Approximation properties of random polytopes associated with Poisson hyperplane processes

Daniel Hug and Rolf Schneider

Abstract

We consider a stationary Poisson hyperplane process with given directional distribution and intensity in d-dimensional Euclidean space. Generalizing the zero cell of such a process, we fix a convex body K and consider the intersection of all closed halfspaces bounded by hyperplanes of the process and containing K. We study how well these random polytopes approximate K (measured by the Hausdorff distance) if the intensity increases, and how this approximation depends on the directional distribution in relation to properties of K.

Keywords: Poisson hyperplane process; zero polytope; approximation of convex bodies; directional distribution

2010 Mathematics Subject Classification: Primary 60D05

1 Introduction

Asymptotic properties of the convex hull of n independent, identically distributed random points in $\mathbb{R}^d$, as n tends to infinity, are an actively studied topic of stochastic geometry; see, for example, Subsection 8.2.4 of the book [8] and the more recent survey by Reitzner [6]. Very often, one studies uniform random points in a given convex body and measures the rate of approximation by the volume difference, or the difference of other global functionals, or one investigates the asymptotic behaviour of combinatorial quantities like face numbers. In contrast, approximation by random polytopes, measured in terms of the Hausdorff metric δ, has been investigated less frequently. We refer to Note 5 for Subsection 8.2.4 in [8] and mention here only the following results. For a convex body K of class C^2_+ (that is, with a twice continuously differentiable boundary with positive Gauss curvature), Bárány [1] (Theorem 6) showed that the Hausdorff distance from K to the convex hull K_n of n i.i.d. uniform random points in K satisfies

$$\mathrm{E}\,\delta(K, K_n) \sim \left(\frac{\log n}{n}\right)^{2/(d+1)}$$

as $n \to \infty$ (here $f(n) \sim g(n)$ means that there are constants c_1, c_2 such that $c_1 g(n) < f(n) < c_2 g(n)$). A result of Dümbgen and Walther [3] (Corollary 1) says that, for an arbitrary convex body K,

$$\delta(K, K_n) = \mathrm{O}\left(\left(\frac{\log n}{n}\right)^{1/d}\right) \quad \text{almost surely.}$$

The second standard approach to convex polytopes, generating them as intersections of closed halfspaces instead of convex hulls of points, has not found equal attention in the study of random polytopes. About the role that duality, either in an exact or a heuristic sense, can play here, we refer to the introduction of [2]. This alternative approach has to offer some new aspects, in particular since random hyperplanes naturally come with some directional distribution, which influences the random polytopes that they generate. This aspect is emphasized in the present article, where we consider random polytopes generated by a stationary Poisson hyperplane process, with an arbitrary directional distribution.

Let X be a stationary nondegenerate (see [8, p. 486]) Poisson hyperplane process in Euclidean space $\mathbb{R}^d$, $d \geq 2$ (with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$). The reader is referred to Chapters 3 and 4 of [8] for an introduction, and also for some notational conventions used here. For a hyperplane H in $\mathbb{R}^d$, not passing through the origin $\boldsymbol{o}$, we denote by $H_{\boldsymbol{o}}^-$ the closed halfspace bounded by H that contains $\boldsymbol{o}$. The random polytope

$$Z_0 := \bigcap_{H \in X} H_{\boldsymbol{o}}^-$$

is called the *zero cell* of X (it is also known as the *Crofton polytope* of X).

A generalization of this notion is obtained as follows. Let $K \subset \mathbb{R}^d$ be a convex body. For a hyperplane H not intersecting K we denote by H_K^- the closed halfspace bounded by H that contains K. Then we define the K-*cell* of X as the random polytope

$$Z_K := \bigcap_{H \in X,\, H \cap K = \emptyset} H_K^-.$$

The almost sure boundedness of Z_K follows as in the proof of [8, Theorem 10.3.2]. In the following we are interested in the question how well K is approximated by Z_K, if the intensity of the process X tends to infinity. Since the intensity is a constant multiple of the expected number of hyperplanes in the process that hit K, the analogy to convex hulls of an increasing number of points is evident.

We consider approximation in sense of the Hausdorff metric δ on the space $\mathcal{K}$ of convex bodies (always with interior points) in $\mathbb{R}^d$. Of course, in order that approximation of K by Z_K be possible at all, the convex body K must somehow be adapted to the directional distribution of the hyperplane process X. For example, a ball K cannot be approximated arbitrarily closely by Z_K if the hyperplane process X has only hyperplanes of finitely many directions. To make this more precise, let N be a closed subset of the unit sphere $\mathbb{S}^{d-1}$, not contained in a closed halfsphere. For a given convex body K, we denote by $\mathcal{P}(K, N)$ the set of all polytopes which are finite intersections of closed halfspaces containing K and with outer unit normal vectors in N.

Proposition 1. The convex body K can be approximated arbitrarily closely by polytopes from $\mathcal{P}(K, N)$ if and only if $\operatorname{supp} S_{d-1}(K, \cdot) \subset N$.

Here supp denotes the support of a measure, and $S_{d-1}(K, \cdot)$ is the surface area measure of K (see [7], for example). We shall give a proof of Proposition 1 in the next section. It serves here only to motivate the assumption (2) made below.

The intensity measure $\Theta = \mathrm{E}X(\cdot)$ of X is assumed, as usual, to be locally finite. It can then be represented in the form (see [8], (4.33))

$$\Theta(A) = 2\gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}_A(H(\boldsymbol{u}, t))\, \mathrm{d}t\, \varphi(\mathrm{d}\boldsymbol{u}) \tag{1}$$

for $A \in \mathcal{B}(\mathcal{H}^d)$, where $\gamma > 0$ is the intensity and φ is the spherical directional distribution of X; the latter is an even probability measure on the unit sphere $\mathbb{S}^{d-1}$ which is not concentrated on a great subsphere. By $\mathcal{H}^d$ we denote the space of hyperplanes in $\mathbb{R}^d$, and $\mathcal{B}(T)$ is the σ-algebra of Borel sets of a topological space T. Further,

$$H(\boldsymbol{u}, t) = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{u} \rangle = t\}$$

for $\boldsymbol{u} \in \mathbb{S}^{d-1}$ and $t > 0$ is the standard parametrization of a hyperplane not passing through the origin $\boldsymbol{o}$. We assume, as usual, that φ is not concentrated on a great subsphere. For convenience (in view of some later estimations of constants), we also assume that $\gamma \geq 1$.

For $K \in \mathcal{K}$, the Hausdorff distance $\delta(K, P)$ of K from a polytope P containing it is the smallest number $\epsilon \geq 0$ such that $P \subset K(\epsilon)$, where $K(\epsilon) = K + \epsilon B^d$ (B^d is the unit ball) denotes the outer parallel body of K at distance ϵ. Therefore, we prescribe a number $\epsilon > 0$ and ask for the probability $\mathrm{P}\{Z_K \not\subset K(\epsilon)\}$. First we give a necessary and sufficient condition that this probability tends to zero if the intensity of the process X tends to infinity; if the condition is satisfied, we obtain that the decay is exponential. Under a slightly stronger assumption, this can then be used to derive our main results, concerning the rate of convergence.

Without loss of generality, we may assume that $\boldsymbol{o} \in \operatorname{int} K$. By the independence properties of the Poisson process, we then have

$$\mathrm{P}\{Z_K \not\subset K(\epsilon)\} = \mathrm{P}\left\{Z_0 \not\subset K(\epsilon) \mid K \subset Z_0\right\}.$$

The conditional probability involving the zero cell is slightly more convenient to handle.

We assume in the following that the surface area measure of the given convex body K satisfies

$$\operatorname{supp} S_{d-1}(K, \cdot) \subset \operatorname{supp} \varphi. \tag{2}$$

By Proposition 1, this assumption is necessary for arbitrarily good approximation of K by Z_K. Theorem 1 shows, in a stronger form, that it is also sufficient.

For $\boldsymbol{y} \in \mathbb{R}^d \setminus K$, let $K^{\boldsymbol{y}} := \operatorname{conv}(K \cup \{\boldsymbol{y}\})$. For $\epsilon > 0$ we define

$$\mu(K, \varphi, \epsilon) := \min_{\boldsymbol{y} \in \operatorname{bd} K(\epsilon)} \int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, u) - h(K, u)] \, \varphi(\mathrm{d}u), \tag{3}$$

where h denotes the support function. Lemma 1, to be proved in the next section, shows that $\mu(K, \varphi, \epsilon) > 0$.

Theorem 1. *Let X be a stationary Poisson hyperplane process in $\mathbb{R}^d$ with intensity γ and directional distribution φ. Let $K \in \mathcal{K}$ be a convex body satisfying* (2)*. There are positive constants $C_1(\epsilon), C_2$ (both depending on K, φ, d) such that the following holds. If $0 < \epsilon \leq 1$, then*

$$\mathrm{P}\left\{Z_K \not\subset K(\epsilon)\right\} \leq C_1(\epsilon) \exp\left[-C_2 \mu(K, \varphi, \epsilon) \gamma\right]. \tag{4}$$

From this estimate we can deduce that Z_K converges to K in the Hausdorff metric almost surely as the intensity goes to infinity. To make this statement precise, we consider an embedding of a Poisson hyperplane process X_τ with intensity $\tau > 0$, directional distribution φ, and intensity measure

$$\mathrm{E}X_\tau(\cdot) = 2\tau \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}\{H(\boldsymbol{u}, t) \in \cdot\} \, \mathrm{d}t \, \varphi(\mathrm{d}\boldsymbol{u}) =: \tau \Theta_1$$

into a Poisson process ξ on $[0,\infty) \times \mathcal{H}^d$ with intensity measure $\lambda \otimes \Theta_1$, where λ denotes Lebesgue measure on $[0,\infty)$. Then $\xi([0,\tau] \times \cdot)$ is a Poisson hyperplane process with intensity measure $\tau\Theta_1$, thus $X_\tau \sim \xi([0,\tau] \times \cdot)$. Let $Z_K^{(\tau)}$ denote the K-cell associated with $\xi([0,\tau] \times \cdot)$. Then we have $K \subset Z_K^{(\sigma)} \subset Z_K^{(\tau)}$ for $\sigma \geq \tau > 0$, and therefore $\delta(K, Z_K^{(\sigma)}) \leq \delta(K, Z_K^{(\tau)})$. This shows that

$$\mathrm{P}\left\{\sup_{\sigma\geq\tau} \delta(K, Z_K^{(\sigma)}) \geq \epsilon\right\} = \mathrm{P}\left\{\delta(K, Z_K^{(\tau)}) \geq \epsilon\right\} \leq C_1(\epsilon) \exp\left[-C_2 \mu(K, \varphi, \epsilon)\tau\right]$$

for all $\epsilon > 0$, and thus

$$\lim_{\tau\to\infty} \delta(K, Z_K^{(\tau)}) = 0$$

holds almost surely.

In order to be able to estimate the rate of convergence, we need a stronger assumption than (2), namely

$$S_{d-1}(K, \cdot) \leq b\,\varphi \tag{5}$$

with some constant b.

We consider a sequence $X_1, X_2, \dots$ of Poisson hyperplane processes as above (defined on a common probability space), with spherical directional distribution φ, where X_n has intensity n. For a given convex body K, the K-cell of X_n is denoted by $Z_K^{(n)}$.

If $(Y_n)_{n\in\mathbb{N}}$ is a sequence of real random variables and $f(n)_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers, we write $Y_n = \mathrm{O}(f(n))$ *almost surely* if there is a constant $C < \infty$ such that with probability one we have $Y_n \leq Cf(n)$ for sufficiently large n. Moreover, we write $Y_n \sim f(n)$ *almost surely* if there are constants $0 < c \leq C < \infty$ such that with probability one we have $cf(n) \leq Y_n \leq Cf(n)$ for all sufficiently large n. A 'ball' in the following is a Euclidean ball of positive radius.

Theorem 2. *Suppose that the convex body K and the directional distribution φ of the stationary Poisson hyperplane processes X_n satisfy* (5)*. There is a constant $\alpha \leq d$, depending only on K, such that*

$$\delta(K, Z_K^{(n)}) = \mathrm{O}\left(\left(\frac{\log n}{n}\right)^{1/\alpha}\right) \quad \textit{almost surely,} \tag{6}$$

as $n \to \infty$. If a ball rolls freely inside K, then (6) *holds with $\alpha = (d+1)/2$, and if K is a polytope, then* (6) *holds with $\alpha = 1$.*

Under stronger assumptions, we can determine the exact asymptotic order of approximation.

Theorem 3. *Let the convex body $K \in \mathcal{K}$ be such that a ball rolls freely inside K and that K rolls freely inside a ball. Suppose that the directional distribution φ of the stationary Poisson hyperplane processes X_n satisfies*

$$a\,\varphi \leq S_{d-1}(K, \cdot) \leq b\,\varphi \tag{7}$$

with some positive constants a, b. Then

$$\delta(K, Z_K^{(n)}) \sim \left(\frac{\log n}{n}\right)^{2/(d+1)} \quad \textit{almost surely,} \tag{8}$$

as $n \to \infty$.

Note that Theorem 3 covers, in particular, the case where K is of class C^2_+ and the hyperplane processes X_n are isotropic, that is, their directional distribution φ is invariant under rotations and thus is equal to the normalized spherical Lebesgue measure. If K is of class C^2_+, the surface area measure $S_{d-1}(K,\cdot)$ has a positive continuous density with respect to spherical Lebesgue measure, so that (7) holds, and the assumptions on K are satisfied by Blaschke's rolling theorem (Corollary 3.2.10 in [7]).

In the next section, we prove some auxiliary results. Theorem 1 is proved in Section 3, and the proofs of Theorems 2 and 3 follow in Section 4.

2 Auxiliary results

Proof of Proposition 1. By [7], Theorem 4.6.3, the support of the area measure $S_{d-1}(K,\cdot)$ is equal to $\mathrm{cl}\,\mathrm{extn}\,K$, the closure of the set of extreme (unit) normal vectors of K.

Suppose now that K can be approximated arbitrarily closely by polytopes from $\mathcal{P}(K,N)$. Let $\boldsymbol{x}$ be a regular boundary point of K, and let $(\boldsymbol{x}_i)_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d \setminus K$ converging to $\boldsymbol{x}$. To each i, there exists a polytope $P_i \in \mathcal{P}(K,N)$ not containing $\boldsymbol{x}_i$, hence there is a closed halfspace H_i^- with outer normal vector $\boldsymbol{u}_i \in N$ containing K but not $\boldsymbol{x}_i$. For $i \to \infty$, the sequence of hyperplanes H_i bounding H_i^- has a convergent subsequence; its limit is the unique supporting hyperplane of K at $\boldsymbol{x}$. It follows that the outer unit normal vector of K at $\boldsymbol{x}$ belongs to the closed set N. A normal vector at a regular boundary point of K is a 0-exposed normal vector. Since $\boldsymbol{x}$ was an arbitrary regular boundary point of K, the set N contains the set of 0-exposed normal vectors of K. The closure of the 0-exposed normal vectors is equal to the closure of the extreme normal vectors (see Theorem 2.2.7 of [7], also for the terminology used here). Hence, $\mathrm{cl}\,\mathrm{extn}\,K \subset N$.

Conversely, suppose that $\mathrm{cl}\,\mathrm{extn}\,K \subset N$. Since the regular boundary points of K are dense in the boundary of K (as follows from [7], Theorem 2.2.4), the body K is the intersection of its supporting halfspaces with a regular point of K in the boundary. The outer unit normal vector of such a halfspace is extreme and hence belongs to N. It follows that K can be approximated arbitrarily closely by polytopes from $\mathcal{P}(K,N)$. □

In the rest of this paper, $c_1, c_2, \dots$ denote positive constants that depend only on K, φ and the dimension d.

Lemma 1. *If* (2) *holds, then* $\mu(K,\varphi,\epsilon) > 0$. *Suppose that* (5) *is satisfied. Let* $0 < \epsilon \le 1$.

(a) *For general* K *and for* $\epsilon \le D(K)$, *where* $D(K)$ *denotes the diameter of* K,

$$\mu(K,\varphi,\epsilon) \ge c_1\epsilon^d. \tag{9}$$

(b) *If a ball rolls freely inside* K, *then*

$$\mu(K,\varphi,\epsilon) \ge c_2\epsilon^{(d+1)/2}. \tag{10}$$

(c) *If* K *is a polytope, then*

$$\mu(K,\varphi,\epsilon) \ge c_3\epsilon. \tag{11}$$

Proof. First, let (2) be satisfied. Let $\boldsymbol{y} \in \mathbb{R}^d \setminus K$. Let V_d denote the volume and V the mixed volume in $\mathbb{R}^d$. Using a formula for mixed volumes ([7], (5.1.18)) and Minkowski's inequality (e.g., [7], (6.2.2)), we get

$$\begin{aligned}
&\frac{1}{d}\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}},\boldsymbol{u}) - h(K,\boldsymbol{u})]\, S_{d-1}(K,\mathrm{d}\boldsymbol{u})\\
&= V(K^{\boldsymbol{y}},K,\dots,K) - V_d(K)\\
&\geq V_d(K^{\boldsymbol{y}})^{\frac{1}{d}} V_d(K)^{\frac{d-1}{d}} - V_d(K)\\
&= V_d(K)^{\frac{d-1}{d}} \left[V_d(K^{\boldsymbol{y}})^{\frac{1}{d}} - V_d(K)^{\frac{1}{d}}\right]\\
&> 0.
\end{aligned}$$

The integrand is nonnegative and continuous as a function of $\boldsymbol{u}$. Since the integral is positive, there exists a neighbourhood (in $\mathbb{S}^{d-1}$) of some point $\boldsymbol{u}_0 \in \operatorname{supp} S_{d-1}(K,\cdot)$ on which the integrand is positive. By (2), $\boldsymbol{u}_0 \in \operatorname{supp}\varphi$, and hence

$$g(\boldsymbol{y}) := \int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}},\boldsymbol{u}) - h(K,\boldsymbol{u})]\, \varphi(\mathrm{d}\boldsymbol{u}) > 0.$$

The function g is continuous, hence on each compact subset of $\mathbb{R}^d \setminus K$ it attains a minimum. This proves that $\mu(K,\varphi,\epsilon) > 0$.

Now suppose that (5) holds. From the preceding estimate we get

$$\begin{aligned}
\frac{b}{d}\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}},\boldsymbol{u}) - h(K,\boldsymbol{u})]\, \varphi(\mathrm{d}\boldsymbol{u}) &\geq \frac{1}{d}\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}},\boldsymbol{u}) - h(K,\boldsymbol{u})]\, S_{d-1}(K,\mathrm{d}\boldsymbol{u})\\
&\geq V_d(K)^{\frac{d-1}{d}} \left[V_d(K^{\boldsymbol{y}})^{\frac{1}{d}} - V_d(K)^{\frac{1}{d}}\right]\\
&\geq c_4 \left[V_d(K^{\boldsymbol{y}}) - V_d(K)\right].
\end{aligned}$$

(a) For the proof of (9), let C be the cone with apex $\boldsymbol{y}$ spanned by K. Let $\boldsymbol{y}'$ be the point in K nearest to $\boldsymbol{y}$. The vector $\boldsymbol{y} - \boldsymbol{y}'$ has length ϵ, and the hyperplane H' orthogonal to it and passing through $\boldsymbol{y}'$ supports K. Let H be the other supporting hyperplane of K parallel to H'. Let Δ be the convex hull of $\boldsymbol{y}$ and $H \cap C$ and Δ' the convex hull of $\boldsymbol{y}$ and $H' \cap C$. Denoting by $D(K)$ the diameter of K and assuming that $\epsilon \leq D(K)$, we have

$$V_d(K^{\boldsymbol{y}}) - V_d(K) \geq V_d(\Delta') \geq \left(\frac{\epsilon}{D(K)+\epsilon}\right)^d V_d(\Delta) \geq \left(\frac{\epsilon}{2D(K)}\right)^d V_d(K).$$

This gives (9).

(b) Suppose that a ball of radius $r > 0$ rolls freely inside K. Since $\mu(\cdot,\varphi,\epsilon)$ is translation invariant, we can assume that K contains the ball $B(\boldsymbol{o},r)$ of radius r centred at $\boldsymbol{o}$. Let $R > 0$ be such that $K \subset B(\boldsymbol{o},R)$. For $s > 0$, the convex body

$$K^s := \{\boldsymbol{x} \in \mathbb{R}^d : V_d(K^{\boldsymbol{x}}) - V_d(K) \leq s\}$$

is known as an illumination body of K (cf. [9, p. 258]; the convexity follows from Satz 4 in Fáry and Rédei [4]). Now let $\boldsymbol{y} \in \operatorname{bd} K(\epsilon)$ and put $\nu := V_d(K^{\boldsymbol{y}}) - V_d(K)$, then $\boldsymbol{y} \in \operatorname{bd} K^{\nu}$.

Let $\boldsymbol{x} \in \mathrm{bd}\, K$ be determined by $\{\boldsymbol{x}\} = [\boldsymbol{o}, \boldsymbol{y}] \cap \mathrm{bd}\, K$, and denote by $N(\boldsymbol{x})$ the unique exterior unit normal vector of K at $\boldsymbol{x}$. Since $B(\boldsymbol{o}, r) \subset K$, we have

$$\langle \boldsymbol{x}, N(\boldsymbol{x}) \rangle \geq r, \qquad \langle \boldsymbol{x}/\|\boldsymbol{x}\|, N(\boldsymbol{x}) \rangle \geq r/R.$$

From $\|\boldsymbol{y}\| - \|\boldsymbol{x}\| \geq \epsilon$ we get $\|\boldsymbol{y}\|^d - \|\boldsymbol{x}\|^d \geq dr^{d-1}\epsilon$. Therefore, Lemma 2 in [9] yields

$$\nu^{2/(d+1)} \geq c_5 r r^{(d-1)/(d+1)} \left(\left(\frac{\|\boldsymbol{y}\|}{\|\boldsymbol{x}\|} \right)^d - 1 \right) \geq c_6 R^{-d} \left(\|\boldsymbol{y}\|^d - \|\boldsymbol{x}\|^d \right) \geq c_7 \epsilon,$$

hence

$$V_d(K^{\boldsymbol{y}}) - V_d(K) \geq c_8\, \epsilon^{(d+1)/2},$$

which gives (10).

(c) Now suppose that K is a polytope. Let $\boldsymbol{y}'$ be the point in K nearest to $\boldsymbol{y}$. Put $\boldsymbol{v} := (\boldsymbol{y} - \boldsymbol{y}')/\|\boldsymbol{y} - \boldsymbol{y}'\|$, and let F denote the unique (proper) face of K which contains $\boldsymbol{y}'$ in its relative interior. Let $F_1, \ldots, F_m$ be the facets of K that contain F, and let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ be their outer unit normal vectors. By [7, p. 74 and Theorem 2.4.9], we have

$$\boldsymbol{v} \in N(K, F) = N(K, \boldsymbol{y}') = \mathrm{pos}\{\boldsymbol{u}_i : i = 1, \ldots, m\},$$

where $N(K, F)$ and $N(K, \boldsymbol{y}')$ are the normal cones of K at F and $\boldsymbol{y}'$, respectively, and pos denotes the positive hull. Hence, there is some $i \in \{1, \ldots, m\}$ such that $\langle \boldsymbol{v}, \boldsymbol{u}_i \rangle > 0$, in particular, we have

$$a(F, \boldsymbol{v}) := \max\{\langle \boldsymbol{v}, \boldsymbol{u}_i \rangle : i = 1 \ldots, m\} = \langle \boldsymbol{v}, \boldsymbol{u}_{i_0} \rangle > 0,$$

for some $i_0 \in \{1, \ldots, m\}$. Since $N(K, F) \cap \mathbb{S}^{d-1}$ is compact, it follows that

$$a(F) := \min\{a(F, \boldsymbol{v}) : \boldsymbol{v} \in N(K, F) \cap \mathbb{S}^{d-1}\} > 0,$$

and thus

$$c_9 := \min\{a(F) : F \text{ is a proper face of } K\} > 0.$$

Therefore, with $c_{10} := \min\{V_{d-1}(F) : F \text{ is a facet of } K\} > 0$, where V_{d-1} denotes the $(d-1)$-dimensional volume, we get

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, S_{d-1}(K, \mathrm{d}\boldsymbol{u}) >& \langle \boldsymbol{y} - \boldsymbol{y}', \boldsymbol{u}_{i_0} \rangle V_{d-1}(F_{i_0}) \\ \geq& \|\boldsymbol{y} - \boldsymbol{y}'\| \cdot c_9 c_{10} = c_{11} \epsilon, \end{aligned}$$

This yields (11). □

Remark. Although in the case of a general convex body K, the derivation of the estimate (9) may seem rather crude, the order of ϵ^d cannot be improved. In fact, if (9) would be replaced by $\mu(K, \varphi, \epsilon) \geq c_1 \epsilon^\alpha$ with $1 < \alpha < d$, then a counterexample would be provided by a body K which in a neighbourhood of some boundary point is congruent to a suitable part of a body of revolution with meridian curve given by $\mu(t) = |t|^r$ with $1 < r < \frac{d-1}{\alpha-1}$.

Lemma 2. *Let the convex body $K \in \mathcal{K}$ be such that a ball rolls freely inside K and K rolls freely inside a ball. Assume further that*

$$a\, \varphi \leq S_{d-1}(K, \cdot) \tag{12}$$

with some positive constant a. Then

$$\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}},\boldsymbol{u}) - h(K,\boldsymbol{u})]\,\varphi(\mathrm{d}\boldsymbol{u}) \le c_{12}\epsilon^{(d+1)/2}$$

for $\epsilon > 0$ and $\boldsymbol{y} \in \operatorname{bd} K(\epsilon)$.

Proof. Since K rolls freely in some ball, say of radius R, there is a convex body L with $K + L = RB^d$ ([7, Theorem 3.2.2]). From the polynomial expansion of $S_{d-1}(K+L,\cdot)$ ([7, (5.1.17)]) it follows that $S_{d-1}(K,\cdot) \le S_{d-1}(RB^d,\cdot) = R^{d-1}\sigma$, where σ denotes the spherical Lebesgue measure on $\mathbb{S}^{d-1}$. Let $\boldsymbol{y} \in \operatorname{bd} K(\epsilon)$. In view of (12) we get

$$\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}},\boldsymbol{u}) - h(K,\boldsymbol{u})]\,\varphi(\mathrm{d}\boldsymbol{u}) \le c_{13}\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}},\boldsymbol{u}) - h(K,\boldsymbol{u})]\,\sigma(\mathrm{d}\boldsymbol{u}).$$

Let $\boldsymbol{y} \in \operatorname{bd} K(\epsilon)$, and let $\boldsymbol{x}$ be the point in K nearest to $\boldsymbol{y}$; then $\boldsymbol{y} = \boldsymbol{x} + \epsilon N(x)$, where $N(x)$ is the outer unit normal vector of K at $\boldsymbol{x}$. By assumption, a ball, say of radius $r > 0$, rolls freely inside K. In particular, some ball B of radius r satisfies $x \in B \subset K$. Let

$$\operatorname{Cap}(\boldsymbol{y},\epsilon) := \left\{\boldsymbol{u} \in \mathbb{S}^{d-1} : \langle \boldsymbol{u}, N(\boldsymbol{x})\rangle \ge \frac{r}{r+\epsilon}\right\}.$$

For $\boldsymbol{u} \in \mathbb{S}^{d-1} \setminus \operatorname{Cap}(\boldsymbol{y},\epsilon)$ we have $h(K^{\boldsymbol{y}},\boldsymbol{u}) - h(K,\boldsymbol{u}) = 0$. If $h(K^{\boldsymbol{y}},\boldsymbol{u}) - h(K,\boldsymbol{u}) \ne 0$, then

$$h(K^{\boldsymbol{y}},\boldsymbol{u}) - h(K,\boldsymbol{u}) \le \langle \boldsymbol{y}-\boldsymbol{x},\boldsymbol{u}\rangle \le \epsilon.$$

With $\alpha(\epsilon) := \arccos r/(r+\epsilon)$ this gives

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}},\boldsymbol{u}) - h(K,\boldsymbol{u})]\,\sigma(\mathrm{d}\boldsymbol{u}) &\le \int_{\operatorname{Cap}(\boldsymbol{y},\epsilon)} \epsilon\,\sigma(\mathrm{d}\boldsymbol{u}) \\
&\le c_{13}\epsilon\sin^{d-1}\alpha(\epsilon) = c_{13}\epsilon\sqrt{1-(r/(r+\epsilon))^2}^{\,d-1} \\
&\le c_{14}\epsilon^{(d+1)/2}.
\end{aligned}$$

This yields the assertion. □

The following lemma is sufficient for our purpose; it does not aim at an optimal order.

Lemma 3. *Let $K \in \mathcal{K}$ be a convex body which rolls freely in some ball. There are constants $c_{15}, c_{16} > 0$ such that the following holds. For $0 < \epsilon < c_{15}$, let $m(\epsilon)$ be the largest number m such that there are m points in $\operatorname{bd} K(\epsilon)$ with the property that each segment connecting any two of them intersects the interior of K. Then*

$$m(\epsilon) \ge c_{16}\epsilon^{-1/2}.$$

Proof. The convex body K contains some ball, without loss of generality the ball rB^d. Let R be such that K rolls freely in a ball of radius R. We put $c_{15} := \min\{2R, (\pi r)^2/64R\}$ and assume that $0 < \epsilon < c_{15}$.

For points $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{bd} K(\epsilon)$, we assert that

$$\|\boldsymbol{x}-\boldsymbol{y}\| \ge 4\sqrt{R\epsilon} \;\Rightarrow\; [\boldsymbol{x},\boldsymbol{y}] \cap \operatorname{int} K \ne \emptyset. \tag{13}$$

For the proof, let $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{bd} K(\epsilon)$ and suppose that $[\boldsymbol{x}, \boldsymbol{y}] \cap \operatorname{int} K = \emptyset$. Let $\boldsymbol{p} \in K$ and $\boldsymbol{q} \in \operatorname{aff}\{\boldsymbol{x}, \boldsymbol{y}\}$ be points of smallest distance. Then the hyperplane H through $\boldsymbol{p}$ orthogonal to $\boldsymbol{q}-\boldsymbol{p}$ supports K. By assumption, there is a ball of radius R, say B, such that $K \subset B$ and $\boldsymbol{p} \in \operatorname{bd} B$. The ball $B + \epsilon B^d$ contains $K(\epsilon)$ and hence the segment $[\boldsymbol{x}, \boldsymbol{y}]$. The line parallel to $[\boldsymbol{x}, \boldsymbol{y}]$ through $\boldsymbol{p}$ lies in H and intersects the ball $B + \epsilon B^d$ in a segment S, which is not shorter than $[\boldsymbol{x}, \boldsymbol{y}]$. Thus, $\|\boldsymbol{x} - \boldsymbol{y}\| \leq \operatorname{length}(S) = 2\sqrt{2R\epsilon + \epsilon^2} < 4\sqrt{R\epsilon}$, since $\epsilon < 2R$. This proves (13).

Let m be the largest integer with

$$m \leq \frac{\pi r}{4\sqrt{R}}\, \epsilon^{-1/2}.$$

Then $m \geq 2$ (by the choice of c_{15}), and there is a constant c_{16} with $m \geq c_{16}/\sqrt{\epsilon}$. Let C be an arbitrary great circle of the ball rB^d. On C, we choose m equidistant points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$. For $i \neq j$ we have $\|\boldsymbol{y}_i - \boldsymbol{y}_j\| \geq 2r \sin(\pi/m) > r\pi/m$. Let $\boldsymbol{x}_i = \lambda_i \boldsymbol{y}_i \in \operatorname{bd} K(\epsilon)$ with $\lambda_i > 0$, then $\lambda_i > 1$ for $i = 1, \ldots, m$ and hence $\|\boldsymbol{x}_i - \boldsymbol{x}_j\| > r\pi/m \geq 4\sqrt{R\epsilon}$ for $i \neq j$. By (13), this completes the proof. $\square$

3 Proof of Theorem 1

We assume that X and K are as in Theorem 1 and satisfy the assumptions mentioned above, that is, φ is not concentrated on a great subsphere, $\gamma \geq 1$, and the inclusion (2) holds. Recall that $\boldsymbol{o} \in \operatorname{int} K$.

For a convex body $L \subset \mathbb{R}^d$ we define

$$\mathcal{H}_L := \{H \in \mathcal{H}^d : H \cap L \neq \emptyset\}$$

and

$$\Phi(L) := \Theta(\mathcal{H}_L).$$

By (1) we have

$$\Phi(L) = 2\gamma \int_{\mathbb{S}^{d-1}} h(L, \boldsymbol{u})\, \varphi(\mathrm{d}\boldsymbol{u}). \tag{14}$$

The following two lemmas use ideas from the proofs of Lemmas 3 and 5 in [5], but the present situation is simpler. As there, we use the abbreviation

$$H_1^- \cap \cdots \cap H_n^- =: P(H_{(n)}),$$

where $H_1, \ldots, H_n$ are hyperplanes not passing through $\boldsymbol{o}$ and H_i^- is the closed halfspace bounded by H_i that contains $\boldsymbol{o}$.

Let $\|\boldsymbol{x}\|_K = \min\{\lambda \geq 0 : \boldsymbol{x} \in \lambda K\}$ for $\boldsymbol{x} \in \mathbb{R}^d$. For a nonempty compact convex set L, we define $\|L\|_K := \max\{\|\boldsymbol{x}\|_K : \boldsymbol{x} \in L\}$. For $\epsilon \geq 0$ and $m \in \mathbb{N}$, let

$$\mathcal{K}_\epsilon(m) := \{L \in \mathcal{K}^d : K \subset L \not\subset K(\epsilon),\ \|L\|_K \in (m, m+1]\}$$

and

$$q_\epsilon(m) := \mathrm{P}\{Z_0 \in \mathcal{K}_\epsilon(m)\}.$$

We abbreviate

$$(m+1)K =: K_m.$$

We have

$$q_\epsilon(m) = \sum_{N=d+1}^{\infty} \mathrm{P}\{X(\mathcal{H}_{K_m}) = N\} p_N \tag{15}$$

with

$$\begin{aligned} p_N &:= \mathrm{P}\{Z_0 \in \mathcal{K}_\epsilon(m) \mid X(\mathcal{H}_{K_m}) = N\} \\ &= \Phi(K_m)^{-N} \int_{\mathcal{H}_{K_m}^N} \mathbf{1}\{P(H_{(N)}) \in \mathcal{K}_\epsilon(m)\}\, \Theta^N(\mathrm{d}(H_1, \dots, H_N)), \end{aligned}$$

the latter by a well-known property of Poisson processes (e.g., [8, Th. 3.2.2(b)]), and

$$\mathrm{P}\{X(\mathcal{H}_{K_m}) = N\} = \frac{\Phi(K_m)^N}{N!} \exp\left[-\Phi(K_m)\right]. \tag{16}$$

Lemma 4. *There exists a number m_0, depending only on K, φ and d, such that*

$$q_0(m) \le c_{17} \exp[-\Phi(K) - c_{18}\gamma m]$$

for $m \ge m_0$.

Proof. We modify and adapt the proof of Lemma 2 in [5]. If $H_1, \dots, H_N \in \mathcal{H}_{K_m}$ and if $P := P(H_{(N)}) \in \mathcal{K}_0(m)$, then P has a vertex $\boldsymbol{v}$ with $m < \|\boldsymbol{v}\|_K \le m+1$. Since $\boldsymbol{v}$ is the intersection of some d facets of P, there exists a d-element set $J \subset \{1, \dots, N\}$ with

$$\{\boldsymbol{v}\} = \bigcap_{j \in J} H_j.$$

We denote the segment $[\boldsymbol{o}, \boldsymbol{v}]$ by $S = S(H_i,\, i \in J)$ (where it is assumed that the hyperplanes H_i, $i \in J$, have linearly independent normal vectors) and note that

$$H_i \cap \operatorname{relint} S = \emptyset \qquad \text{for} \qquad i = 1, \dots, N.$$

For any segment $S = [\boldsymbol{o}, \boldsymbol{v}]$ with $\|\boldsymbol{v}\|_K \ge m$ we have

$$\Phi(S) = 2\gamma \int_{\mathbb{S}^{d-1}} \langle \boldsymbol{v}, \boldsymbol{u} \rangle^+ \varphi(\mathrm{d}\boldsymbol{u}) \ge 2c_{19}\gamma m$$

with a positive constant c_{19}. This follows from the fact that the function

$$\boldsymbol{v}_1 \mapsto \int_{\mathbb{S}^{d-1}} \langle \boldsymbol{v}_1, \boldsymbol{u} \rangle^+ \varphi(\mathrm{d}\boldsymbol{u}), \qquad \boldsymbol{v}_1 \in \mathbb{S}^{d-1},$$

is positive (since φ is not concentrated on a great subsphere) and continuous. Let m_0 be the smallest integer $\ge (2/c_{19}) \int_{\mathbb{S}^{d-1}} h(K, u)\, \varphi(\mathrm{d}\boldsymbol{u})$. For $m \ge m_0$ we then have

$$\Phi(S) \ge \Phi(K) + c_{19}\gamma m,$$

and hence

$$\int_{\mathcal{H}_{K_m}} \mathbf{1}\{H \cap S = \emptyset\}\, \Theta(\mathrm{d}H) = \Phi(K_m) - \Phi(S) \le \Phi(K_m) - \Phi(K) - c_{19}\gamma m,$$

where we used that $S \subset K_m$, since $\|v\|_K \le m+1$. Now we obtain

$$\begin{aligned}
p_N &\le \binom{N}{d} \Phi(K_m)^{-N} \int_{\mathcal{H}^d_{K_m}} \mathbf{1}\left\{\|S(H_j,\, j \in \{1,\dots,d\})\|_K \ge m\right\} \\
&\quad \int_{\mathcal{H}^{N-d}_{K_m}} \mathbf{1}\left\{H_i \cap S(H_j,\, j \in \{1,\dots,d\}) = \emptyset \text{ for } i = d+1,\dots,N\right\} \\
&\quad \times \Theta^{N-d}(\mathrm{d}(H_{d+1},\dots,H_N))\, \Theta^d(\mathrm{d}(H_1,\dots,H_d)) \\
&\le \binom{N}{d} \Phi(K_m)^{-N} \int_{\mathcal{H}^d_{K_m}} [\Phi(K_m) - \Phi(K) - c_{19}\gamma m]^{N-d}\, \Theta^d(\mathrm{d}(H_1,\dots,H_d)) \\
&= \binom{N}{d} \Phi(K_m)^{d-N} [\Phi(K_m) - \Phi(K) - c_{19}\gamma m]^{N-d}.
\end{aligned}$$

With (15) (for $\epsilon = 0$) and (16) this gives

$$\begin{aligned}
&q_0(m) \\
&\le \sum_{N=d+1}^{\infty} \frac{\Phi(K_m)^N}{N!} \exp[-\Phi(K_m)] \binom{N}{d} \Phi(K_m)^{d-N} [\Phi(K_m) - \Phi(K) - c_{19}\gamma m]^{N-d} \\
&= \frac{1}{d!} \Phi(K_m)^d \exp[-\Phi(K_m)] \sum_{N=d+1}^{\infty} \frac{1}{(N-d)!} [\Phi(K_m) - \Phi(K) - c_{19}\gamma m]^{N-d} \\
&\le \frac{1}{d!} \Phi(K_m)^d \exp[-\Phi(K) - c_{19}\gamma m] \\
&= \frac{1}{d!} \left(2\gamma(m+1) \int_{\mathbb{S}^{d-1}} h(K,\boldsymbol{u})\, \varphi(\mathrm{d}\boldsymbol{u})\right)^d \exp[-\Phi(K) - c_{19}\gamma m] \\
&\le c_{17} \exp[-\Phi(K) - c_{18}\gamma m]
\end{aligned}$$

with $c_{18} = c_{19}/2$, say. □

According to (14) and (3), we have

$$\Phi(K^{\boldsymbol{y}}) - \Phi(K) \ge 2\gamma\mu(K,\varphi,\epsilon) \tag{17}$$

for $0 < \epsilon \le 1$ and $\boldsymbol{y} \in \operatorname{bd} K(\epsilon)$, if (2) is satisfied. In the following lemma, we assume that (17) holds.

Lemma 5. *Let* $0 < \epsilon \le 1$ *and suppose that* (17) *holds whenever* $\boldsymbol{y} \in \operatorname{bd} K(\epsilon)$. Then, for $m \in \mathbb{N}$,

$$q_\epsilon(m) \le c_{20}(\gamma m)^d \exp[-\Phi(K) - 2\gamma\mu(K,\varphi,\epsilon)].$$

Proof. Suppose that $H_1,\dots,H_N \in \mathcal{H}_{K_m}$ are such that $P := P(H_{(N)}) \in \mathcal{K}_\epsilon(m)$. Then P has a vertex $\boldsymbol{x} \in K_m \setminus K(\epsilon)$. This vertex is the intersection of d facets of P. Hence, there exists an index set $J \subset \{1,\dots,N\}$ with d elements such that

$$\{\boldsymbol{x}\} = \bigcap_{j \in J} H_j.$$

There exists a point $\boldsymbol{y} \in \operatorname{bd} K(\epsilon)$ such that

$$\Phi(\operatorname{conv}(K \cup \{\boldsymbol{x}\}) \geq \Phi(\operatorname{conv}(K \cup \{\boldsymbol{y}\}) = \Phi(K^{\boldsymbol{y}}) \geq \Phi(K) + 2\gamma\mu(K, \varphi, \epsilon),$$

where (17) was used, together with the monotonicity of Φ. This gives

$$\begin{aligned} \int_{\mathcal{H}_{K_m}} \mathbf{1}\{H \cap \operatorname{conv}(K \cup \{\boldsymbol{x}\}) = \emptyset\} \, \Theta(\mathrm{d}H) &= \Phi(K_m) - \Phi(\operatorname{conv}(K \cup \{\boldsymbol{x}\}) \\ &\leq \Phi(K_m) - \Phi(K) - 2\gamma\mu(K, \varphi, \epsilon). \end{aligned}$$

We write $\boldsymbol{x} = \boldsymbol{x}(H_1, \ldots, H_d)$ for the intersection point of the hyperplanes $H_1, \ldots, H_d$ (supposed in general position) and obtain

$$\begin{aligned} p_N &\leq \binom{N}{d} \Phi(K_m)^{-N} \int_{\mathcal{H}_{K_m}^d} \mathbf{1}\{\boldsymbol{x}(H_1, \ldots, H_d) \in K_m \setminus K(\epsilon)\} \\ &\quad \int_{\mathcal{H}_{K_m}^{N-d}} \mathbf{1}\{H_i \cap \operatorname{conv}(K \cup \{\boldsymbol{x}(H_1, \ldots, H_d)\}) = \emptyset \text{ for } i = d+1, \ldots, N\} \\ &\quad \times \Theta^{N-d}(\mathrm{d}(H_{d+1}, \ldots, H_N)) \, \Theta^d(\mathrm{d}(H_1 \ldots, H_d)) \\ &\leq \binom{N}{d} \Phi(K_m)^{d-N} \left[\Phi(K_m) - \Phi(K) - 2\gamma\mu(K, \varphi, \epsilon)\right]^{N-d}. \end{aligned}$$

Similarly as in the proof of Lemma 2, summation over N gives

$$q_\epsilon(m) \leq \frac{1}{d!} \Phi(K_m)^d \exp\left[-\Phi(K) - 2\gamma\mu(K, \varphi, \epsilon)\right] \leq c_{20}(\gamma m)^d \exp\left[-\Phi(K) - 2\gamma\mu(K, \varphi, \epsilon)\right].$$

□

Proof of Theorem 1. We have

$$\mathrm{P}\{Z_0 \not\subset K(\epsilon) \mid K \subset Z_0\} = \frac{\mathrm{P}\{K \subset Z_0, \ Z_0 \not\subset K(\epsilon)\}}{\mathrm{P}\{K \subset Z_0\}} = \frac{\sum_{m=1}^{\infty} q_\epsilon(m)}{\exp\left[-\Phi(K)\right]}.$$

To estimate the last numerator, we choose m_0 according to Lemma 4 and use Lemma 5 for $m \leq m_0$ and Lemma 4 together with $q_\epsilon(m) \leq q_0(m)$ for $m > m_0$. Since assumption (17) is satisfied, Lemma 5 can be applied, and we obtain

$$\mathrm{P}\{Z_0 \not\subset K(\epsilon) \mid K \subset Z_0\} \leq \sum_{m=1}^{m_0} c_{20}(\gamma m)^d \exp[-2\gamma\mu(K, \varphi, \epsilon)] + \sum_{m > m_0} c_{17} \exp[-c_{18}\gamma m].$$

The first sum can be estimated by

$$\begin{aligned} &\sum_{m=1}^{m_0} c_{20}(\gamma m)^d \exp[-2\gamma\mu(K, \varphi, \epsilon)] \qquad (18) \\ &\leq c_{20} m_0^{d+1} \gamma^d \exp\left[-\gamma\mu(K, \varphi, \epsilon)\right] \exp\left[-\gamma\mu(K, \varphi, \epsilon)\right] \\ &\leq c_{21}(\epsilon) \exp\left[-c_{22}\gamma\mu(K, \varphi, \epsilon)\right]. \end{aligned}$$

The second sum can be estimated by

$$\sum_{m > m_0} c_{17} \exp[-c_{18}\gamma m] \leq c_{17} \exp[-c_{18}\gamma] \sum_{m > m_0} \exp[-c_{18}(m-1)] \leq c_{23} \exp\left[-c_{18}\gamma\right],$$

since $\gamma \geq 1$ (by assumption) and the last sum converges. Both estimates together yield (4). □

4 Proofs of Theorems 2 and 3

Under the stronger assumption (5), we can conclude from Lemma 1 that $\mu(K,\varphi,\epsilon) \geq c_{24}\epsilon^{\alpha}$ with some $\alpha \leq d$. Therefore, in estimating (18) we can use that

$$\gamma^d \exp\left[-\gamma\mu(K,\varphi,\epsilon)\right] \leq \gamma^d \exp\left(-\gamma c_{24}\epsilon^{\alpha}\right) \leq c_{25}\epsilon^{-d\alpha}.$$

This gives

$$\sum_{m=1}^{m_0} c_{20}(\gamma m)^d \exp[-2\gamma\mu(K,\varphi,\epsilon)] \leq c_{26}\epsilon^{-d\alpha} \exp\left(-c_{27}\gamma\epsilon^{\alpha}\right).$$

The estimation of the second sum above remains unchanged. Hence, under the assumptions of Theorem 2 and with $\gamma = n$, we can conclude that

$$\mathrm{P}\left\{\delta(K, Z_K^{(n)}) > \epsilon\right\} \leq c_{28}\epsilon^{-d\alpha} \exp\left(-c_{29}n\epsilon^{\alpha}\right).$$

We choose

$$C > \frac{d+1}{c_{29}}$$

and put

$$\epsilon_n := \left(\frac{C\log n}{n}\right)^{1/\alpha}.$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty} \mathrm{P}\left\{\delta(K, Z_K^{(n)}) > \epsilon_n\right\} &\leq \sum_{n=1}^{\infty} c_{28}\left(\frac{n}{C\log n}\right)^d \exp\left(-c_{29}C\log n\right) \\ &= c_{30}\sum_{n=1}^{\infty} (\log n)^{-d} n^{d-c_{29}C} < \infty. \end{aligned} \qquad (19)$$

The Borel–Cantelli lemma gives

$$\mathrm{P}\left\{\delta(K, Z_K^{(n)}) > \epsilon_n \text{ for infinitely many } n\right\} = 0,$$

hence

$$\mathrm{P}\left\{\delta(K, Z_K^{(n)}) \leq \left(\frac{C\log n}{n}\right)^{1/\alpha} \text{ for sufficiently large } n\right\} = 1.$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. Let $0 < \epsilon < c_{15}$ (with c_{15} as in Lemma 3). According to Lemma 3, we can choose

$$m = m(\epsilon) \geq c_{16}\epsilon^{-1/2}$$

points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in \mathrm{bd}\, K(\epsilon)$ such that the segment joining any two of them intersects the interior of K. Let $n \in \mathbb{N}$. Suppose that $\delta(K, Z_K^{(n)}) < \epsilon$. Then each point $\boldsymbol{x}_i$ is separated from K by some hyperplane from X_n. Let $\mathcal{A}_i \subset \mathcal{H}^d$ be the set of hyperplanes separating $\boldsymbol{x}_i$ and K. By the choice of the points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$, the sets $\mathcal{A}_1, \dots, \mathcal{A}_m$ are pairwise disjoint. Since

X_n is a Poisson process, the processes $X_n \llcorner \mathcal{A}_1, \dots, X_n \llcorner \mathcal{A}_m$ are stochastically independent (e.g., [8, Theorem 3.2.2]). It follows that

$$\begin{aligned} \mathrm{P}\{\delta(K, Z_K^{(n)}) < \epsilon\} &\leq \mathrm{P}\{X_n(\mathcal{A}_i) \geq 1 \text{ for } i = 1, \dots, m\} \\ &= \prod_{i=1}^m \mathrm{P}\{X_n(\mathcal{A}_i) \geq 1\} = \prod_{i=1}^m [1 - \mathrm{P}\{X_n(\mathcal{A}_i) = 0\}] \\ &= \prod_{i=1}^m (1 - \exp[-\Theta_n(\mathcal{A}_i)]) \end{aligned}$$

where Θ_n is the intensity measure of X_n. Since the assumptions on K in Lemma 2 are satisfied, we can conclude that

$$\begin{aligned} \Theta_n(\mathcal{A}_i) &= \Theta_n(\mathcal{H}_{K^{\boldsymbol{x}_i}}) - \Theta_n(\mathcal{H}_K) \\ &= 2n \int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{x}_i}, \boldsymbol{u}) - h(K, \boldsymbol{u})] \, \varphi(\mathrm{d}\boldsymbol{u}) \\ &\leq 2nc_{12}\epsilon^{(d+1)/2}. \end{aligned}$$

This gives

$$\mathrm{P}\left\{\delta(K, Z_K^{(n)}) < \epsilon\right\} \leq \left[1 - \exp\left(-2c_{12}n\epsilon^{(d+1)/2}\right)\right]^{m(\epsilon)}.$$

Now we choose

$$\epsilon_n^{(d+1)/2} = \frac{c \log n}{n}$$

with

$$0 < c < \frac{1}{4c_{12}(d+1)}.$$

Then

$$\mathrm{P}\left\{\delta(K, Z_K^{(n)}) < \epsilon_n\right\} \leq \left(1 - n^{-2c_{12}c}\right)^{m(\epsilon_n)}$$

with

$$m(\epsilon_n) \geq c_{16}\epsilon_n^{-1/2} = c_{16}\left(\frac{n}{c \log n}\right)^{1/(d+1)} > c_{31}n^{1/(2d+2)}$$

for sufficiently large n. With $p := 2c_{12}c$ and $q := 1/(2d+2)$ we have $q > p$ and

$$\left(1 - n^{-2c_{12}c}\right)^{m(\epsilon_n)} < \left(1 - \frac{1}{n^p}\right)^{c_{31}n^q} = \left[\left(1 - \frac{1}{n^p}\right)^{n^p \cdot n^{q-p}}\right]^{c_{31}} \leq (\mathrm{e}^{-c_{31}})^{n^{q-p}}.$$

It follows that

$$\sum_{n=1}^{\infty} \mathrm{P}\left\{\delta(K, Z_K^{(n)}) < \left(\frac{c \log n}{n}\right)^{\frac{2}{d+1}}\right\} < \infty.$$

From the Borel–Cantelli lemma we conclude that

$$\mathrm{P}\left\{\delta(K, Z_K^{(n)}) < \left(\frac{c \log n}{n}\right)^{\frac{2}{d+1}} \text{ for infinitely many } n\right\} = 0$$

and hence

$$\mathrm{P}\left\{\delta(K, Z_K^{(n)}) \geq \left(\frac{c \log n}{n}\right)^{\frac{2}{d+1}} \text{ for almost all } n\right\} = 1.$$

Together with Theorem 2, this completes the proof of Theorem 3. □

References

[1] Bárány, I., Intrinsic volumes and f-vectors of random polytopes. *Math. Ann.* **285** (1989), 671–699.

[2] Böröczky, K. J., and Schneider, R., The mean width of circumscribed random polytopes. *Canad. Math. Bull.* **53** (2010), 614–628.

[3] Dümbgen, L. and Walther, G., Rates of convergence for random approximations of convex sets. *Adv. Appl. Prob. (SGSA)* **28** (1996), 384–393.

[4] Fáry, I. and Rédei, L., Der zentralsymmetrische Kern und die zentralsymmetrische Hülle von konvexen Körpern. *Math. Ann.* **122** (1950), 205–220.

[5] Hug, D. and Schneider, R., Asymptotic shapes of large cells in random tessellations. *Geom. Funct. Anal.* **17** (2007), 156–191.

[6] Reitzner, M., Random polytopes. In *New Perspectives in Stochastic Geometry* (eds. W. S. Kendall, I. Molchanov), pp. 45–76, Oxford University Press 2010.

[7] Schneider, R., *Convex Bodies – the Brunn–Minkowski Theory.* Cambridge University Press, Cambridge 1993.

[8] Schneider, R. and Weil, W., *Stochastic and Integral Geometry.* Springer, Berlin 2008.

[9] Werner, E., Illumination bodies and affine surface area. *Studia Math.* **110** (1994), 257–269.