The mean width of random polytopes circumscribed around a convex body

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ABSTRACT

Let $K$ be a $d$-dimensional convex body, and let $K^{(n)}$ be the intersection of $n$ halfspaces containing $K$ whose bounding hyperplanes are independent and identically distributed. Under suitable distributional assumptions, we prove an asymptotic formula for the expectation of the difference of the mean widths of $K^{(n)}$ and $K$, and another asymptotic formula for the expectation of the number of facets of $K^{(n)}$. These results are achieved by establishing an asymptotic result on weighted volume approximation of $K$ and by “dualizing” it using polarity.

1. Introduction

Let $K$ be a convex body (compact convex set with nonempty interior) in $d$-dimensional Euclidean space $\mathbb{R}^d$. The convex hull $K^{(n)}$ of $n$ independent random points in $K$ chosen according to the uniform distribution is a common model of a random polytope contained in $K$. The famous four-point problem of Sylvester [40] is the starting point of an extensive investigation of random polytopes of this type. Beside specific probabilities as in Sylvester’s problem, important objects of study are expectations, variances and distributions of various geometric functionals associated with $K^{(n)}$. Typical examples of such functionals are volume, other intrinsic volumes, and the number of $i$-dimensional faces. In their ground-breaking papers [30] and [31], Rényi and Sulanke considered random polytopes in the Euclidean plane and proved asymptotic results for the expectations of basic functionals of random polytopes in a convex domain $K$ in the cases where $K$ is sufficiently smooth or a convex polygon. Since then most results have been in the form of asymptotic formulae as the number $n$ of random points tends to infinity.

In the last three decades, much effort has been devoted to exploring the properties of this particular model of a random polytope contained in a $d$-dimensional convex body $K$. For instance, for a sufficiently smooth convex body $K$, asymptotic formulae were proved for the expectation of the mean width difference $W(K) - W(K^{(n)})$ by Schneider and Wieacker [37], and for the volume difference $V(K) - V(K^{(n)})$ by Bárány [1]. The assumption of smoothness was relaxed in the case of the mean width by Böröczky, Fodor, Reitzner and Vígh [6], and removed by Schütt [38] in the case of the volume, general intrinsic volumes are treated in [7] under a weak smoothness assumption. Recently, even variance estimates, laws of large numbers, and central limit theorems have been proved in different settings in a sequence of contributions, for instance by Bárány, Reitzner, and Vu [3], [4], [28], [29], [41].

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For more details on the current state-of-the-art of this line of research, see the survey papers by Weil and Wieacker [43], Gruber [15] and Schneider [35], and the recent monograph of Schneider and Weil [36].

In a third paper, Rényi and Sulanke [32] suggested a model which is “dual” to the model of a random polytope contained in a given convex body \( K \) (a random inscribed polytope), that is, they considered a random polytope containing a given convex body (a random circumscribed polytope). Subsequently, this approach has not received as much attention as the “inscribed case”, although it is closely related to linear optimization (cf. [10], [33], §6). There are various ways of producing circumscribing random polytopes containing a given convex body. In this paper, we consider a model in which the circumscribing polytope arises as an intersection of closed halfspaces whose bounding hyperplanes are randomly chosen hyperplanes. The rough description of the probability model is the following, it is described more precisely in Section 2, a more general setting is provided in Section 5. In Euclidean space \( \mathbb{R}^d \), we consider hyperplanes that intersect the radius one parallel domain of a given convex body \( K \) but miss the interior of \( K \), and we use the restriction of the (suitably normalized) Haar measure on the set of hyperplanes in \( \mathbb{R}^d \) to provide an associated probability measure. For \( n \) independent random hyperplanes \( H_1, \ldots, H_n \) chosen according to this distribution, the intersection of the closed halfspaces bounded by \( H_1, \ldots, H_n \) and containing \( K \) determines a circumscribed random polyhedral set containing \( K \) (which might be unbounded). The main goal of this article is to find asymptotic formulae for the expectation of the difference of the mean widths of a random circumscribed polytope and the given convex body \( K \), and for the expectation of the number of facets of a circumscribed random polyhedral set. These (and more general) results will be achieved by establishing general results on weighted volume approximation of a given convex body by inscribed random polytopes. In all these results, no regularity or curvature assumptions on \( K \) are required.

As for earlier results, we mention the paper [47] by Ziezold who investigated circumscribed polygons in the plane, and the doctoral dissertation [21] of Kaltenbach who proved asymptotic formulae for the expectation of the volume difference and for the expectation of the number of vertices of circumscribed random polytopes around a convex body, under the assumption that the boundary of the reference body \( K \) is sufficiently smooth. Recently, Böröczky and Schneider [9] established upper and lower bounds for the expectation of the mean width difference for a general convex body \( K \). Furthermore, they also proved asymptotic formulae for the expected number of vertices and facets of \( K^{(n)} \), and an asymptotic formula for the expectation of the mean width difference, under the assumption that the reference body \( K \) is a simplicial polytope with \( r \) facets.

In [8], Böröczky and Reitzner discuss a different model of a random circumscribed polytope where \( n \) independent random points are chosen from the boundary of a given smooth convex body \( K \), and the intersection of the supporting halfspaces of \( K \) at these points is the random polyhedral set under consideration. This framework is again dual to the one considered by Schütt and Werner (see [36]) who study the expected volume of the convex hull of \( n \) independent random points chosen from the boundary of a convex body satisfying a weak regularity assumption.

2. The probability space and the main goal

Let us first describe the setting for stating our results on circumscribed random polyhedral sets. Throughout this article, \( K \) will denote a compact convex set with interior points (a convex body) in \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) \((d \geq 2)\). We write \( \langle \cdot, \cdot \rangle \) for the scalar product and \( \| \cdot \| \) for the norm in \( \mathbb{R}^d \). For background on convexity, we refer to the monographs by Schneider [34] or by Gruber [16]. Let \( V \) denote volume, and let \( \mathcal{H}^j \) denote the \( j \)-dimensional Hausdorff measure. The unit ball of \( \mathbb{R}^d \) with center at the origin \( o \) is denoted by \( B^d \), and \( S^{d-1} \) is its boundary. We put \( \alpha_d := V(B^d) \) and \( \omega_d := \mathcal{H}^d(S^{d-1}) = d\alpha_d \). The parallel body of \( K \) of radius \( 1 \) is \( K_1 := K + B^d \). Let \( \mathcal{H} \) denote the space (with its usual topology) of hyperplanes in \( \mathbb{R}^d \), and let \( \mathcal{H}_K \) be the subspace of hyperplanes meeting \( K_1 \) but not the interior of \( K \). For \( H \in \mathcal{H}_K \), the closed halfspace bounded by \( H \) that contains
$K$ is denoted by $H^-$. Let $\mu$ denote the motion invariant Borel measure on $\mathcal{H}$, normalized so that $\mu(\{H \in \mathcal{H} : H \cap K \neq \emptyset\})$ is the mean width $W(M)$ of $M$, for every convex body $M \subset \mathbb{R}^d$. Let $2\mu_K$ be the restriction of $\mu$ to $\mathcal{H}_K$. Since $\mu(\mathcal{H}_K) = W(K + B^d) - W(K) = W(B^d) = 2$, the measure $\mu_K$ is a probability measure. For $n \in \mathbb{N}$, let $H_1, \ldots, H_n$ be independent random hyperplanes in $\mathbb{R}^d$, i.e., independent $\mathcal{H}$-valued random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, each with distribution $\mu_K$. The possibly unbounded intersection

$$K^{(n)} := \bigcap_{i=1}^n H_i^-$$

of the halfspaces $H_i^-$, with $H_i \in \mathcal{H}_K$ for $i = 1, \ldots, n$, is a random polyhedral set. A major aim of the present work is to investigate $\mathbb{E}W(K^{(n)} \cap K_1)$, where $\mathbb{E}$ denotes mathematical expectation. The intersection with $K_1$ is considered, since $K^{(n)}$ is unbounded with positive probability. Instead of $\mathbb{E}W(K^{(n)} \cap K_1)$, we could consider $\mathbb{E}_1 W(K^{(n)})$, the conditional expectation of $W(K^{(n)})$ under the condition that $K^{(n)} \subset K_1$. Since $\mathbb{E}W(K^{(n)} \cap K_1) = \mathbb{E}_1 W(K^{(n)}) + O(\gamma^n)$ with $\gamma \in (0, 1)$ (cf. [9]), there is no difference in the asymptotic behaviors of both quantities, as $n \to \infty$. We also remark that, for the asymptotic results, the parallel body $K_1$ could be replaced by any other convex body containing $K$ in its interior; this would only affect some normalization constants.

Let $\partial K$ denote the boundary of $K$. We call $\partial K$ twice differentiable in the generalized sense at a boundary point $x \in \partial K$ if there exists a quadratic form $Q$ on $\mathbb{R}^{d-1}$, the second fundamental form of $K$ at $x$, with the following property: If $K$ is positioned in such a way that $x = 0$ and $\mathbb{R}^{d-1}$ is a support hyperplane of $K$ at $o$, then in a neighborhood of $o$, $\partial K$ is the graph of a convex function $f$ defined on a $(d-1)$-dimensional ball around $o$ in $\mathbb{R}^{d-1}$ satisfying

$$f(z) = \frac{1}{2} Q(z) + o(\|z\|^2), \quad (2.1)$$

as $z \to o$. Alternatively, we call $x$ a normal boundary point of $K$. If this is the case, we write $\kappa(x) = \det(Q)$ to denote the generalized Gaussian curvature of $K$ at $x$. Writing $\kappa(x)$, we always assume that $\partial K$ is twice differentiable in the generalized sense at $x \in \partial K$. According to a classical result of Alexandrov (see [34], [16]), $\partial K$ is twice differentiable in the generalized sense almost everywhere with respect to the boundary measure of $K$ ($\mathcal{H}^{d-1}$-almost all boundary points are normal boundary points). Finally, we define the constant

$$c_d = \frac{(d^2 + d + 2)(d^2 + 1)}{2(d + 3) \cdot (d + 1)!}, \quad \frac{(d^2 + 1)}{d + 1} \cdot \alpha_{d-1}^{2/(d+1)} \quad (2.2)$$

(cf. J.A. Wieacker) [46], which will appear in the statements of our main results. In the following, we simply write $dx$ instead of $\mathcal{H}^d(dx)$.

The main asymptotic result concerning the expected difference of the mean widths of $K^{(n)}$ and $K$ is the following theorem. Generalizations of Theorem 2.1, and also of Theorem 2.2 below, which hold under more general distributional assumptions, are provided in Section 5. There we also indicate the connection to the $p$-affine surface area of a convex body.

**THEOREM 2.1.** If $K$ is a convex body in $\mathbb{R}^d$, then

$$\lim_{n \to \infty} n^{\frac{d-1}{d}} \mathbb{E}(W(K^{(n)} \cap K_1) - W(K)) = 2 c_d \omega_d^{-\frac{d+1}{d}} \int_{\partial K} \kappa(x) \frac{d\mathcal{H}^{d-1}}{\pi^d dx}. \quad (3.1)$$

Let $f_i(P)$, $i \in \{0, \ldots, d-1\}$, denote the number of $i$-dimensional faces of a polyhedral set $P$. In the statement of the following theorem, $K^{(n)}$ could be replaced by the intersection of $K^{(n)}$ with a fixed polytope containing $K$ in its interior without changing the right-hand side. Alternatively, instead of $\mathbb{E}(f_{d-1}(K^{(n)}))$ we could consider the conditional expectation of $f_{d-1}(K^{(n)})$ under the assumption that $K^{(n)}$ is contained in $K_1$. 

**Theorem 2.2.** If $K$ is a convex body in $\mathbb{R}^d$, then

$$
\lim_{n \to \infty} n^{-\frac{d+1}{2d}} \mathbb{E}(f_{d-1}(K^{(n)})) = c_d \omega_d \int_{\partial K} \kappa(x) \frac{d}{d+1} \mathcal{H}^{d-1}(dx).
$$

Both theorems will be deduced from a “dual” result on weighted volume approximation of convex bodies by inscribed random polytopes which is stated in the subsequent section. The usefulness of duality in random or best approximation has previously been observed e.g. in [21], [14], [11].

### 3. Weighted volume approximation by inscribed polytopes

For a given convex body, we introduce a class of inscribed random polytopes. Let $C$ be a convex body in $\mathbb{R}^d$, let $\varrho$ be a bounded, nonnegative, measurable function on $C$, and let $\mathcal{H}^d|_C$ denote the restriction of $\mathcal{H}^d$ to $C$. Assuming that $\int_C \varrho(x) \mathcal{H}^d(dx) > 0$, we choose random points from $C$ according to the probability measure $P_{\varrho,C} := \left( \int_C \varrho(x) \mathcal{H}^d(dx) \right)^{-1} \varrho \mathcal{H}^d|_C$. Expectation with respect to $P_{\varrho,C}$ is denoted by $\mathbb{E}_{\varrho,C}$. The convex hull of $n$ independent and identically distributed random points with distribution $P_{\varrho,C}$ is denoted by $C(n)$ if $\varrho$ is clear from the context. This yields a general model of an inscribed random polytope.

Generalizing a result by C. Schütt [38], we prove the following theorem.

**Theorem 3.1.** For a convex body $K$ in $\mathbb{R}^d$, a probability density function $\varrho$ on $K$, and an integrable function $\lambda : K \to \mathbb{R}$ such that, on a neighborhood of $\partial K$ relative to $K$, $\lambda$ and $\varrho$ are continuous and $\varrho$ is positive,

$$
\lim_{n \to \infty} n^{\frac{d}{d+1}} \mathbb{E}_{\varrho,K} \int_{K \setminus K^{(n)}} \lambda(x) \, dx = c_d \int_{\partial K} \varrho(x) \frac{1}{d+1} \lambda(x) \kappa(x) \frac{d}{d-1} \mathcal{H}^{d-1}(dx)
$$

(3.1)

where $c_d$ is defined in (2.2).

The limit on the right-hand side of (3.1) depends only on the values of $\varrho$ and $\lambda$ on the boundary of $K$. In particular, we may prescribe any continuous, positive function $\varrho$ on $\partial K$. Then any continuous extension of $\varrho$ to a probability density on $K$ (there always exists such an extension) will satisfy Theorem 3.1 with the prescribed values of $\varrho$ on the right-hand side.

Our proof of Theorem 3.1 is inspired by the argument in C. Schütz [38] who considered the special case $\varrho \equiv \lambda \equiv 1$. We note that for Lemma 2 in [38], which is crucial for the proof in [38], no explicit proof is provided, but reference is given to an analogous result in an unpublished note by M. Schmuckenschläger. Besides a missing factor $\frac{1}{2}$, Lemma 2 does not hold in the generality stated in [38]. For instance, it is not true for simplices. Most probably, this gap can be overcome, but still our approach to prove Theorem 3.1, where Lemma 2 in [38] is replaced by the elementary Lemma 4.2, might be of some interest.

The present partially new approach to Theorem 3.1 involves also some other interesting new features. In particular, we do not need the concept of a Macbeath region. An outline of the proof is given below. It should also be emphasized that the generality of Theorem 3.1 is needed for our study of circumscribed random polyhedral sets via duality.

A classical argument going back to Efron shows that

$$
\mathbb{E}_{\varrho,K} \left( f_0(K^{(n)}) \right) = n \cdot \mathbb{E}_{\varrho,K} \int_{K \setminus K^{(n-1)}} \varrho(x) \, dx,
$$

where $f_0$ is the characteristic function of $K$.
which yields the following consequence of Theorem 3.1.

**Corollary 3.2.** For a convex body $K$ in $\mathbb{R}^d$, and for a probability density function $\varrho$ on $K$ which is continuous and positive in a neighborhood of $\partial K$ relative to $K$,

$$
\lim_{n \to \infty} n^{-\frac{d+1}{d+2}} \mathbb{E}_\varrho(K(f_n(K(n)))) = c_d \int_{\partial K} \varrho(x)^{\frac{d-1}{d+2}} \kappa(x) \frac{1}{H^{d-1}} \, dx,
$$

where $c_d$ is defined in (2.2).

The proof of Theorem 3.1 is obtained through the following intermediate steps. Details are provided in Section 4. Since the convex body $K$ is fixed, we write $\mathbb{E}_\varrho$ and $\mathbb{P}_\varrho$ instead of $\mathbb{E}_\varrho(K)$ and $\mathbb{P}_\varrho(K)$, respectively. The basic observation to prove Theorem 3.1 is that

$$
\mathbb{E}_\varrho \int_{K \setminus K(n)} \lambda(x) \, dx = \int_K \mathbb{P}_\varrho(x \notin K(n)) \lambda(x) \, dx,
$$

which is an immediate consequence of Fubini’s theorem. Throughout the proof, we may assume that $\varrho \in \text{int}(K)$. The asymptotic behavior, as $n \to \infty$, of the right-hand side of (3.2) is determined by points $x \in K$ which are sufficiently close to the boundary of $K$. In order to give this statement a precise meaning, scaled copies of $K$ are introduced as follows. For $t \in (0, 1)$, we define $K_t := (1-t)K$ and $y_t := (1-t)y$ for $y \in \partial K$. In Lemma 4.3, we show that

$$
\lim_{n \to \infty} n^{\frac{d+1}{d+2}} \mathbb{P}_\varrho \left( x \notin K(n) \right) \lambda(x) \, dx = 0.
$$

This limit relation is based on a geometric estimate of $\mathbb{P}_\varrho(x \notin K(n))$, provided in Lemma 4.1, and on a disintegration result stated as Lemma 4.2.

For $y \in \partial K$, we write $u(y)$ for some exterior unit normal of $K$ at $y$. This exterior unit normal is uniquely determined for $H^{d-1}$-almost all boundary points of $K$. Applying the disintegration result again and using Lebesgue’s dominated convergence result, we finally get

$$
\lim_{n \to \infty} n^{\frac{d+1}{d+2}} \mathbb{P}_\varrho \int_{K \setminus K(n)} \lambda(x) \, dx = \int_{\partial K} \lambda(y)J_\varrho(y) H^{d-1}(dy),
$$

where

$$
J_\varrho(y) = \lim_{n \to \infty} \int_0^{n^{-\frac{1}{d+2}}} n^{\frac{d+1}{d+2}} \langle y, u(y) \rangle \mathbb{P}_\varrho \left( y_t \notin K(n) \right) \, dt
$$

for $H^{d-1}$-almost all $y \in \partial K$. For the subsequent analysis, it is sufficient to consider a small cap of $K$ at a normal boundary point $y \in \partial K$. The case $\kappa(y) = 0$ is treated in Lemma 4.4. The main case is $\kappa(y) > 0$. Here we reparametrize $y_t$ as $\tilde{y}_s$, in terms of the probability content of a small cap of $K$ whose bounding hyperplane passes through $y_t$. This implies that

$$
J_\varrho(y) = (d+1)^{-\frac{1}{d+2}} \alpha_{d-1}^{\frac{2}{d+2}} \varrho(y)^{\frac{d-1}{d+2}} \kappa(y) \frac{\lambda(y)}{H^{d-1}} \lim_{n \to \infty} \int_0^{n^{-1/2}} n^{\frac{d+1}{d+2}} \mathbb{P}_\varrho \left( \tilde{y}_s \not\in K(n) \right) s^{-\frac{d+1}{d+2}} \, ds,
$$

cf. (4.26). It is then a crucial step in the proof to show that the remaining integral asymptotically is independent of the particular convex body $K$, and thus the limit of the integral is the same as for a Euclidean ball (see Lemma 4.6). To achieve this, the integral is first approximated, up to a prescribed error of order $\varepsilon > 0$, by replacing $\mathbb{P}_\varrho(\tilde{y}_s \not\in K(n))$ by the probability of an event that depends only on a small cap of $K$ at $y$ and on a small number of random points. This important step is accomplished in Lemma 4.5. For the proofs of Lemmas 4.5 and 4.6 it is essential that the boundary of $K$ near the normal boundary point $y$ can be suitably approximated by the osculating paraboloid of $K$ at $y$. 

4. Proof of Theorem 3.1

To start with the actual proof, we fix some further notation. For \( y \in \partial K \) and \( t \in (0, 1) \), we define the cap \( C(y, t) := \{ x \in K : \langle u(y), x \rangle \geq \langle u(y), y_t \rangle \} \), whose bounding hyperplane passes through \( y_t \) and has normal \( u(y) \). For \( u \in \mathbb{R}^d \setminus \{ \phi \} \) and \( t \in \mathbb{R} \), we define the hyperplane \( H(u, t) := \{ x \in \mathbb{R}^d : \langle x, u \rangle = t \} \), and the closed halfspaces \( H^+(u, t) := \{ x \in \mathbb{R}^d : \langle x, u \rangle \geq t \} \) and \( H^-(u, t) := \{ x \in \mathbb{R}^d : \langle x, u \rangle \leq t \} \) bounded by \( H(u, t) \). We denote by \( h(K, \cdot) = h_K \) the support function of \( K \), that is \( h(K, u) := \max\{ \langle x, u \rangle : x \in K \} \) for \( u \in \mathbb{R}^d \).

For \( y \in \partial K \), the maximal number \( r \geq 0 \) such that \( y - r u(y) + r B^d \subset K \) is denoted by \( r(y) \). This number is called the interior reach of the boundary point \( y \). It is well known that \( r(y) > 0 \) for \( H^{d-1} \)-almost all \( y \in \partial K \). If \( r(y) > 0 \), there is a unique tangent plane of \( K \) at \( y \). In particular, \( r(y) \leq r(K) \) where \( r(K) \) is the inradius of \( K \). The convex hull of subsets \( X_1, \ldots, X_r \subset \mathbb{R}^d \) and points \( z_1, \ldots, z_s \in \mathbb{R}^d \) is denoted by \( \hat{K}(X_1, \ldots, X_r, z_1, \ldots, z_s) \).

For real functions \( f \) and \( g \) defined on the same space \( I \), we write \( f \ll g \) or \( f = O(g) \) if there exists a positive constant \( \gamma \), depending only on \( K \), \( g \) and \( \lambda \), such that \( |f| \leq \gamma \cdot g \) on \( I \). In general, we write \( \gamma_0, \gamma_1, \ldots \) to denote positive constants depending only on \( K \), \( g \) and \( \lambda \). The Landau symbol \( o(\cdot) \) is defined as usual. We further put \( \mathbb{R}^+ := [0, \infty) \).

Finally, we observe that there exists a constant \( \gamma_0 \in (0, 1) \) such that for \( y \in \partial K \), we have
\[
|\langle y, u(y) \rangle| \geq \gamma_0 \| y \|, \quad \text{and hence} \quad \| y \| u(y)^\perp \leq \sqrt{1 - \gamma_0^2} \cdot \| y \|, \tag{4.1}
\]
where \( y/u^\perp \) denotes the orthogonal projection of \( y \) onto the orthogonal complement of the vector \( u \in \mathbb{R}^d \setminus \{ \phi \} \). Subsequently, we always assume that \( n \in \mathbb{N} \).

**Lemma 4.1.** There exists a constant \( \delta > 0 \), depending on \( K \) and \( \varphi \), such that if \( y \in \partial K \) and \( t \in (0, \delta) \), then
\[
P_\varphi(y_t \not\in K_{(n)}) \ll \left(1 - \gamma_1 r(y)^{-\frac{1}{2}} \left(1 + t^{-\frac{1}{2}} \right)^\frac{2d}{d-1} \right)^n.
\]

**Remarks.**

(i) In addition, we may assume that on \( K \setminus \text{int}(K) \), both functions \( \varphi, \lambda \) are continuous, \( \varphi \) is positive and \( \gamma_1 r(K)^{\frac{d-1}{2}} \delta^{-\frac{d-1}{2}} < 1 \).

(ii) In the following, we will use the notion of a “coordinate corner”. Given an orthonormal basis in a linear \( r \)-dimensional subspace \( L \), the corresponding \((i-1)\)-dimensional coordinate planes cut \( L \) into \( 2^r \) convex cones, which we call coordinate corners (with respect to \( L \) and the given basis).

**Proof of Lemma 4.1.** If \( r(y) = 0 \), then there is nothing to prove. So let \( r(y) > 0 \), hence \( u(y) \) is uniquely determined. Choose an orthonormal basis in \( u(y)^\perp \), and let \( \Theta_1', \ldots, \Theta_{2^{d-1}}' \) be the corresponding coordinate corners in \( u(y)^\perp \). For \( i = 1, \ldots, 2^{d-1} \) and \( t \in [0, 1] \), we define
\[
\Theta_{i,t} := C(y, t) \cap \left(y_t + \left[ \Theta_i', \mathbb{R}^+ y \right]\right).
\]

If \( \delta > 0 \) is small enough to ensure that \( \varphi > 0 \) is positive and continuous in a neighborhood (relative to \( K \)) of \( \partial K \), then
\[
\int_{\Theta_{i,t}} \varphi(x) \, dx \geq \gamma_2 V(\Theta_{i,t}).
\]

If \( y_t \not\in K_{(n)} \) and \( o \in K_{(n)} \), then there exists a hyperplane \( H \) through \( y_t \), bounding the halfspaces \( H^- \) and \( H^+ \), for which \( K_{(n)} \subset H^- \). Moreover, there is some \( i \in \{1, \ldots, 2^{d-1}\} \) such that \( \Theta_{i,t} \subset H^+ \).
Therefore
\[ \mathbb{P}_e \left( y_t \notin K_{(n)}, o \in K_{(n)} \right) \ll \sum_{i=1}^{2^{d-1}} \left( 1 - \gamma_2 V(\Theta_{i,t}) \right)^n. \] (4.2)

Finally, we prove
\[ V(\Theta_{i,t}) \gg r(y)^{d-1} t^{\frac{d+1}{2}}, \] (4.3)
for \( i = 1, \ldots, 2^{d-1} \). According to (4.1), there exist positive constants \( \gamma_3, \gamma_4 \) with \( \gamma_3 \leq 1 \) such that if \( t \leq \gamma_3 r(y) \), then \( \{y_t + \Theta_{j} \cap K\} \) contains a \((d-1)\)-ball of radius at least
\[ \gamma_4 \sqrt{r(y)^2 - (r(y)-t)^2} \geq \gamma_4 \sqrt{r(y)t}, \]
and we are done. On the other hand, if \( t \geq \gamma_3 r(y) \), then
\[ V(\Theta_{i,t}) \gg r(y)^{d-1} t^{\frac{d+1}{2}}. \]

To deal with the case \( o \notin K_{(n)} \), we observe that there exists a positive constant \( \gamma_5 \in (0,1) \) such that the probability measure of each of the \( 2^n \) coordinate corners of \( \mathbb{R}^d \) is at least \( \gamma_5 \). If \( o \notin K_{(n)} \), then \( \{x_1, \ldots, x_n\} \) is disjoint from one of these coordinate corners, and hence
\[ \mathbb{P}_e(o \notin K_{(n)}) \leq 2^d(1 - \gamma_5)^n. \] (4.4)

Now the assertion follows from (4.2), (4.3) and (4.4).

Subsequently, the estimate of Lemma 4.1 will be used, for instance, to restrict the domain of integration on the right-hand side of (3.2) (cf. Lemma 4.3) and to justify an application of Lebesgue's dominated convergence theorem (see (4.9)). For these applications, we also need that if \( c > 0 \) is such that \( \omega := c \delta^{\frac{d+1}{2}} < 1 \), then
\[ \int_0^{\delta} \left( 1 - c t^{\frac{d+1}{2}} \right)^n dt = \frac{2}{d+1} c^{\frac{n}{2}} \int_0^\omega s^{\frac{n-1}{2}} (1 - s)^n ds \ll c^{\frac{n}{2}} \cdot n^{\frac{n}{2}}, \] (4.5)
where we use that \((1 - s)^n \leq e^{-ns} \) for \( s \in [0,1] \) and \( n \in \mathbb{N} \).

The next lemma will allow us to decompose integrals in a suitable way.

**Lemma 4.2.** If \( 0 < t_0 \leq t_1 < \delta \) and \( h : K \to [0,\infty) \) is a measurable function, then
\[ \int_{K_{t_0} \setminus K_{t_1}} h(x) \, dx = \int_{\partial K} \int_{t_0}^{t_1} (1 - t)^{d-1} (y,u(y)) h(y_t) \, dt \, \mathcal{H}^{d-1}(dy). \]

**Proof.** The map \( T : \partial K \times [t_0,t_1] \to K_{t_0} \setminus K_{t_1}, (y,t) \mapsto (1-t)y \), provides a bilipschitz parametrization of \( K_{t_0} \setminus K_{t_1} \) with \((1-t)y = y_t \in \partial K_t\). The Jacobian of \( T \), for \( \mathcal{H}^{d-1} \)-almost all \( y \in \partial K \) and \( t \in [t_0,t_1] \), is given by \( JT(y,t) = (1-t)^{d-1}(y,u(y)) \), where \( u(y) \) is the (\( \mathcal{H}^{d-1} \)-almost everywhere) unique exterior unit normal of \( \partial K \) at \( y \). The assertion now follows from Federer’s area/coarea theorem (see [13]).

In the following, we will use the important fact that, for \( \alpha > -1 \),
\[ \int_{\partial K} r(y)^\alpha \mathcal{H}^{d-1}(dy) < \infty, \] (4.6)
which is a result due to C. Schütt and E. Werner [39].

By decomposing \( \lambda \) in its positive and its negative part, we can henceforth assume that \( \lambda \) is a nonnegative, integrable function.
LEMMA 4.3. As $n$ tends to infinity, 
\[ \int_K \frac{P_n}{n^{d+1}} \mathbb{P}_\omega(x \not\in K(n)) \lambda(x) \, dx = o \left( n^{-\frac{d}{d+1}} \right) . \]

**Proof.** Let $\delta > 0$ be chosen as in Lemma 4.1 and the subsequent remark. First, we consider a point $x$ in $K_\delta$. Let $\omega$ be the minimal distance between the points of $\partial K$ and $K_\delta$, and let $z_1, \ldots, z_k$ be a maximal family of points in $K \setminus \text{int}(K_\delta)$ such that $\|z_i - z_j\| \geq \frac{\delta}{4}$ for $i \neq j$. We define $p_0 > 0$ by 
\[ p_0 := \min \left\{ \mathbb{P}_\omega(z_i + \frac{\omega}{4} B^d) : i = 1, \ldots, k \right\} . \]
Let $x \in K_\delta$. If $x \not\in K(n)$, then there exists a hyperplane $H(u, t)$ such that $x \in \text{int}(H^+(u, t))$ and $K(n) \subset H^-(u, t)$. Since $x \in K_\delta$, there exists a supporting hyperplane $H(u, h(K_\delta, u))$ of $K_\delta$ for which $K(n) \subset \text{int}(H^-(u, h(K_\delta, u)))$. If $z \in H(u, h(K_\delta, u)) \cap \partial K_\delta$, then 
\[ z + \frac{\omega}{2} u + \frac{\omega}{4} B^d \subset K \cap H^+(u, h(K_\delta, u)) . \]
By the maximality of the set $\{z_1, \ldots, z_k\}$, we have 
\[ \{z_1, \ldots, z_k\} \cap \left( z + \frac{\omega}{2} u + \frac{\omega}{4} B^d \right) \neq \emptyset . \]
Let $z_j$ lie in the intersection. Then $z_j + \frac{\omega}{2} B^d \subset H^+(u, h(K_\delta, u))$, and hence $x \not\in z_j + \frac{\omega}{2} B^d$ for $i = 1, \ldots, n$. This implies that, for $x \in K_\delta$, 
\[ \mathbb{P}_\omega(x \not\in K(n)) \leq k(1 - p_0)^n . \tag{4.7} \]
Put $\varepsilon := (2(d^2 - 1))^{-1}$ and let $n \geq \delta^{-(d+1)}$. For $y \in \partial K$ we show that 
\[ \int_{n^{d+1}}^{\delta} \mathbb{P}_\omega(y_t \not\in K(n)) \, dt \ll r(y)^{-\frac{d}{d+1}} n^{-\frac{d}{d+1} - \varepsilon} . \tag{4.8} \]
In fact, if $r(y) \leq n^{-\frac{1}{d+1}}$, then Lemma 4.1 and (4.5) yield 
\[ \int_{n^{d+1}}^{\delta} \mathbb{P}_\omega(y_t \not\in K(n)) \, dt \leq \int_{n^{d+1}}^{\delta} \left( 1 - \gamma_1 r(y)^{\frac{d+1}{d}} t^{\frac{d}{d+1}} \right)^n dt \ll r(y)^{-\frac{d}{d+1}} n^{-\frac{d}{d+1} - \varepsilon} , \]
where the assumption on $r(y)$ is used for the last estimate.

If $r(y) \geq n^{-\frac{1}{d+1} + \varepsilon}$ and $n \geq n_0$, where $n_0$ depends on $K$, $\theta$ and $\lambda$, then Lemma 4.1 implies for all $t \in (n^{d+1}, \delta)$ that 
\[ \mathbb{P}_\omega(y_t \not\in K(n)) \ll \left( 1 - \gamma_1 n^{-\frac{d^2}{d+1}} t^{-\frac{d}{d+1}} \right)^n = (1 - \gamma_1 n^{-3/4} n^{\frac{1}{d+1}} n^{-\frac{d}{d+1}}) \leq e^{-\gamma_1 n^{1/4}} \leq r(K)^{-\frac{d}{d+1}} n^{-\frac{d}{d+1} - \varepsilon} , \]
which again yields (4.8). In particular, writing $I$ to denote the integral in Lemma 4.3, we obtain from Lemma 4.2, (4.7), (4.8) and (4.6) that 
\[ I \ll \int_{K_\delta} \mathbb{P}_\omega(x \not\in K(n)) \lambda(x) \, dx + \int_{\partial K} \int_{n^{d+1}}^{\delta} \mathbb{P}_\omega(y_t \not\in K(n)) \, dt \, H^{d-1}(dy) \ll k(1 - p)^n + \int_{\partial K} r(y)^{-\frac{d}{d+1}} n^{-\frac{d}{d+1} - \varepsilon} \, H^{d-1}(dy) \ll n^{-\frac{d}{d+1} - \varepsilon} , \]
where we also used that $\lambda$ is integrable on $K$ and bounded on $K \setminus K_\delta$. This is the required estimate. \[ \square \]
It follows from (3.2), Lemma 4.3 and Lemma 4.2 that
\[
\lim_{n \to \infty} n^{\frac{-2}{d+1}} \mathbb{E}_\theta \int_{K \setminus K_{(n)}} \lambda(x) \, dx \\
= \lim_{n \to \infty} n^{\frac{-2}{d+1}} \int_{K} P_\theta (x \not\in K_{(n)}) \lambda(x) \, dx \\
= \lim_{n \to \infty} \int_{\partial K} \int_{0}^{n^{\frac{-2}{d+1}}} n^{\frac{-2}{d+1}} (1 - t)^{d-1} \langle y, u(y) \rangle \mathbb{P}_\theta (y_t \not\in K_{(n)}) \lambda(y_t) \, dt \, \mathcal{H}^{d-1}(dy).
\]

Lemma 4.1 and (4.5) imply that if \( y \in \partial K \) and \( r(y) > 0 \), then
\[
\int_{0}^{n^{\frac{-2}{d+1}}} n^{\frac{-2}{d+1}} \mathbb{P}_\theta (y_t \not\in K_{(n)}) \langle y, u(y) \rangle \lambda(y_t) \, dt \ll r(y)^{-\frac{d+1}{d+1}}.
\]

Therefore, by (4.6) and since \( \lambda \) is bounded and continuous in a neighborhood of \( \partial K \) we may apply Lebesgue’s dominated convergence theorem, and thus we conclude
\[
\lim_{n \to \infty} n^{\frac{-2}{d+1}} \mathbb{E}_\theta \int_{K \setminus K_{(n)}} \lambda(x) \, dx = \int_{\partial K} \lambda(y) J_\theta(y) \mathcal{H}^{d-1}(dy),
\]
where
\[
J_\theta(y) := \lim_{n \to \infty} \int_{0}^{n^{\frac{-2}{d+1}}} n^{\frac{-2}{d+1}} \langle y, u(y) \rangle \mathbb{P}_\theta (y_t \not\in K_{(n)}) \, dt,
\]
for \( \mathcal{H}^{d-1} \)-almost all \( y \in \partial K \).

**Lemma 4.4.** If \( y \in \partial K \) is a normal boundary point of \( K \) with \( \kappa(y) = 0 \), then \( J_\theta(y) = 0 \).

**Proof.** In view of the estimate (4.4), it is sufficient to prove that for any given \( \varepsilon > 0 \),
\[
\int_{0}^{n^{\frac{-2}{d+1}}} n^{\frac{-2}{d+1}} \mathbb{P}_\theta (y_t \not\in K_{(n)}, o \in K_{(n)}) \, dt \ll \varepsilon,
\]
(4.10)
if \( n \) is sufficiently large. We choose the coordinate axes in \( u(y)^\perp \) parallel to the principal curvature directions of \( K \) at \( y \), and denote by \( \Theta_1', \ldots, \Theta_{2^{d-1}}' \) the corresponding coordinate corners. For \( i = 1, \ldots, 2^{d-1} \) and \( t \in (0, n^{\frac{-2}{d+1}}) \), let
\[
\Theta_{i,t} := C(y, t) \cap (y_t + [\Theta_1', \mathbb{R}^+ y]),
\]
and hence, if \( n \) is large enough, then
\[
\int_{\Theta_{i,t}} \varphi(x) \, dx \gg V(\Theta_{i,t}),
\]
since \( \varphi \) is continuous and positive near \( \partial K \). If \( y_t \not\in K_{(n)} \) and \( o \in K_{(n)} \), then there exists a halfspace \( H^- \) which contains \( K_{(n)} \) and for which \( y_t \in \partial H^- \). Moreover, for some \( i \in \{1, \ldots, 2^{d-1}\} \) the interior of \( H^- \) is disjoint from \( \Theta_{i,t} \). Hence, as in the proof of Lemma 4.1,
\[
\mathbb{P}_\theta (y_t \not\in K_{(n)}, o \in K_{(n)}) \ll \sum_{i=1}^{2^{d-1}} (1 - \gamma_0 V(\Theta_{i,t}))^n.
\]
(4.11)
Since \( \partial K \) is twice differentiable in the generalized sense at \( y \), we have \( r(y) > 0 \). By assumption, \( \kappa(y) = 0 \), therefore one principal curvature at \( y \) is zero, and hence less than \( \varepsilon^{d+1} r(y)^d - 2 \). In particular, there exists \( \delta' \in (0, \delta) \), which by (4.1) depends only on \( y \) and \( \varepsilon \), such that if \( i \in \{1, \ldots, 2^{d-1}\} \) and
\( t \in (0, \delta^\prime) \), then

\[
\mathcal{H}^{d-1}((y_t + \Theta_s') \cap K) \gg \sqrt{t e^{-(d+1)r(y) - (d-2)}} \cdot \sqrt{tr(y)}^{d-2}.
\]

We deduce \( V(\Theta_s, t) \gg \varepsilon^{-\frac{d+1}{2}t^{d+1}} \). Therefore (4.10) follows from (4.5) and (4.11).

Next we consider the case of a normal boundary point \( y \in \partial K \) with \( \kappa(y) > 0 \). First, we prove that \( J_y(y) \) depends only on the random points near \( y \) (see Lemma 4.5). In a second step, we compare the simplified expression obtained for \( J_y(y) \) with the corresponding expression which is obtained if \( K \) is a ball.

We start by reparametrizing \( y_t \) in terms of the probability measure of the corresponding cap. For \( t \in (0, \frac{n}{n_0 + 1}) \), where \( n \geq n_0 \) is sufficiently large so that \( \varrho \) is positive and continuous on \( C(y, t) \), for all \( y \in \partial K \), we put

\[
\tilde{y}_s := y_t
\]

where for given \( s > 0 \) (sufficiently small) the corresponding \( t = t(s) \) is determined by the relation

\[
s = \int_{C(y, t)} \varrho(x) \, dx. \tag{4.12}
\]

It is easy to see that the right-hand side of (4.12) is a continuous and strictly increasing function \( s = s(t) \) of \( t \), if \( t > 0 \) is sufficiently small. This implies that for a given \( s > 0 \) (sufficiently small) there is a unique \( t(s) \) such that (4.12) is satisfied.

Moreover, observe that

\[
\frac{ds}{dt} = \langle \varrho(y), y \rangle \int_{H(y, t) \cap K} \varrho(x) \mathcal{H}^{d-1}(dx) \tag{4.13}
\]

for \( t \in (0, \frac{n}{n_0 + 1}) \). We further define

\[
\tilde{C}(y, s) := C(y, t) \quad \text{and} \quad \tilde{H}(y, s) := \{ x \in \mathbb{R}^d : \langle \varrho(y), x \rangle = \langle \varrho(y), \tilde{y}_s \rangle \},
\]

where \( t = t(s) \).

Let \( Q \) denote the second fundamental form of \( \partial K \) at \( y \) (cf. (2.1)), considered as a function on \( u(y)^\perp \). We define

\[
E := \{ z \in u(y)^\perp : Q(z) \leq 1 \}.
\]

and put \( u := u(y) \). Choosing a suitable orthonormal basis \( v_1, \ldots, v_{d-1} \) of \( u(y)^\perp \), we have

\[
Q(z) = \sum_{i=1}^{d-1} k_i(y) z_i^2,
\]

where \( k_i(y), i = 1, \ldots, d - 1 \), are the generalized principal curvatures of \( K \) at \( y \) and where \( z = z_1 v_1 + \cdots + z_{d-1} v_{d-1} \). Since \( y \) is a normal boundary point of \( K \), there is a nondecreasing function \( \mu : (0, \infty) \to \mathbb{R} \) with \( \lim_{r \to 0^+} \mu(t) = 1 \) such that

\[
\frac{\mu(r)^{-1}}{\sqrt{2r}} (K(u, r) + ru - y) \subset E \subset \frac{\mu(r)}{\sqrt{2r}} (K(u, r) + ru - y), \tag{4.14}
\]

where \( K(u, r) := K \cap H(u, h(K, u) - r) \). In the following, \( \mu_i : (0, \infty) \to \mathbb{R}, i = 1, 2, \ldots \) always denote nondecreasing functions with \( \lim_{r \to 0^+} \mu(t) = 1 \). Applying (4.14) and Fubini’s theorem, we get

\[
V(K \cap H^+(u, h(K, u) - r)) = \mu_1(r) \frac{(2r)^{d+1}}{d+1} \alpha_{d-1} \kappa(y)^{-\frac{1}{2}} g(y),
\]

which yields

\[
s(t) = \mu_2(t) \frac{(2t(y, u))^{d+1}}{d+1} \alpha_{d-1} \kappa(y)^{-\frac{1}{2}} g(y), \tag{4.15}
\]
since $g$ is continuous at $y$. Moreover, defining

$$\eta := (d + 1) \frac{2}{d + 1} \alpha_d^{-1} \frac{1}{d - 1} \varrho(y) \frac{2}{d + 1} \kappa(y) \frac{1}{d + 1},$$

we obtain

$$\lim_{s \to 0^+} s^{\frac{d}{d + 1}}[\tilde{H}(y, s) \cap K] - \tilde{y}_s] = \eta \cdot E \quad (4.16)$$

in the sense of the Hausdorff metric on compact convex sets (see Schneider [34] or Gruber [16]). Here we also use that

$$\lim_{s \to 0^+} s^{\frac{d}{d + 1}}(\tilde{y}_s - \langle \tilde{y}_s, u \rangle) = o. \quad (4.17)$$

Now it follows from (4.13) and (4.16) that (4.9) turns into

$$J_\varrho(y) = (d + 1)^{\frac{2}{d + 1}} \alpha_d^{-1} \varrho(y) \frac{2}{d + 1} \kappa(y) \frac{1}{d + 1} \lim_{n \to \infty} \int_0^{\frac{g(n, y)}{n^{d + 1}}} \varphi(K, y, \varrho, \varepsilon, s) s^{\frac{d}{d + 1}} ds,$$

where

$$\lim_{n \to \infty} n^{\frac{d}{d + 1}} g(n, y) = (d + 1)^{-1} \alpha_d^{-1} \varrho(y) (2 \langle u(y), y \rangle)^{\frac{d+1}{d+1}} \kappa(y)^{-\frac{1}{d+1}}.$$

The rest of the proof is devoted to identifying the asymptotic behavior of the integral. First, we adjust the domain of integration and the integrand in a suitable way. In a second step, the resulting expression is compared to the case where $K$ is the unit ball. We recall that $x_1, \ldots, x_n$ are random points in $K$, and we put $\Xi_n := \{x_1, \ldots, x_n\}$, and hence $K_n = [\Xi_n]$. Let $\#X$ denote the cardinality of a finite set $X \subset \mathbb{R}^d$.

**Lemma 4.5.** For $\varepsilon \in (0, 1)$, there exist $\alpha, \beta > 1$ and an integer $k > 1$, depending only on $\varepsilon$ and $d$, with the following property. If $y \in \partial K$ is a normal boundary point of $K$ with $\kappa(y) > 0$ and if $n > n_0$, where $n_0$ depends only on $\varepsilon$, $K$, $\varrho$, then

$$\int_0^{\frac{g(n, y)}{n^{d + 1}}} \varphi(K, y, \varrho, \varepsilon, s) s^{\frac{d}{d + 1}} ds = \int_0^{\frac{2}{n^{d + 1}}} \varphi(K, y, \varrho, \varepsilon, s) s^{\frac{d}{d + 1}} ds + O\left(\frac{\varepsilon}{n^{d+1}}\right),$$

where

$$\varphi(K, y, \varrho, \varepsilon, s) = \varphi\left(\tilde{y}_s \notin [\tilde{C}(y, \beta s) \cap \Xi_n]\right) \text{ and } \left(\#(\tilde{C}(y, \beta s) \cap \Xi_n) \leq k\right).$$

**Proof.** Let $Q$ be the second fundamental form of $\partial K$ at the normal boundary point $y$, and let $v_1, \ldots, v_{d-1}$ be an orthonormal basis of $u(y)^\perp$ with respect to $Q$, as described above. Let $\Theta'_1, \ldots, \Theta'_{d-1}$ be the corresponding coordinate corners, and, for $i = 1, \ldots, 2^{d-1}$ and for $s \in (0, n^{-1/2})$, put

$$\tilde{\Theta}_{i,s} := \tilde{C}(y, s) \cap (\tilde{y}_s + [\Theta'_i, \mathbb{R}^+ y]).$$

Let $A_s$, $s > 0$, be the affine map of $\mathbb{R}^d$ with $A_s(y) = y$ for which the associated linear map $\tilde{A}_s$ is determined by $\tilde{A}_s(v) = s^{\frac{d}{d + 1}} v$, for $v \in u^\perp$, and $\tilde{A}_s(u) = s^{\frac{d}{d + 1}} u$. Then $\det(\tilde{A}_s) = s$ and $A_{s^{-1}}(\tilde{C}(y, s))$ converges in the Hausdorff metric as $s \to 0^+$ to the cap $\tilde{C}(y)$ of the osculating paraboloid of $K$ at $y$ having volume $\varrho(y)^{-\frac{1}{d+1}}$. Here we use that $g$ is continuous at $y$, $\varrho(y) > 0$ and relation (4.12). Let $\lambda > 0$ be such that $\tilde{y} := y - \lambda u \in \partial \tilde{C}(y)$. Then $A_{s^{-1}}(\tilde{\Theta}_{i,s})$ converges in the Hausdorff metric as $s \to 0^+$ to $\tilde{C}(y) \cap (\tilde{y} + [\Theta'_i, \mathbb{R}^+ u])$, since (4.17) is satisfied. Using again that $g$ is continuous and positive at $y$, we
deduce that
\[
\lim_{s \to 0^+} s^{-1} \int_{\Theta_{1,s}} \varrho(x) \, dx = \lim_{s \to 0^+} s^{-1} V(\tilde{\Theta}_{1,s}) \varrho(y) \\
= \lim_{s \to 0^+} V(A_{s-1}(\tilde{\Theta}_{1,s})) \varrho(y) \\
= V(\tilde{C}(y) \cap (\tilde{y} + [\Theta_1^\prime, \mathbb{R}^d + u])) \varrho(y) \\
= 2^{-(d-1)} V(\tilde{C}(y)) \varrho(y) \\
= 2^{-(d-1)} \lim_{s \to 0^+} s^{-1} V(\tilde{C}(y, s)) \varrho(y) \\
= 2^{-(d-1)} \lim_{s \to 0^+} s^{-1} \int_{\tilde{C}(y, s)} \varrho(x) \, dx \\
= 2^{-(d-1)},
\]
that is
\[
\lim_{s \to 0^+} s^{-1} \int_{\Theta_{1,s}} \varrho(x) \, dx = 2^{-(d-1)}. \tag{4.18}
\]
Let \( \alpha > 1 \) be chosen such that
\[
2^{d-1} e^{-2(d+1)/\alpha} \leq \epsilon.
\]
Then we first choose \( \beta \geq (16(d-1))^{d+1} \) such that
\[
2^{d-1} e^{-2(d+1)} \beta^{d+1} \frac{e^{\alpha^{d+1}}}{\alpha^{d+1}} \leq \frac{\epsilon}{\alpha^{d+1}},
\]
and then we fix an integer \( k > 1 \) such that
\[
\frac{(\alpha \beta)^k}{k!} \leq \frac{\epsilon}{\alpha^{d+1}}.
\]
Lemma 4.5 follows from the following three statements, which we will prove assuming that \( n \) is sufficiently large.

(i) \[
\int_0^{(n,y)} \mathbb{P}_y \left( \tilde{y}_s \not\in K_n \right) s^{-\frac{d+1}{d+1}} \, ds = \int_0^{\frac{n}{e^{(d+1)/2}}} \mathbb{P}_y \left( \tilde{y}_s \not\in K_n \right) s^{-\frac{d+1}{d+1}} \, ds + O \left( \frac{\epsilon}{n^{d+1}} \right).
\]

(ii) If \( \frac{e^{(d+1)/2}}{n} < s < \frac{n}{2} \), then
\[
\mathbb{P}_y \left( \# \left( \tilde{C}(y, \beta s) \cap \Xi_n \right) \geq k \right) = O \left( \frac{\epsilon}{\alpha^{d+1}} \right).
\]

(iii) If \( \frac{e^{(d+1)/2}}{n} < s < \frac{n}{2} \), then
\[
\mathbb{P}_y \left( \tilde{y}_s \not\in K_n \right) = \mathbb{P}_y \left( \tilde{y}_s \not\in \left[ \tilde{C}(y, \beta s) \cap \Xi_n \right] \right) + O \left( \frac{\epsilon}{\alpha^{d+1}} \right).
\]
To prove (i), we first observe that
\[
\int_0^{(d+1)/2} \mathbb{P}_y \left( \tilde{y}_s \not\in K_n \right) s^{-\frac{d+1}{d+1}} \, ds \leq \int_0^{(d+1)/2} s^{-\frac{d+1}{d+1}} \, ds \ll \frac{\epsilon}{n^{d+1}}.
\]
If $\frac{2}{n} < s < q(n, y)$, $o \in K(n)$, $\tilde{y}_s \not\in K(n)$ and if $n$ is sufficiently large, then there is some $i \in \{1, \ldots, 2^{d-1}\}$ such that $\hat{\Theta}_{i,s} \cap K(n) = \emptyset$, and hence (4.4) and (4.18) yield

$$\mathbb{P}_e \left( \tilde{y}_s \not\in K(n) \right) \ll 2^{d-1}(1 - 2^{-d}s)^n \leq 2^{d-1}e^{-2^{-d}ns}. \quad (4.19)$$

Therefore, by the definition of $\alpha$, we get

$$\int_{\alpha}^{\beta(n,y)} \mathbb{P}_e \left( \tilde{y}_s \not\in K(n) \right) s^{-4n+1} ds \ll 2^{d-1} \int_{\alpha}^{\infty} e^{-2^{-d}ns} s^{\frac{2}{d+1} - 1} ds \leq \varepsilon n^{-\frac{2}{d+1}},$$

which verifies (i).

Next (ii) simply follows from (4.12) as if $s < \frac{2}{n}$, then

$$\mathbb{P}_e \left( \# \left( \hat{C}(y, \beta s) \cap \Xi_n \right) \geq k \right) = \left( \begin{array}{c} n \\ k \end{array} \right) (\beta s)^k \leq \left( \begin{array}{c} n \\ k \end{array} \right) \left( \frac{\alpha \beta}{n} \right)^k < \left( \frac{\alpha \beta}{k!} \right)^k \leq \frac{\varepsilon}{\alpha^{\frac{2}{d+1}}}.$$

Now we prove (iii). To this end, for $s$ in the given range, our plan is to construct sets $\tilde{\Omega}_1, \ldots, \tilde{\Omega}_{2^{d-1}-1}, K$ such that

$$\int_{\tilde{\Omega}_i} \varrho(x) d x \geq d^{-1}2^{-(d+3)}\beta^{\frac{1}{d+1}} s, \quad \text{for } i = 1, \ldots, 2^{d-1}, \quad (4.20)$$

and if $\tilde{y}_s \in K(n)$ but $\tilde{y}_s \not\in \left( \hat{C}(y, \beta s) \cap \Xi_n \right)$, then $\Xi_n \cap \tilde{\Omega}_1 = \emptyset$ for some $i \in \{1, \ldots, 2^{d-1}\}$.

For $i = 1, \ldots, 2^{d-1}$, let $w_i \in \Theta'_i$ be the vector whose coordinates (up to sign) in the basis $v_1, \ldots, v_{d-1}$ are

$$w_i := \left( \sqrt{\beta s} \right)^{\frac{1}{d+1}} \frac{\eta}{2\sqrt{d-1}} \left( \pm \frac{1}{\sqrt{k_1(y)}}, \ldots, \pm \frac{1}{\sqrt{k_{d-1}(y)}} \right).$$

Further, for $i = 1, \ldots, 2^{d-1}$ we define

$$\tilde{\Omega}_i = \left[ \tilde{y}_s \sqrt{\beta s} + w_i, K \cap (\tilde{y}_s + \Theta'_i) \right].$$

Then, if $s > 0$ is small enough, $\tilde{y}_s \sqrt{\beta s} + w_i \in K$, and hence $\tilde{\Omega}_i \subset K$. Here we use that

$$w_i \in \left( \sqrt{\beta s} \right)^{\frac{1}{d+1}} \frac{1}{2} \eta E$$

and therefore by (4.16)

$$\tilde{y}_s \sqrt{\beta s} + w_i \in \hat{H}(y, \sqrt{\beta s}) \cap K \subset K.$$

Using that $\tilde{y}_s = (1 - t)y$, where $s$ and $t$ are related by (4.15), and if $s, t > 0$ are sufficiently small, we obtain

$$\langle u(y), \tilde{y}_s - \tilde{y}_s \sqrt{\beta s} \rangle > \frac{\beta^{\frac{1}{d+1}}}{2} - \frac{1}{2} \langle u(y), y - \tilde{y}_s \rangle > \frac{\beta^{\frac{1}{d+1}}}{4} \langle u(y), y - \tilde{y}_s \rangle, \quad (4.21)$$

since $\beta \geq 2^{d+1}$. Moreover, we have

$$\langle u(y), y - \tilde{y}_s \rangle : \mathcal{H}^{d-1}(K \cap (\tilde{y}_s + \Theta'_i)) \geq V(\tilde{\Theta}_{i,s}). \quad (4.22)$$
Combining (4.21), (4.22), (4.18) and the continuity of \( \varrho \) at \( y \) with \( \varrho(y) > 0 \), we deduce (4.20), that is

\[
\int_{\tilde{\Omega}_{i,s}} \varrho(x) \, dx \geq \frac{1}{\sqrt{2d}} \frac{1}{\varrho(y)} \langle u(y), \bar{y}_s - \bar{y}_{i,s} \rangle \mathcal{H}^{d-1}(K \cap (\bar{y}_s + \Theta'_i))
\]

\[
\geq \frac{\beta \pi^{\frac{d}{2}}}{4} \sqrt{2d} V(\tilde{\Omega}_{i,s})
\]

\[
\geq \frac{\beta \pi^{\frac{d}{2}}}{4} \int_{\tilde{\Omega}_{i,s}} \varrho(x) \, dx
\]

\[
\geq \frac{\beta \pi^{\frac{d}{2}}}{8d^{\frac{2}{d}}}
\]

It is still left to prove that if \( \bar{y}_s \in K(n) \) but \( \bar{y}_s \notin \tilde{C}(y, \beta s) \), then \( \Xi_n \cap \tilde{\Omega}_{i,s} = 0 \) for some \( i \in \{1, \ldots, 2^{d-1}\} \). So we assume that \( \bar{y}_s \in K(n) \) but \( \bar{y}_s \notin \tilde{C}(y, \beta s) \). Then there exist \( a \in [\tilde{C}(y, \beta s) \cap \Xi_n] \) and \( b \in K(n) \setminus \tilde{C}(y, \beta s) \) such that \( \bar{y}_s \in [a, b] \), and hence there exists a hyperplane \( H \) containing \( \bar{y}_s \) bounding the halfspaces \( H^+ \) and \( H^- \) such that \( \tilde{C}(y, \beta s) \cap \Xi_n \subset \text{int}(H^+) \) and \( b \in \text{int}(H^-) \).

Next we show that there exists \( q \in [\bar{y}_s, b] \) such that

\[
q \in H^- \cap \left( \bar{y}_s \sqrt{\pi s} + \frac{\eta}{2\sqrt{d}-1} \sqrt{\beta s} \pi^{\frac{d}{2}} E \right).
\]

In fact, define \( q := [\bar{y}_s, b] \cap \tilde{H}(y, \sqrt{\beta s}) \) and \( q' := [\bar{y}_s, b] \cap \tilde{H}(y, \beta s) \). Since \( a \in H^+ \) and \( \bar{y}_s \in H \), it follows that \( q \in H^- \). From (4.16) we get

\[
\tilde{H}(y, \beta s) \cap K \subset \bar{y}_s + 2\beta \pi^{\frac{d}{2}} \sqrt{s} \pi^{\frac{d}{2}} \eta E.
\]

Applying (4.15), we deduce

\[
\langle u(y), \bar{y}_s - \bar{y}_{i,s} \rangle < \frac{\beta \pi^{\frac{d}{2}}}{\beta \pi^{\frac{d}{2}} - 1} \cdot \frac{\beta \pi^{\frac{d}{2}} - 1}{\beta \pi^{\frac{d}{2}} - \beta \pi^{\frac{d-1}{2}}} \langle u(y), \bar{y}_{\sqrt{\pi s}} - \bar{y}_{i,s} \rangle
\]

\[
< \frac{\beta \pi^{\frac{d}{2}}}{\beta \pi^{\frac{d}{2}} - 1} \langle u(y), \bar{y}_{\sqrt{\pi s}} - \bar{y}_{i,s} \rangle.
\]

Furthermore, elementary geometry yields

\[
\frac{\| q - \bar{y}_{\sqrt{\pi s}} \|}{\| q' - \bar{y}_{i,s} \|} = \frac{\langle u, \bar{y}_s - \bar{y}_{\sqrt{\pi s}} \rangle}{\langle u, \bar{y}_s - \bar{y}_{i,s} \rangle}.
\]

Then (4.24) and (4.25) imply that

\[
q \in \bar{y}_{\sqrt{\pi s}} + \frac{\langle u, \bar{y}_s - \bar{y}_{\sqrt{\pi s}} \rangle}{\langle u, \bar{y}_s - \bar{y}_{i,s} \rangle} \cdot 2(\beta s) \pi^{\frac{d}{2}} \eta E
\]

\[
\subset \bar{y}_{\sqrt{\pi s}} + \left( 1 - \frac{\langle u, \bar{y}_s - \bar{y}_{i,s} \rangle}{\langle u, \bar{y}_s - \bar{y}_{i,s} \rangle} \right) \cdot 2\beta \pi^{\frac{d}{2}} \sqrt{s} \pi^{\frac{d}{2}} \eta E
\]

\[
\subset \bar{y}_{\sqrt{\pi s}} + \frac{1}{2\sqrt{d}-1} (\sqrt{\beta s} \pi^{\frac{d}{2}}) \eta E,
\]

where \( \beta \geq (16(d-1))^{d+1} \) is used for the last inclusion. Now there exists some \( i \in \{1, \ldots, 2^{d-1}\} \) such that \( \bar{y}_s + \Theta'_i \subset H^- \), and hence \( q + \Theta'_i \subset H^- \). By (4.23) this finally yields

\[
\bar{y}_{\sqrt{\pi s}} + w_i \subset q + \Theta'_i \subset H^-.
\]

Therefore we obtain \( \tilde{\Omega}_{i,s} \cap \Xi_n = 0 \).
Finally, (iii) follows as if \( \frac{d+1}{n} < s < \frac{d}{n} \), then

\[
0 \leq P_\rho \left( \tilde{y}_s \not\in \left[ \tilde{C}(y, \beta s) \cap \Xi_n \right] \right) - P_\rho \left( \tilde{y}_s \not\in K(n) \right) \leq \sum_{i=1}^{2^{d-1}} (1 - \int_{\Omega_{t,s}} g(x) dx)^n \\
\leq \sum_{i=1}^{2^{d-1}} e^{-n \int_{\Omega_{t,s}} \rho(x) dx} \\
\leq 2^{d-1} e^{-d-1-\frac{d+1}{2}} \beta^{\frac{d}{d-1}} \leq \epsilon^\alpha \frac{d+1}{n},
\]

by the choice of \( \beta \).

\[ \square \]

**Remark.** As a consequence of the proof of Lemma 4.5, it follows that

\[
J_\rho(y) = (d + 1)^{\frac{d-1}{n+1}} \alpha_{d-1} \frac{\rho(y)^{\frac{d-1}{n+1}}}{\pi^{\frac{d-1}{n+1}}} \kappa(y)^{\frac{1}{n+1}} \lim_{n \to \infty} \int_0^{n^{\frac{d+1}{n+1}}} \rho \left( \tilde{y}_s \not\in K(n) \right) s^{\frac{d}{n+1}} ds. \tag{4.26}
\]

In fact, since \( g(n, y) \ll n^{-1/2} \), it is sufficient to show that

\[
\lim_{n \to \infty} n^{\frac{d}{n+1}} \int_{c_1n^{-1/2}}^{c_2n^{-1/2}} P_\rho \left( \tilde{y}_s \not\in K(n) \right) s^{\frac{d}{n+1}} ds = 0
\]

for any two constants \( 0 < c_1 \leq c_2 < \infty \). Since the estimate (4.19) can be applied, we get

\[
n^{\frac{d}{n+1}} \int_{c_1n^{-1/2}}^{c_2n^{-1/2}} P_\rho \left( \tilde{y}_s \not\in K(n) \right) s^{\frac{d}{n+1}} ds \ll n^{\frac{d}{n+1}} \int_{c_1n^{-1/2}}^{c_2n^{-1/2}} e^{-2^{-d} n^s \pi^{\frac{d}{d+1}} - 1} ds \\
\ll \int_{2^{-d}c_1n^{1/2}}^{2^{-d}c_2n^{1/2}} e^{-t} t^{\frac{d}{d+1}} dt,
\]

from which the conclusion follows.

Subsequently, we write 1 to denote the constant one function on \( \mathbb{R}^d \). For the unit ball \( B_1^d \), we recall that \( B_0^d(n) \) denotes the convex hull of \( n \) random points distributed uniformly and independently in \( B_1^d \).

We fix a point \( w \in \partial B_1^d \), and for \( s \in (0, \frac{1}{2}) \), define \( \tilde{w}_s := t \cdot w \), where \( t \in (0, 1) \) is chosen such that

\[
s = \alpha_{d-1} \cdot V(\{ x \in B_1^d : \langle x, w \rangle \geq \langle \tilde{w}_s, w \rangle \}).
\]

A classical result due to J.A. Wieacker [46] is that

\[
\lim_{n \to \infty} n^{\frac{d}{n+1}} \int_0^{n^{\frac{d}{n+1}}} P_{1,B^d} \left( \tilde{w}_s \not\in S_{B_0^d(n)} \right) s^{\frac{d-1}{n+1}} ds = \omega_d \alpha_d \frac{\omega_{d-1}^2}{\pi^{\frac{d}{d+1}}},
\]

where the constant \( \omega_d \) is given in (2.2). It follows from (4.9), (4.26) and the preceding remark that

\[
\lim_{n \to \infty} n^{\frac{d}{n+1}} \int_0^{n^{\frac{d}{n+1}}} P_{1,B^d} \left( \tilde{w}_s \not\in S_{B_0^d(n)} \right) s^{\frac{d-1}{n+1}} ds = \omega_d \alpha_d \frac{\omega_{d-1}^2}{\pi^{\frac{d}{d+1}}}. \tag{4.27}
\]

We are now going to show that the same limit is obtained if \( B_1^d \) is replaced by the convex body \( K \) and if a normal boundary point \( y \) of \( K \) with positive Gauss curvature is considered instead of \( w \in \partial B_1^d \).

**Lemma 4.6.** If \( y \in \partial K \) is a normal boundary point of \( K \) satisfying \( \kappa(y) > 0 \), then

\[
\lim_{n \to \infty} \int_0^{n^{\frac{d}{n+1}}} P_\rho \left( \tilde{y}_s \not\in K(n) \right) s^{\frac{d-1}{n+1}} ds = \omega_d \alpha_d \frac{\omega_{d-1}^2}{\pi^{\frac{d}{d-1}}}. \tag{4.27}
\]
The same formula is obtained for $K$. We conclude from (4.28) and (4.29) that
\[ \frac{\tilde{s}(\beta, s)}{V(C(\beta))} = \frac{\tilde{\Psi}(y, \beta, s)}{V(C(y, \beta))}. \]

Then (4.12) implies that
\[ \tilde{s}(\beta, s) = \frac{\beta s}{\mu(\beta, s)\rho(y)V(C(1))} = \frac{s}{\mu(\beta, s)\rho(y)V(C(1))}, \]
where $\mu(\beta, s) \to 1$ as $s \to 0^+$. Let $A_s$, $s > 0$, denote the affinity of $\mathbb{R}^d$ with $A_s(y) = y$ for which the associated linear map $A_s$ satisfies $A_s(v) = s^{\frac{d-1}{2}} v$ for $v \in p^+$ and $A_s(u) = s^{\frac{d-1}{2}} u$. Then the image under $A_s$ of a cap of $K$ at $y$ converges in the Hausdorff metric as $s \to 0^+$ to a cap of the osculating paraboloid of $K$ at $y$. For a more explicit statement, let $A$ be a volume preserving affinity of $\mathbb{R}^d$ such that $A(y) = o$ and $A(y - u) = p$, which maps the osculating paraboloid of $K$ at $y$ to $\Psi$. Then $\Phi_{s,\beta} := A \circ A_s(\beta, s) - 1$ is an affinity satisfying
\[ \Phi_{s,\beta}(y) = o, \quad \det(\Phi_{s,\beta}) = \tilde{s}(\beta, s)^{-1} = \frac{V(C(\beta))}{V(C(y, \beta))}, \]
and, consequently, $\Phi_{s,\beta}(\tilde{\Psi}(y, \beta, s)) \to C(\beta)$ in the Hausdorff metric as $s \to 0^+$. Moreover, we have
\[ \lim_{s \to 0^+} \Phi_{s,\beta}(\tilde{y}_s) = \lim_{s \to 0^+} \Phi_{s,1}(\tilde{y}_s) = p, \]
since $\mu(\beta, s) \to 1$ and $\mu(1, s) \to 1$ as $s \to 0^+$, $\tilde{y}_s \in \partial \tilde{C}(y, s)$ and $\Phi_{s,1}(\tilde{y}_s) \in \partial C(1)$, and by (4.17). Since $\rho$ is continuous at $y$, the properties of $\Phi_{s,\beta}$ imply that, for $i = 0, \ldots, k$,
\[ \lim_{s \to 0^+} \mathbb{P}_e(\tilde{y}_s \notin \tilde{C}(y, \beta, s(i)) = \mathbb{P}_1, C(\beta)(p \notin C(\beta(i)). \]

We conclude from (4.28) and (4.29) that
\[ \int_0^{n^{-1/2}} \mathbb{P}_e(\tilde{y}_s \notin K(n)) s^{-\frac{d-1}{2}} ds = O \left( \frac{e}{n^{\frac{d-1}{2}}} \right) + \sum_{i=0}^k \binom{n}{i} \left( \int_0^{n^{-1/2}} \left( \beta s \right)^{i(1 - \beta)} s^{-n-i} \right. \]}
\[ \times \mathbb{P}_1, C(\beta)(p \notin C(\beta(i))) s^{-\frac{d-1}{2}} ds, \]

The same formula is obtained for
\[ \int_0^{n^{-1/2}} \mathbb{P}_{1, B^d}(\tilde{w}_s \notin B^d(n)) s^{-\frac{d-1}{2}} ds, \]
since $C(\beta)$ is independent of $K$. Since $e \in (0, 1)$ was arbitrary, we conclude
\[ \lim_{n \to \infty} \int_0^{n^{-1/2}} n^{\frac{d-1}{2}} \mathbb{P}_e(\tilde{y}_s \notin K(n)) s^{-\frac{d-1}{2}} ds = \lim_{n \to \infty} \int_0^{n^{-1/2}} n^{\frac{d-1}{2}} \mathbb{P}_{1, B^d}(\tilde{w}_s \notin B^d(n)) s^{-\frac{d-1}{2}} ds. \]

Now (4.27) yields Lemma 6.4. □
5. Polarity and the proof of Theorem 2.1

In this section, we deduce Theorem 2.1 and Theorem 2.2 from Theorem 3.1 and Corollary 3.2, respectively. In order to obtain more general results, for not necessarily homogeneous or isotropic hyperplane distributions, we start with a description of the basic setting.

Let $K \subset \mathbb{R}^d$ be a convex body with origin $o \in \text{int}(K)$, as usual let $K^* := \{ z \in \mathbb{R}^d : \langle x, z \rangle \leq 1 \text{ for all } x \in K \}$ denote the polar body of $K$, and put $K_1 := K + B^d$. Let $H_K$ denote the set of all hyperplanes $H$ in $\mathbb{R}^d$ for which $H \cap \text{int}(K) = \emptyset$ and $H \cap K_1 \neq \emptyset$. The model of a random polytope (random polyhedral set) described in the introduction is based on random hyperplanes with distribution $\mu_K := 2^{-1}(\mu, H_K)$. More generally, we now consider random hyperplanes with distribution

$$\mu_q := \int_{S^{d-1}} \int_0^\infty 1\{H(u, t) \in \cdot\} q(t, u) \, dt \, \sigma(du), \quad (5.1)$$

where $\sigma$ is the rotation invariant probability measure on the unit sphere $S^{d-1}$. The model of a random polytope (random polyhedral set) described in the introduction is based on random hyperplanes with distribution $\mu_K := 2^{-1}(\mu, H_K)$. More generally, we now consider random hyperplanes with distribution

$$\mu_q := \int_{S^{d-1}} \int_0^\infty 1\{H(u, t) \in \cdot\} q(t, u) \, dt \, \sigma(du), \quad (5.1)$$

where $q : [0, \infty) \times S^{d-1} \to [0, \infty)$ is a measurable function which is

- (q1) concentrated on $D_K := \{(t, u) \in [0, \infty) \times S^{d-1} : h(K, u) \leq t \leq h(K_1, u)\}$,
- (q2) positive and continuous in a neighborhood of $\{(t, u) \in [0, \infty) \times S^{d-1} : t = h(K, u)\}$ relative to $D_K$,
- (q3) and satisfies $\mu_q(H_K) = 1$.

The intersection of $n$ halfspaces $H_i^-$ containing the origin $o$ and bounded by $n$ independent random hyperplanes $H_i$ with distribution $\mu_q$ is denoted by $K^{(n)} := \cap_{i=1}^n H_i^-$. Probabilities and expectations with respect to $\mu_q$ are denoted by $\mathbb{P}_{\mu_q}$ and $\mathbb{E}_{\mu_q}$, respectively. The special example $q \equiv 1_{D_K}$ ($q$ is the characteristic function of $D_K$) covers the situation discussed in the introduction.

In the following, beside the support function, we will also need the radial function $\rho(L, \cdot)$ of a convex body $L$ with origin $o \in \text{int}(L)$. Let $F$ be a nonnegative measurable functional on convex polyhedral sets in $\mathbb{R}^d$. Using (5.1) and Fubini’s theorem, we get

$$\mathbb{E}_{\mu_q}(F(K^{(n)})) \equiv \int_{A(d,d-1)^n} F\left(\prod_{i=1}^n H_i^-\right) \mu_q^{\otimes n}(d(H_1, \ldots, H_n))$$

$$= \int_{(S^{d-1})^n} \int_{h(K_1, u_1)}^{h(K, u_1)} \cdots \int_{h(K_1, u_n)}^{h(K, u_n)} F\left(\prod_{i=1}^n H_i^-\right) \prod_{i=1}^n q(t_i, u_i)$$

$$\times dt_1 \cdots dt_n \, \sigma^{\otimes n}(du_1, \ldots, u_n).$$

For $t_1, \ldots, t_n > 0$, we have

$$\prod_{i=1}^n H_i^-\left(u_i, t_i\right) = \left[t_1^{-1} u_1, \ldots, t_n^{-1} u_n\right]^*.$$
Using the substitution \( s_i = 1/t_i \), \( \rho(L^*, u_i) = h(L, u_i)^{-1} \) for \( L \in K^n \) with \( o \in \text{int}(L) \), and polar coordinates, we obtain

\[
\mathbb{E}_{\mu}(F(K^{(n)})) = \frac{1}{\omega_d} \int_{(K^*)^n} F([x_1, \ldots, x_n]^*) \prod_{i=1}^{n} \tilde{q}(x_i)||x_i||^{-(d+1)} d(x_1, \ldots, x_n)
\]

with \( K^*_i := (K_1)^* \) and

\[
\tilde{q}(x) := q \left( \frac{1}{||x||}, \frac{x}{||x||} \right), \quad x \in K^* \setminus \{o\}.
\]

The case \( n = 1 \) and \( F \equiv 1 \) yields

\[
\frac{1}{\omega_d} \int_{K^* \setminus K^*_1} \tilde{q}(x)||x||^{-(d+1)} dx = 1,
\]

hence

\[
\rho(x) := \begin{cases} 
\omega_d^{-1} \tilde{q}(x)||x||^{-(d+1)}, & x \in K^* \setminus K^*_1, \\
0, & x \in K^*_1,
\end{cases}
\]

is a probability density with respect to \( \mathcal{H}^d, K^* \) which is positive and continuous in a neighborhood of \( \partial K^* \) relative to \( K^* \). Thus we conclude that

\[
\mathbb{E}_{\mu}(F(K^{(n)})) = \int_{(K^*)^n} F([x_1, \ldots, x_n]^*) \prod_{i=1}^{n} \rho(x_i) d(x_1, \ldots, x_n) = \mathbb{E}_{\nu_{K^*}} \left( F((K^{(n)}_0)^*) \right),
\]

where \( K^{(n)}_0 := (K^{(n)}_0) \).

**Proposition 5.1.** Let \( K \subset \mathbb{R}^d \) be a convex body with \( o \in \text{int}(K) \), and let \( q \) and \( \rho \) be defined as above. Then the random polyhedral sets \( K^{(n)} \) and \( (K^{(n)}_0)^* \) are equal in distribution.

For a first application, let

\[
F(P) := 1\{P \subset K_1\} (W(P) - W(K)),
\]

for a polyhedral set \( P \subset \mathbb{R}^d \), with the convention \( 0 \cdot \infty := 0 \). For \( x_1, \ldots, x_n \in K^* \setminus K^*_1 \), we have \( K \subset [x_1, \ldots, x_n]^{**} \) and, arguing as before,

\[
F([x_1, \ldots, x_n]^*) = 1\{[x_1, \ldots, x_n]^* \subset K_1\} (W([x_1, \ldots, x_n]^*) - W(K)) = 2 \cdot 1\{[x_1, \ldots, x_n]^* \subset K_1\} \int_{K^* \setminus [x_1, \ldots, x_n]} \lambda(x) dx,
\]

where

\[
\lambda(x) := \begin{cases} 
\omega_d^{-1}||x||^{-(d+1)}, & x \in K^* \setminus K^*_1, \\
0, & x \in K^*_1.
\end{cases}
\]

Note that if \( [x_1, \ldots, x_n]^* \subset K_1 \), then the set \( [x_1, \ldots, x_n]^* \) is bounded, hence \( o \in \text{int}([x_1, \ldots, x_n]) \), and therefore \( K^*_1 \subset [x_1, \ldots, x_n]^{**} = [x_1, \ldots, x_n]^* \).

As in [9], it can be shown that \( P_{\mu}(K^{(n)} \not\subset K_1) \ll \alpha^n \), for some \( \alpha \in (0, 1) \) depending on \( K \) and \( q \). By Proposition 5.1, we also get

\[
\mathbb{P}_{\nu_{K^*}} \left( (K^{(n)}_0)^* \not\subset K_1 \right) = \mathbb{P}_{\nu_{K}} \left( K^{(n)} \not\subset K_1 \right) \ll \alpha^n.
\]
Hence
\[
\mathbb{E}_{\mu_q} \left( W(K^{(n)} \cap K_1) - W(K) \right) \\
= \mathbb{E}_{\mu_q} \left( 1\{K^{(n)} \subset K_1\} \left( W(K^{(n)}) - W(K) \right) \right) + O(\alpha^n) \\
= 2 \cdot \mathbb{E}_{\mu_q, K^*} \left( 1\{(K^{(n)})^* \subset K_1\} \int_{K^{(n)} \setminus K_1^*} \lambda(x) \, dx \right) + O(\alpha^n) \\
= 2 \cdot \mathbb{E}_{\mu_q, K^*} \left( \int_{K^{(n)} \setminus K_1^*} \lambda(x) \, dx \right) + O(\alpha^n),
\]
where we used that \( \lambda \) is integrable. Therefore, Theorem 3.1 implies
\[
\lim_{n \to \infty} n^{\frac{2}{d+1}} \mathbb{E}_{\mu_q} \left( W(K^{(n)} \cap K_1) - W(K) \right) \\
= 2 \cdot \lim_{n \to \infty} n^{\frac{2}{d+1}} \mathbb{E}_{\mu_q, K^*} \int_{K^{(n)} \setminus K_1^*} \lambda(x) \, dx \\
= 2 c_d \int_{\partial K^*} q(x)^{-\frac{2}{d+1}} \lambda(x) \kappa^* (x) \frac{1}{d+1} \mathcal{H}^{d-1}(dx) \\
= 2 c_d \omega_d^{-\frac{2}{d+1}} \int_{\partial K} q(x)^{-\frac{2}{d+1}} \| x \|^{-d+1} \kappa^* (x) \frac{1}{d+1} \mathcal{H}^{d-1}(dx),
\]
where \( \kappa^* \) denotes the generalized Gauss curvature of \( K^* \). In the following, for \( x \in \partial K \), let \( \sigma_K(x) \) denote an exterior unit normal vector of \( K \) at \( x \). It is unique for \( \mathcal{H}^{d-1} \)-almost all \( x \in \partial K \).

**Theorem 5.2.** Let \( K \subset \mathbb{R}^d \) be a convex body with \( o \in \text{int}(K) \), and let \( q : [0, \infty) \times S^{d-1} \to [0, \infty) \) be a measurable function satisfying (q1)–(q3). Then
\[
\lim_{n \to \infty} n^{\frac{2}{d+1}} \mathbb{E}_{\mu_q} \left( W(K^{(n)} \cap K_1) - W(K) \right) \\
= 2 c_d \omega_d^{-\frac{2}{d+1}} \int_{\partial K} q(h(K, \sigma_K(x)), \sigma_K(x))^{-\frac{2}{d+1}} \kappa(x)^{\frac{2}{d+1}} \mathcal{H}^{d-1}(dx). \tag{5.2}
\]

The proof is completed in Section 6 by providing Lemma 6.2.

**Example.** Observe that if \( q : \{(h(K, u), u) \in (0, \infty) \times S^{d-1} : u \in S^{d-1}\} \to (0, \infty) \) is positive and continuous, then \( q \) can be extended to \( [0, \infty) \times S^{d-1} \) such that (q1)–(q3) are satisfied. For any such extension, the right-hand side of (5.2) remains unchanged. As an example, we may choose \( q_1 \) such that \( q_1(t, u) = t^{(d-1)/2} \) for \( t = h(K, u) \) and \( u \in S^{d-1} \). Then the integral in (5.2) turns into
\[
\int_{\partial K} \frac{\kappa(x)^{\frac{2}{d+1}}}{\langle x, \sigma_K(x) \rangle^{\frac{d-1}{d+1}}} \mathcal{H}^{d-1}(dx) = \Omega_d(K),
\]
where
\[
\Omega_p(K) := \int_{\partial K} \frac{\kappa(x)^{\frac{2}{d+1}}}{\langle x, \sigma_K(x) \rangle^{\frac{d-1}{d+1}}} \mathcal{H}^{d-1}(dx)
\]
is the \( p \)-affine surface area of \( K \) (see [26], [17], [18], [22], [44], [45], [23], [24]). It has been shown that \( \Omega_d(K) = \Omega_1(K^*) \); see [18]. Moreover, for a convex body \( L \subset \mathbb{R}^d \), the equiaffine isoperimetric inequality states that
\[
\Omega_1(L) \leq d \omega_d \Omega_d(V(L))^{\frac{d-1}{d+1}}.
\]
with equality if and only if $L$ is an ellipsoid (cf. [27], [25], [26], [17], [5]). Thus we get
\[
\lim_{n \to \infty} n^{\frac{d+1}{d}} \mathbb{E}_{\mu_n}(W(K^{(n)} \cap K_1) - W(K)) \leq 2dc_{d} \omega_d^{-\frac{d+1}{d}} \alpha_d^{\frac{d+1}{d}} V(K^*)^{\frac{d+1}{d}}
\]
with equality if and only if $K^*$ is an ellipsoid, that is, if and only if $K$ is an ellipsoid. This can be interpreted as saying that among all convex bodies for which the volume of the polar body is fixed, ellipsoids are worst approximated asymptotically by circumscribed random polytopes (with respect to the density $q_1$) in the sense of the mean width.

For another application, we define
\[
F(P) := f_{d-1}(P),
\]
for a convex polyhedral set $P \subset \mathbb{R}^d$. It is well known that $f_0(P) = f_{d-1}(P^*)$ for a convex polytope $P \subset \mathbb{R}^d$ with $\alpha \in \text{int}(P)$. Thus, from Proposition 5.1 we get
\[
\mathbb{E}_{\mu_n}(f_{d-1}(K^{(n)})) = \mathbb{E}_{\nu,K^*}(f_{d-1}(K^{(n)}))^{*}) = \mathbb{E}_{\nu,K^*}(1\{(K^{(n)}_1)^* \subset K_1\})f_{d-1}(K^{(n)}_1)^*) + \mathbb{E}_{\nu,K^*}(1\{(K^{(n)}_1)^* \not\subset K_1\})f_{d-1}(K^{(n)}_1)^*)
\]
\[
= \mathbb{E}_{\nu,K^*}(1\{(K^{(n)}_1)^* \subset K_1\})f_0(K^{(n)}_1) + O(n \cdot \alpha^n)
\]
\[
= \mathbb{E}_{\nu,K^*}(f_0(K^{(n)}_1)) + O(n \cdot \alpha^n),
\]
where $\alpha \in (0, 1)$ is a suitable constant.

The following Theorem 5.3 generalizes Theorem 2.1 in the same way as Theorem 5.2 extends Theorem 2.2.

**Theorem 5.3.** Let $K \subset \mathbb{R}^d$ be a convex body with $\alpha \in \text{int}(K)$, and let $q : [0, \infty) \times S^{d-1} \to [0, \infty)$ be a measurable function satisfying (q1)–(q3). Then
\[
\lim_{n \to \infty} n^{-\frac{d+1}{d}} \mathbb{E}_{\mu_n}(f_{d-1}(K^{(n)})) = c_d \omega_d^{-\frac{d+1}{d}} \int_{\partial K} q(h(K, \sigma_K(x)), \sigma_K(x)) \frac{d+1}{d} r(x)^{\frac{d+1}{d}} \mathcal{H}^{d-1}(dx).
\]

The proof follows by applying Corollary 3.2 and Lemma 6.2.

### 6. Polarity and an integral transformation

In this section, we establish the required integral transformation involving the generalized Gauss curvatures of a convex body and its polar body. The main difficulty of the proof is due to the fact that we do not make any smoothness assumptions on the convex bodies that are considered.

Let $L \subset \mathbb{R}^d$ be a convex body. If the support function $h_L$ of $L$ is differentiable at $u \neq 0$, then the gradient $\nabla h_L(u)$ of $h_L$ at $u$ is equal to the unique boundary point of $L$ having $u$ as an exterior normal vector. In particular, the gradient of $h_L$ is a function which is homogeneous of degree zero. Note that $h_L$ is differentiable at $\mathcal{H}^{d-1}$-almost all unit vectors. We write $D_{d-1}h_L(u)$ for the product of the principal radii of curvature of $L$ in direction $u \in S^{d-1}$, whenever the support function $h_L$ is twice differentiable in the generalized sense at $u \in S^{d-1}$. Note that this is the case for $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$. The Gauss map $\sigma_L$ is defined $\mathcal{H}^{d-1}$-almost everywhere on $\partial L$. If $\sigma_L$ is differentiable in the generalized sense at $x \in \partial L$, which is the case for $\mathcal{H}^{d-1}$-almost all $x \in \partial L$, then the product of the eigenvalues of the differential is the Gauss curvature $\sigma_L(x)$. The connection to curvatures defined on the generalized normal bundle $\mathcal{N}(L)$ of $L$ will be used in the following proof (cf. [19]).
Lemma 6.1. Let $L \subset \mathbb{R}^d$ be a convex body containing the origin in its interior. If $g : \partial L \to [0, \infty]$ is measurable, then
\[
\int_{\partial L} g(x)\kappa_L(x) \frac{dx}{\mathcal{H}^{d-1}} = \int_{S^{d-1}} g(\nabla h_L(u)) D_{d-1} h_L(u) \frac{d\pi}{\mathcal{H}^{d-1}} \mathcal{H}^{d-1}(du).
\]

Proof. In the following proof, we use results and methods from [19], to which we refer for additional references and detailed definitions. Let $\mathcal{N}(L)$ denote the generalized normal bundle of $L$, and let $k_i(x, u) \in [0, \infty]$, $i = 1, \ldots, d - 1$, be the generalized curvatures of $L$, which are defined for $\mathcal{H}^{d-1}$-almost all $(x, u) \in \mathcal{N}(L)$. Expressions such as
\[
\frac{k_i(x, u)}{\sqrt{1 + k_i(x, u)^2}} \quad \text{or} \quad \frac{k_i(x, u)}{\sqrt{1 + k_i(x, u)^2}}
\]
with $k_i(x, u) = \infty$ are understood as limits as $k_i(x, u) \to \infty$, and yield 0 or 1, respectively in the two given examples. As is common in measure theory, the product $0 \cdot \infty$ is defined as 0.

Our starting point is the expression
\[
I := \int_{\mathcal{N}(L)} g(x) \prod_{i=1}^{d-1} \frac{k_i(x, u)}{\sqrt{1 + k_i(x, u)^2}} \mathcal{H}^{d-1}(d(x, u)),
\]
which will be evaluated in two different ways. A comparison of the resulting expressions yields the assertion of the lemma.

First, we rewrite $I$ in the form
\[
I = \int_{\mathcal{N}(L)} g(x) \left( \prod_{i=1}^{d-1} k_i(x, u) \right)^{-\frac{d}{\pi d}} J_{d-1} \pi_2(x, u) \mathcal{H}^{d-1}(d(x, u)),
\]
where
\[
J_{d-1} \pi_2(x, u) = \prod_{i=1}^{d-1} \frac{k_i(x, u)}{\sqrt{1 + k_i(x, u)^2}},
\]
for $\mathcal{H}^{d-1}$-almost all $(x, u) \in \mathcal{N}(L)$, is the (approximate) Jacobian of the map $\pi_2 : \mathcal{N}(L) \to S^{d-1}$, $(x, u) \mapsto u$. To check (6.2), we distinguish the following cases. If $k_i(x, u) = 0$ for some $i$, then the integrands on the right-hand sides of (6.1) and of (6.2) are zero, since $0 \cdot \infty = 0$ and $J_{d-1} \pi_2(x, u) = 0$. If $k_i(x, u) \neq 0$ for all $i$ and $k_j(x, u) = \infty$ for some $j$, then again both integrands are zero. In all other assertions is the case is clear.

For $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$, $\nabla h_L(u) \in \partial L$ is the unique boundary point of $L$ which has $u$ as an exterior unit normal vector. Then the coarea formula yields
\[
I = \int_{S^{d-1}} g(\nabla h_L(u)) \left( \prod_{i=1}^{d-1} k_i(\nabla h_L(u), u) \right)^{-\frac{d}{\pi d}} \mathcal{H}^{d-1}(du).
\]

Using Lemma 3.4 in [19], we get
\[
I = \int_{S^{d-1}} g(\nabla h_L(u)) D_{d-1} h_L(u) \frac{d\pi}{\mathcal{H}^{d-1}} \mathcal{H}^{d-1}(du).
\]
for $H^{d-1}$-almost all $(x, u) ∈ N(L)$. A similar argument as before yields
\[
I = \int_{N(L)} g(x) \left( \prod_{i=1}^{d-1} k_i(x, u) \right)^{1/2} J_{d-1} \pi_1(x, u) H^{d-1}(d(x, u)) \]
\[
= \int_{\partial L} g(x) \left( \prod_{i=1}^{d-1} k_i(x, \sigma_L(x)) \right)^{1/2} \mathcal{H}^{d-1}(dx).
\]
By Lemma 3.1 in [19], we finally also get
\[
I = \int_{\partial L} g(x) \kappa_L(x) \frac{1}{\pi^{d}} \mathcal{H}^{d-1}(dx). \tag{6.4}
\]

A comparison of equations (6.3) and (6.4) gives the required equality.

**Remark.** An alternative argument can be based on arguments similar to those used in [17] for the proof of the equality of two representations of the affine surface area of a convex body.

**Lemma 6.2.** Let $K ⊂ \mathbb{R}^d$ be a convex body with $o ∈ \text{int}(K)$. If $f : [0, \infty) × S^{d-1} → [0, \infty)$ is a measurable function and $\tilde{f}(x) := f(\|x\|^{-1}, \|x\|^{-1}x), x ∈ \partial K^*$, then
\[
\int_{\partial K^*} \tilde{f}(x) \|x\|^{-d+1} \kappa^*_x \mathcal{H}^{d-1}(dx) = \int_{\partial K} f(h(K, \sigma_K(x)), \sigma_K(x)) \kappa_x(x) \frac{d}{\pi^d} \mathcal{H}^{d-1}(dx).
\]

**Proof.** We apply Lemma 6.1 with $L = K^*$ and $g(x) = \tilde{f}(x) \|x\|^{-d+1}, x ∈ \partial K^*$, and thus we get
\[
\int_{\partial K^*} \tilde{f}(x) \|x\|^{-d+1} \kappa^*_x \mathcal{H}^{d-1}(dx)
= \int_{S^{d-1}} \tilde{f}(\nabla h_{K^*}(u)) \|\nabla h_{K^*}(u)\|^{-d+1} D_{d-1} h_{K^*}(u) \frac{d}{\pi^d} \mathcal{H}^{d-1}(du).
\]
Next we apply Theorem 2.2 in [18] (or the second part of Corollary 5.1 in [20]). Thus, using the fact that, for $\mathcal{H}^{d-1}$-almost all $u ∈ S^{d-1}$, $h_{K^*}$ is differentiable in the generalized sense at $u$ and $ρ(K, u)u$ is a normal boundary point of $K$, we have
\[
D_{d-1} h_{K^*}(u) \frac{d}{\pi^d} = \kappa(u) \frac{d}{\pi^d} (u, \sigma_K(x))^{-d},
\]
where $x = ρ(K, u)u ∈ \partial K$ and $u = \|x\|^{-1}x ∈ S^{d-1}$. Thus we obtain
\[
\int_{\partial K^*} \tilde{f}(x) \|x\|^{-d+1} \kappa^*_x \mathcal{H}^{d-1}(dx)
= \int_{S^{d-1}} \tilde{f}(\nabla h_{K^*}(u)) \frac{\|\nabla h_{K^*}(u)\|^{-d+1}}{\kappa(ρ(K, u)u)} \kappa(ρ(K, u)u) \frac{d}{\pi^d} \mathcal{H}^{d-1}(dx).
\]
The bijective and bilipschitz transformation $T : S^{d-1} → \partial K, u → ρ(K, u)u$, has the Jacobian
\[
JT(u) = \frac{\|\nabla h_{K^*}(u)\|}{h_{K^*}(u)^d}.
\]
for $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$ (see the proof of Lemma 2.4 in [18]). Thus

$$
\int_{\partial K^\ast} f(x)||x||^{-d+1}K^\ast(x)\frac{d}{\partial ||x||}\mathcal{H}^{d-1}(dx)
$$

$$
= \int_{\partial K} \int \tilde{f}(\nabla h_{K^\ast}(x)) \left(\frac{\nabla h_{K^\ast}(x)}{||x||} \right)^{-d} h_{K^\ast}(x) \kappa(x) \frac{d}{\partial ||x||} \mathcal{H}^{d-1}(dx)
$$

$$
= \int_{\partial K} \int f(||\nabla h_{K^\ast}(x)||^{-1}, \nabla h_{K^\ast}(x)/||\nabla h_{K^\ast}(x)||) \kappa(x) \frac{d}{\partial ||x||} \mathcal{H}^{d-1}(dx),
$$

since $h_{K^\ast}(x) = 1$ for $x \in \partial K$ and $x^\ast := \nabla h_{K^\ast}(x)$ satisfies $||x^\ast||^{-1} = \langle x, \kappa(x) \rangle$ and $x^\ast/||x^\ast|| = \kappa(x)$, for $\mathcal{H}^{d-1}$-almost all $x \in \partial K$.

\[ \square \]

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\section*{References}
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