} \) in eq. (29) are integrals over the individually parametrised triangles

\[
I_{T,i,k} = \int_{\partial T} \mathbf{n}_{T,k} \, d\Omega
\]

\[
i \quad \tilde{f}_i \quad I_{i,k} \quad n_k
\]

\[
x \quad (0, xy, 0)^t \quad I_{x,y} \quad n_y
\]

\[
y \quad (0, 0, yz)^t \quad I_{y,z} \quad n_z
\]

\[
z \quad (xz, 0, 0)^t \quad I_{x,z} \quad n_x
\]

\[
i \quad \tilde{f}_i \quad J_{i,j,k} \quad n_k
\]

\[
x, x \quad (0, 0, xz)^t \quad J_{x,x,z} \quad n_z
\]

\[
y, y \quad (0, 0, yz)^t \quad J_{y,y,z} \quad n_z
\]

\[
z, z \quad (0, zy, 0)^t \quad J_{z,z,y} \quad n_y
\]

\[
x, y \quad (0, 0, xy)^t \quad J_{x,y,z} \quad n_z
\]

\[
x, z \quad (0, xz, 0)^t \quad J_{x, z,y} \quad n_y
\]

\[
y, z \quad (xyz, 0, 0)^t \quad J_{y, z,x} \quad n_x
\]

**TABLE III**

**Utility functions for computing the \( \nu = 0 \) Minkowski tensors.**

the functional determinant of each is \( 2AT \). \( I_{T,i,k} \) can be written in terms of the triangle vertices and centers of mass as

\[
I_{T,i,k} = \frac{9}{24} c_{T,i} c_{T,k} + \frac{1}{24} \sum_{j=1}^{3} c_{T,j} c_{T,j,k}.
\]

**E. Surface and curvature centroids \( W_0^{1,0} \)**

The remaining integrals \( W_0^{1,0} \) with \( \nu = 1, 2, 3 \) are evaluated similarly to the integrals in \( W_0^{2,0} \) and \( W_0 \). The integrals \( W_0^{1,0} \) involving surface normals vanish for arbitrary bodies (with closed bounding surfaces).

**F. Volume integral \( W_0^{2,0} \)**

The volume integral \( W_0^{2,0}(K) \) can be computed in a similar way as \( W_0^{1,0}(K) \). Using

\[
J_{T,i,j,k} = \int_0^1 \int_0^{1-a} db \left[ c_{T,1} + a(c_{T,2} - c_{T,1}) + b(c_{T,3} - c_{T,1}) \right]_i \times \left[ c_{T,1} + a(c_{T,2} - c_{T,1}) + b(c_{T,3} - c_{T,1}) \right]_j \times \left[ c_{T,1} + a(c_{T,2} - c_{T,1}) + b(c_{T,3} - c_{T,1}) \right]_k,
\]

the tensor may be expressed as

\[
W_0^{2,0}(K) = \sum_{T \in F_2} 2AT \cdot J_{T,i,j,k} \cdot \mathbf{n}_{T,k}.
\]

Again, the index \( k \) is not a summation index but given in Table III. This derivation applies equally to convex and non-convex bodies \( K \).

**G. Surface integrals \( W_0^{1,2} \) and \( W_2^{0,2} \)**

The computation of \( W_1^{0,2} \) results in a simple sum of integrals over triangular facets, resulting in the formulae in Tab. II, both for convex and non-convex bodies.

The tensor \( W_2^{0,2} \) is calculated by a parallel body construction, first demonstrated for convex bodies. Consider a convex body
K, and the corresponding parallel body \( K_\epsilon \). The integral over the parallel surface is split up into integrals over flat facets, cylindrical edge segments and spherical vertex caps. Out of these only the cylindrical edge segments contribute, for the same reasons as for the scalar measure \( W \). The remaining contribution is calculated for \( \epsilon \to 0 \) using the following representation for the normal vectors on the cylindrical caps. Given an edge \( e \) with facet normals \( \mathbf{n}_{e,T} \) and \( \mathbf{n}_{e,T'} \) of the adjacent triangles. One obtains (also representing a special case of eq. (14))

\[
W_2^{0,2}(K) = \frac{1}{6} \sum_{e \in \mathcal{F}_1} |e| \int_{-\alpha_e/2}^{\alpha_e/2} n^2 d\phi \tag{34}
\]

with \( \mathbf{e} = \mathbf{e}/|\mathbf{e}| \), \( n'' = (n_{e,1} + n_{e,2})/(n_{e,1} + n_{e,2}) \) and \( \mathbf{n}' = \mathbf{e} \times n'' \). For a given edge, \( \mathbf{n} \) can be written as \( \mathbf{n} = \cos \theta \mathbf{n}' + \sin \theta \mathbf{n}'' \) and the individual edge integral simplifies to \((1/12)|e|((\alpha_e + \sin \alpha_e)n''^2 + (\alpha_e - \sin \alpha_e)n'^2)\), cf. Fig. 5. This yields the formula in Tab. II. The validity of this formula for non-convex bodies follows from similar considerations as for \( W_2 \).

**H. Curvature-weighted surface integrals \( W_2^{2,0} \) and \( W_3^{2,0} \)**

The mean and Gaussian curvature weighted surface integrals \( W_2^{2,0} \) and \( W_3^{2,0} \) over position vectors can be evaluated as the limit \( \epsilon \to 0 \) of the parallel body construction, for convex bodies. The validity for non-convex shapes follows from similar considerations as for \( W_2 \) and \( W_3 \).

**I. Open Bodies, Labelled domains and local Minkowski Maps**

The analysis presented thus far has been derived for compact bodies in \( \mathbb{E}^3 \) with a closed bounding surface – and inherits strong robustness from its integral nature. For some analyses the requirement of closed bodies needs to be stringent. For example, experimental datasets of percolating or periodic structures, both of which extend infinitely through space, always represent finite subsets of the structure with components that traverse the dataset boundaries. Similarly, an analysis of a periodic model may be restricted to a translational unit cell, see Fig. 8. Furthermore a “local” Minkowski tensor analysis, termed a Minkowski map [7], [20], can be useful to quantify variations throughout the probe where a grid is superposed on the body \( K \), and the Minkowski tensors are computed separately for each grid domain \( Q \). Such Minkowski maps can be useful to analyse spatial heterogeneity of anisotropy or orientation at the length scale given by the size of \( Q \). In these situations the Minkowski tensors are computed for the subset \( S_Q \) of the whole bounding surface \( S = \partial K \) that is contained in a box \( Q \). In general, \( S_Q \) is not a closed surface.

It is evidently possible to take the subset \( K \cap Q \) of \( K \) that is contained in \( Q \), and consider \( \partial(K \cap Q) \) as the bounding surface. However this introduces bounding surface patches (e.g. solid/void interfaces if \( K \) is a porous medium) that are not part of the bounding surface \( \partial K \) of \( K \). For physical analyses one may want to avoid such boundary effects, i.e. not consider the contributions of these additional bounding surface patches. This motivates the introduction of Minkowski measures for open bodies, i.e. bodies without a closed bounding surface.

In lieu of providing a definition for Minkowski functionals for open bodies, we define a component-wise analysis of Minkowski tensors and allow for components that are not disjoint. For example we consider a situation where the part \( K_A \) of \( K \) that is inside a box \( Q \) is labelled with a label \( a \) and the rest \( K_B \) as \( b \). We assume that the triangulation is adjusted such that a triangle is associated either fully with \( a \) or fully with \( b \). The aim is to compute a value of \( W(K_A) \) of \( W^{r,s} \) (with \( r = 1,2,3 \)) of that part of \( K \) with bounding surface \( \partial Q \).

A label-wise analysis for the surface tensors \( W_2^{2,0} \) and \( W_1^{0,2} \) is straightforward as the contributing components are the surface triangles that are unambiguously assigned to a label. By contrast, for \( W := W_2^{0,2} \) or \( W_2^{2,0} \) an edge shared by two triangles with different labels \( a \) and \( b \) contributes equally with a factor 0.5 to \( W(K_A) \) and \( W(K_B) \). For \( W_3^{2,0} \) the relative contribution to \( W_3^{2,0}(K_A) \) for a given label \( A \) of a vertex adjacent to triangles with different labels is given by a label factor

\[
f_A(\mathbf{e}) := \frac{\sum_{T \in \mathcal{F}_2(\mathbf{e}) \cap \mathcal{F}_2(A)} \beta_{T,e}}{\sum_{T \in \mathcal{F}_e} \beta_{T,e}}, \tag{35}
\]

where \( \mathcal{F}_2(A) \) is the set of all triangles labelled with label \( A \). Hence \( f_A(\mathbf{e}) \) is the sum of angles at \( \mathbf{e} \) of those triangles adjacent to \( \mathbf{e} \) and are labelled \( A \) divided by sum of these angles of all adjacent triangles. The vertex contribution to \( W(K_A) \) is \( f_\text{m}(\mathbf{e})w_2^{2,0} \) with \( w_2^{2,0} = (2\pi - \sum \beta_{T,e}) c_e \), see Tab. II.

For the volume tensor \( W_0^{2,0} \) a label-wise analysis is only well-defined if the body \( K \) is subdivided (and not only the bounding surface \( \partial K \)).

**III. Anisotropy Measures**

Based on eigenvalue ratios of Minkowski tensors robust measures of intrinsic anisotropy can be defined that are sufficiently sensitive to capture subtle anisotropy effects and that are applicable to arbitrary microstructure. The usefulness and versatility of this approach is demonstrated by three examples representing three different types of structures – a cellular partition, a percolating porous material and a network structure. Essentially the anisotropy indices \( \beta_{r,s}^{l,j} \) are defined as the minimal-to-maximal eigenvalue ratio of the Minkowski tensors.
\[ W^r_s. \]

Our particular focus is on the characterisation of intrinsic anisotropy of connected percolating structures that extend infinitely and that are, on some length scale, homogeneous. Typically a subset, or observation window, of the structure is available for analysis. We derive measures that characterize the *intrinsic* anisotropy, i.e. their values do not depend on the size, aspect ratio or position of the observation window provided that the structure is homogeneous on length scales corresponding to the size of the observation window.

A measure that fulfills these requirements is derived from the distributions of normal vectors, and is now shown to be the same as the ratio of largest to smallest eigenvalues of \( W^{0.2}_1 \). The distribution of normal vectors \( w_1(n') = \int_{\partial K} \delta(n - n') \, d\Omega \) is the sum of the area of all surface patches with normal vector \( n' \). The Minkowski tensor \( W^{0.2}_1 \) can be expressed for an arbitrary body \( K \) as

\[
W^{0.2}_1(K) = \frac{1}{3} \int_{S^2} d\Omega \, w_1(n) \, n \otimes n \tag{36}
\]

with the unit sphere \( S^2 \). Therefore a robust integral measure for the degree of variation in occurrence between surface patches with given normal direction is afforded by

\[
\beta_1^{0.2} = \frac{\mu_{\min}}{\mu_{\max}} \tag{37}
\]

where \( \mu_{\max} \) and \( \mu_{\min} \) are the maximal and the minimal eigenvalue of the corresponding Minkowski tensor \( W^{0.2}_1 \). In the same way we define also \( \gamma_1^{0.2} \) as the ratio of the medium eigenvalue to the maximal eigenvalue.

For example, if \( K \) is a sphere, then \( w_1(n) \) is constant and \( \beta_1^{0.2} = 1 \) as expected. For the rectangular box \([0,a] \times [0,b] \times [0,c]\) the function \( w_1(n) = 0 \) for all \( n \), except for \( w_1(\pm e_x) = c d \delta(n \pm e_x) \), and analogous for \( e_y \) and \( e_z \), quantifying the total surface area with a given normal direction. The resulting anisotropy measure is \( bc/ab \) for \( a \geq b \geq c \).

It is instructive to express the second translation-invariant Minkowski tensor \( W^{0.2}_2 \) by a distribution of normals and curvatures. The function

\[
w_2(n', G^2_2) = \int_{\partial K} \delta(n - n') \delta(G_2 - G^2_2) \, d\Omega \tag{38}
\]

gives the sum of the area of all surface patches that have normal direction \( n' \) and mean curvature \( G^2_2 \).

\[
W^{0.2}_2(K) = \frac{1}{3} \int_{-\infty}^{\infty} dG^2_2 \int_{S^2} d\Omega \, w_2(n, G_2, G^2_2) \, n \otimes n. \tag{39}
\]

If the function \( w_2 \) can be written as a product \( w_2(n, G_2) = \tilde{w}_2(G_2) w_1(n) \), the anisotropy characteristics \( \beta_1^{0.2} \) and \( \beta_2^{0.2} \), defined as the ratio of the largest to the smallest eigenvalue of \( W^{0.2}_2 \), are identical. In this sense, \( \beta_2^{0.2} \) provides a higher order anisotropy measure that quantifies anisotropy of the curvature distribution.

The eigenvalue ratios of the translation-covariant tensors \( W^{2.0}_v \) (and of the tensor of inertia) capture different aspects of the anisotropy of a shape compared to \( W^{0.2}_v \), see also section 1-C. However, their usefulness is dependent on whether or not a natural definition of the origin is available for the system. For example, for the analysis of convex cells one may choose the center of mass \( W^{1.0}_v/W_0 \) or the corresponding curvature centroid \( W^{1.0}_v/W_0 \) as the origin. For Voronoi cells and related partitions the center point of the generating particle may be a generic choice of origin [57]. However, especially for percolating or periodic bodies, for which the analysis is always restricted to a finite window of observation, the choice of origin is crucial and often not naturally determined.

An additional problem for such structures is that the measures \( \beta^{2.0}_v \) derived from translation covariant tensors \( W^{2.0}_v \), as opposed to the translation-invariant measures \( \beta^{0.2}_v \), crucially depend on the shape and size of the window of observation, because of the second power of the position vector (assuming that the origin is within the window of observation).

### A. Triply-Periodic Minimal Surface Labyrinthine Bodies

Triply-periodic minimal surfaces (TPMS) are oriented surfaces in \( \mathbb{E}^3 \) that have constant vanishing mean curvature \( G_2 = 0 \) and that are periodic with three linearly independent lattice vectors \([58]–[62] \). Any TPMS \( S \) divides \( \mathbb{E}^3 \) into two domains \( K \) and \( \overline{K} := \mathbb{E}^3 \setminus K \) in positive and negative normal direction from \( S \), respectively. Both domains are continuous (i.e. connected), hence the term *bicontinuous*, see Fig. 9. Particularly well-known are three TPMS of cubic symmetry, the (G)yroid [58], the (Diamond [62] and the (P)rimitive [62] surfaces. In general, \( K \) and \( \overline{K} \) may not be congruent, although for the specific examples considered in this article they are.

TPMS have been shown to describe the spatial structure of complex mesophases in both lipids [63], [64], copolymers [65]– [68] and other biological systems [69]–[71]. The recent finding of anisotropic (i.e. in particular non-cubic) TPMS in polymers [72] and mesoporous silica systems [73] underlines the relevance of studying anisotropy measures for these models.

The surfaces studied here are one-parameter families of TPMS with a free parameter that alters their shape and in particular...
their anisotropy and lattice vectors of the symmetry group; nevertheless these surfaces are bicontinuous, non-selfintersecting and minimal for all values of this free parameter. Specifically we analyse the one-parameter TPMS-families \(tP, tD\) and \(tG\) of tetragonal, \(rG\) and \(rPD\) of rhombohedral and \(H\) of hexagonal symmetry [59], [74]. Triangulations of almost arbitrary vertex coordinate precision can be obtained from the Weierstrass representation for minimal surfaces. We use the common notation that \(r_0\) is the free parameter of \(tP, tD\) and \(rPD\), \(\phi_0\) the free parameter of the \(tG\) and \(rG\) families and \(A\) the free parameter of the \((H)\)exagonal surface. For certain values of the free parameters the rhombohedral and tetragonal TPMS family members correspond to cubic TPMS.

The symmetry groups of these minimal surfaces imply that the eigendirections correspond to lattice directions. In particular, for the cases studied here, the tensor \(W_{1}^{0,2}\) has an eigendirection corresponding to the \(z\)-axis and a degenerate eigenspace corresponding to the \(xy\)-plane. Instead of the ratio of smallest to largest eigenvalue, the anisotropy of these structures is better elucidated by the anisotropy index

\[
\sigma_{1}^{0,2} = \frac{\mu_{z}}{\mu_{z}} \tag{40}
\]

with \(\mu_{z}\) and \(\mu_{x}\) the eigenvalues of \(W_{1}^{0,2}\) corresponding to the \(z\)-axis resp. the \(xy\)-plane. For an isotropic normal distribution (note that this includes cubic symmetry) \(\sigma_{1}^{0,2} = 1\).

One could also extend this definitions to any other rank-2 Minkowski tensors to obtain \(\sigma_{i}^{0,2}\) and \(\sigma_{j}^{0,2}\).

For minimal surfaces, the space of translation-invariant valuations of rank two reduces to \(W_{1}^{0,2}\) and \(W_{0}^{0,0}\) since \(W_{2}^{0,2} = 0\) as the mean curvature vanishes for minimal surfaces. Figure 11 shows the dependence of eigenvalue ratios of \(W_{2}^{0,2}\) of the free parameter of the surface family for the surface families listed above, as well as the ratio of lattice parameters \(c/a\) of the symmetry groups of the respective surfaces, see [74]. The \(c/a\)-ratios provide an inherent anisotropy measure for periodic structures (that e.g. determines the tensor of inertia and \(W_{1}^{0,2}\)). Our data show that the eigenvalue ratios of the translation-invariant tensor \(W_{1}^{0,2}\) is a more sensitive anisotropy index than the \(c/a\) ratios. This suggests that for porous materials, ordered or random, the ratios of eigenvalues of \(W_{1}^{0,2}\) may be a sensitive measure of anisotropy that is also easy to compute. The diagrams also contain exact representations of the Minkowski tensors derived from the Weierstrass representation, see also the appendix. We refrain from an analysis of the translation-covariant tensors \(W_{\nu}^{2,0}\) because of the origin dependence.

An interesting question with physical relevance for bicontinuous lipid phases arises from this analysis: The member of the hexagonal \(H\) surface for which the ratio of \(W_{1}^{0,2}\) eigenvalues is unity (this is the member with \(A \approx 0.679\) and \(c/a = 0.832\)) may be a candidate for an isotropic hexagonal bicontinuous lipid mesophase, somewhat similar to the ubiquitous cubic \(P, D\) and \(G\) surfaces. We note that this isotropic \(H\) surface is also the
analysed for different segmentation parameter between both plots are the standard deviation of the distribution of the quantities when aligns with the shear direction when large shear is applied. The error bars in and the direction of applied shear; its decay to 0 indicates that the network to approximately cylindrical tubes. The insert shows the alignment angle $\phi$ of the eigenvector corresponding to the minimal eigenvalue $\mu_{\text{min}}$ of $W^{1/2}_{1/2}$ and the direction of applied shear; its decay to 0 indicates that the network aligns with the shear direction when large shear is applied. The error bars in both plots are the standard deviation of the distribution of the quantities when analysed for different segmentation parameter between $0.95 < \Lambda < 0.99$. The illustrations above sketch the experimental setup with confocal microscopy (the small confocal microscopy images are taken from [75]).

For biopolymer networks made of actin or collagen fibers, the relationship between anisotropy and alignment on the one-hand side and elastic or visco-elastic properties on the other hand can be addressed with shear experiments [75]. Three-dimensional datasets can be obtained by confocal microscopy; from these a line representation of the fibers is extracted. Minkowski tensors are more direct measures of anisotropy than measures based on two-dimensional projections or on directional distributions of fiber tangent vectors.

The datasets analyzed here represent actin fiber networks reconstituted from rabbit actin biopolymer networks with actin concentration of 1.2mg/ml cross-linked with filamin A. These are imaged using confocal microscopy for different shear-rates, see the explanation in Fig. 12. The data sets are the same as those analysed in [75]. The gray-scale dataset is converted into a binary dataset with 1 corresponding to actin and 0 corresponding to the surrounding fluid by standard threshold segmentation with threshold $I_C$. The threshold $I_C$ is chosen such that only the brightest and hence thickest fibers are retained. The medial axis of the 1 phase is computed using distance-ordered homotopic thinning [76], [77] and is used as the one-voxel thick line representation of the actin fiber network. A triangulated representation is obtained using the Marching Cubes algorithm [78]. For more details of the analysis of biopolymers see ref. [79]

Figure 12 shows the ratio of the largest to the smallest and of the intermediate to the largest eigenvalue of the tensor $W^{1/2}_{1/2}$, evaluated on the whole network (that consists essentially in a single component). It shows that the distribution of normal directions of the fiber bounding surface becomes less isotropic with increasing shear, indicating alignment of the fibers. The angle between the eigenvector to the minimal eigenvalue (corresponding approximately to the dominant tangent direction) and the direction of shear decreases to 0, indicating the alignment of the fibers with the direction of shear. This is commensurate with the results published in [75].

C. Anisotropy of Free-Volume Cells of Random Bead Packs

The distributions of volumes of the Voronoi cells of disordered sphere packs with packing fractions from 0.55 (random loose packing) to 0.64 (random close packing) has recently attracted interest for the study of granular systems [80], motivated by a possible statistical mechanics description of granular systems [81], [82]. However, higher order shape measures of the Voronoi cells, such as their anisotropy, have only recently been investigated demonstrating their relevance for granular media [57].

Experimental and simulated random close-packed configurations of approximately $10^5 - 10^6$ monodisperse spheres were generated using three different protocols [83]–[85]. The center coordinates of all beads (in the experimental datasets) were extracted by X-ray microtomography and a deconvolution technique [86], [87]. The Voronoi cells of the sphere centers are computed using the program qhull [88]. For a point $p$ of a set $\mathcal{P}$ of points, its Voronoi cell is the convex polytope that contains all points of $\mathbb{E}^3$ closer to $p$ than to any other point in $\mathcal{P}$ [89].

Figure 13 shows the result of our analysis of the average anisotropy of these Voronoi cells. The ratio $\beta_1^{1/2}$ of the largest to smallest eigenvalue of the tensor $W^{1/2}_{1/2}$ of a Voronoi cell (averaged over all Voronoi cells of each bead pack) shows a clear decay of anisotropy as a function of the packing density $\eta$ [57]. This demonstrates that anisotropy measures based on Minkowski tensors are applicable to cellular structures with subtle geometric anisotropy.

The characterisation of the shape of convex polygonal cells is also relevant in various other contexts, e.g. for the relationship between structure and dynamics in glass-forming liquids [90] or for packing entropy of the hard micellar cores in supramolecular micellar materials [91], for the understanding of physical properties of foams, both dynamic [92] and static members of the $H$ family with close to minimal packing and bending frustration [74].

B. Alignment of Actin Biopolymer networks under shear

For biopolymer networks made of actin or collagen fibers, the relationship between anisotropy and alignment on the one-hand side and elastic or visco-elastic properties on the other hand can be addressed with shear experiments [75]. Three-dimensional datasets can be obtained by confocal microscopy; from these a line representation of the fibers is extracted. Minkowski tensors are more direct measures of anisotropy than measures based on two-dimensional projections or on directional distributions of fiber tangent vectors.

The datasets analyzed here represent actin fiber networks reconstituted from rabbit actin biopolymer networks with actin concentration of 1.2mg/ml cross-linked with filamin A. These are imaged using confocal microscopy for different shear-rates, see the explanation in Fig. 12. The data sets are the same as those analysed in [75]. The gray-scale dataset is converted into a binary dataset with 1 corresponding to actin and 0 corresponding to the surrounding fluid by standard threshold segmentation with threshold $I_C$. The threshold $I_C$ is chosen such that only the brightest and hence thickest fibers are retained. The medial axis of the 1 phase is computed using distance-ordered homotopic thinning [76], [77] and is used as the one-voxel thick line representation of the actin fiber network. A triangulated representation is obtained using the Marching Cubes algorithm [78]. For more details of the analysis of biopolymers see ref. [79]

For a given segmentation threshold $I_C$ the integrated intensity of all voxels of the fluid phase is $\Lambda = (\sum I(p))^{-1} \sum I(p)$ where $I(p)$ is the intensity of voxel $p$ in the original intensity data set, $\sum$ the sum over all voxels of the data set and $\sum_p^\ast$ the sum over all voxels of the fluid phase, i.e. those voxels that are set to 0 by the segmentation process. The values of $I_C$ chosen here correspond to $0.95 < \Lambda < 0.99$. 

3For a given segmentation threshold $I_C$ the integrated intensity of all voxels of the fluid phase is $\Lambda = (\sum I(p))^{-1} \sum I(p)$ where $I(p)$ is the intensity of voxel $p$ in the original intensity data set, $\sum$ the sum over all voxels of the data set and $\sum_p^\ast$ the sum over all voxels of the fluid phase, i.e. those voxels that are set to 0 by the segmentation process. The values of $I_C$ chosen here correspond to $0.95 < \Lambda < 0.99$. 

$\frac{1}{\beta_1} \left( \frac{2}{3} \right)^{\frac{1}{2}}$
These analyses are mostly concerned with (moments of the) distributions of cell volumes, number of neighbors or face areas. More advanced cell shape measures capable of quantifying intrinsic or global anisotropy are needed, as assumptions of isotropy can often not be verified for systems with inherent anisotropy, such as dense granular flows under gravity [94] or shear-zones in granular matter [95]. This emphasizes the role that the shape characterisation by Minkowski tensors will play in the future.

IV. Conclusion

We have described linear-time algorithms to compute Minkowski tensors of three-dimensional triangulated bodies. We have shown that robust measures of intrinsic anisotropy can be derived from these tensors. These have been used to quantify the anisotropy of an experimental model system for granular matter, of confocal microscopy images of sheared biopolymer networks and of triply-periodic minimal surface models for amphiphilic self-assembly.

The availability of this algorithm is a prerequisite for applying Minkowski tensor analyses more widely for the quantitative description of anisotropic structural features in physical systems, e.g. of mechanical, conductive or flow properties of anisotropic metal foams or porous rocks and electron- or X-ray tomography images. Simultaneously a deeper theoretical understanding of the Minkowski tensors, that has begun to emerge with applications to density functional theory of non-spherical liquid crystal systems [15] and to biomolecular motors [17], has to develop further. This will be facilitated by analyses of Minkowski tensors of various stochastic geometry models, e.g. of orientation-biased line processes or Poisson-Voronoi tessellations, as well as of ordered structures. Except for the simplest models much of this work will require the numerical algorithms described in this paper.

SOFTWARE

Minkowski tensor software is available at www.theorie1.physik.uni-erlangen.de/karambola.

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V. Appendix: Weierstrass Formulae for $W_1^{0,2}$

Weierstrass parameterisations are explicit surface parameterisations of minimal surfaces via path integrals in the complex plane [96], [97]. For triply-periodic minimal surfaces the Weierstrass parameterisation maps a $n$-fold Riemann-surface $\mathbb{C}^*$ onto the primitive translational unit cell of the TPMS in $\mathbb{E}^3$, see the illustration for the so-called I-WP surface with $n = 3$ in Fig. 14. All other TPMS discussed in this article have $n = 2^4$.

$$S : \mathbb{C}^* \rightarrow \mathbb{E}^3$$

$$\omega \mapsto p_0 + \text{Re} \left[ e^{i\theta} \int_{\omega_0}^{\omega} \left( \frac{1 - \omega'^2}{\sqrt{1 + \omega'^2}} \right) d\omega' \right]$$

The analytic non-vanishing function $R(\omega)$ is called the Weierstrass-function and $\theta$ the Bonnet angle, both specific to the parametrised surface (Re is the real part). The Weierstrass-functions for the surfaces discussed here are given in Tab. IV. The Weierstrass parameterisation is the inversion of the Gauss-map composed with the stereographic projection, see refs. [97], [98]. The Gaussian curvature of $S$ at point $p = p(\omega)$ is

$$G_3(\omega) = -4(1 + |\omega|^2)^{-4} |R(\omega)|^{-2},$$

the area element is

$$d\mathcal{O} = d^2\omega(1 + |\omega|^2)^2 |R(\omega)|^2.$$
for a fraction $S_{\text{TPUC}}$ of the surface that corresponds to the primitive translational unit cell.\(^6\)

For convenience (in particular to avoid the cumbersome treatment of branch cuts of $C^+$) we use the Weierstrass formalism only for the domain $C \subset C^+$ that corresponds to an asymmetric patch of the symmetric surface. We then apply all symmetry group operations (listed in Tab. IV) to obtain the tensor for the whole translational unit cell. If $W_{\nu}^{0,2}(S_{\text{ASY}})$ is the Minkowski tensor of a single asymmetric patch $S_{\text{ASY}}$ and $G$ the symmetry group of the body bounded by the periodic surface, the tensor of the translational unit cell is given by

$$W_{\nu}^{0,2}(K) = \sum_{g \in G} g W_{\nu}^{0,2}.$$  (46)

The symmetry group is that of the body $K$ bounded by the TPMS, i.e. the same as that of the oriented bounding surface with a specified normal orientation and that is called the symmetry group of the oriented (i.e. colored) surface, and not that of the unoriented bounding surface. The latter may contain operations that map the bounding surface onto itself with reversed normals, i.e. a reversal of inside and outside.\(^6\)

For the TPMS discussed here, the asymmetric unit patch corresponds to a wedge of the complex plane given by the points $r \exp(i\varphi)$ with $r \in [0,1]$ and $\varphi \in [\varphi_a, \varphi_b]$. The values of $\varphi_a$ and $\varphi_b$ and the range of permissible values for the single free parameter of these surface families, denoted $\varphi_0$ or $\psi_0$, are given in Tab. IV. The expression for $W_{1}^{0,2}$ is then given by integrals of the type

$$W_{1}^{0,2}(S_{\text{ASY}}) = \frac{1}{2\pi} \int_{\varphi_a}^{\varphi_b} r (1 + r^2)^2 |R(\omega)|^2 n(\omega)^2 d\varphi dr$$  (47)

with $\omega = r \exp(i\varphi)$. The curves in Fig. 11 are plots of the explicit solution of eqs. (46) and (47). The Weierstrass functions and symmetry groups for all surfaces discussed in Fig. 11 are shown in Tab. IV. Note that the expressions cannot be given in closed form, but numerical integration yields values with high accuracy.

In principle, eq. (46) can be further simplified as the symmetric structure of all TPMS imposes constraints on the Minkowski tensors. General principles valid for all rank-2 tensors, expressed e.g. [99], lead to the following relationship for the eigenvalues and eigenvectors: If a structure has an $n$-fold symmetry axis with $n = 3, 4, 6$, two of the eigenvalues become degenerate (identical) and the corresponding eigenspace is the plane vertical to the rotation axis; the third eigenvector is by definition parallel to the rotation axis. If a body has two or more $n$-fold rotation axis with $n = 3, 4, 6$ that are not parallel all eigenvalues are identical and the structure is called isotropic; this is specifically the case for all structures with cubic symmetry. This requirement implies that trigonal, monoclinic and rhomboic systems have up to three distinct eigenvalues, tetragonal, hexagonal and rhombohedral structures up to two and cubic structures only a single eigenvalue. Note that specifically for the translation-invariant tensors $W_{1}^{0,2}$ and $W_{2}^{0,2}$, a two-fold rotation axis always represents a multiplication of the tensors with a factor of 2.

We note that $W_{1}^{0,2}(K) = W_{1}^{0,2}(K')$ for two bodies whose bounding surfaces $\partial K$ and $\partial K'$ are isometric (see e.g. [98] for the definition). This is e.g. the case for the cubic Primitive, Diamond and Gyroid surfaces with the length scale that results from eq. (41) that are related by the so-called Bonnet-transformation [98], [100], [101]; in eq. (41) this transformation corresponds to adjusting the Bonnet angle $\theta$. This is evidenced by the fact that $\theta$ does no occur in eqs. (44) and (43). However, rescaling the length scale, e.g. to yield constant surface-to-volume ratio, lifts that degeneracy.

The same formalism can also be used to compute the scalar motion-invariant Minkowski functionals, and in principle also the translation-covariant Minkowski tensor valuations. A problem is that the domain $C$ that corresponds to the translational unit cells used in section III-A is not readily known (and the translation-covariant tensors depend on the precise form of the translational unit cell, in contrast to the translation-invariant tensors).

References

TABLE IV

<table>
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<tr>
<th>free parameter</th>
<th>Weierstrass function ( R(\omega, \omega_0) )</th>
<th>([\phi_0, \phi_0 + \pi]_\phi )</th>
<th>colored space group</th>
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<td>( tG )</td>
<td>( \frac{1}{\sqrt{\omega^4 - \omega_0^4}} )</td>
<td>([\phi_0, \phi_0 + \pi]_\phi )</td>
<td>( I_4122 )</td>
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<td>( I41/amd )</td>
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<td>([0, 2\pi/3]_\phi )</td>
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<td>( P63/mmc )</td>
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</tbody>
</table>

Details of the TPMS families analysed in section III-A: Conventional name of the free parameter, Weierstrass function \( R(\omega) \), limits \( \phi_0 \) and \( \phi_0 \) of the integration domain \( r \exp(\phi) \) with \( 0 \leq r \leq 1 \) and \( \phi_0 \leq \phi \leq \phi_0 \) and symmetry group of the oriented TPMS (i.e., of the solid body that results from filling one of the two labyrinthine domains with a solid). The Bonnet angle \( \theta \) (which is irrelevant for the Minkowski tensors) and further detail are given in refs. [59]–[61].

Convex Bodies: The Brunn-Minkowski Theory


