# Flag measures for convex bodies

Daniel Hug*, Ines Türk and Wolfgang Weil

Karlsruhe Institute of Technology, Department of Mathematics

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**Abstract**

Measures on flag manifolds have been recently used to describe local properties of convex bodies and more general sets in $\mathbb{R}^d$. Here, we provide a systematic account of flag measures for convex bodies, we collect various properties of flag measures and we prove some new results. In particular, we discuss mixed flag measures for several bodies and we present formulas for (mixed) flag measures of generalized zonoids.

## 1 Introduction (with historical remarks)

The classical Brunn-Minkowski theory of convex geometry is based on the notion of mixed volumes. Special cases of these multilinear expressions lead to the basic functionals of compact convex sets (convex bodies), the intrinsic volumes (quermassintegrals, Minkowski functionals) $V_j(K)$, $j = 0, \dots, d$, of a convex body $K$ in $\mathbb{R}^d$. Already in the work leading to the celebrated Alexandrov-Fenchel inequalities, local variants of the intrinsic volumes, the area measures $\Psi_j(K, \cdot)$, $j = 0, \dots, d-1$, and their mixed versions played an important role. The area measures are finite Borel measures on the unit sphere $S^{d-1}$. In the full dimensional case, they describe the convex body $K$ uniquely, up to translations. In the case of polytopes and for $j = d-1$ this fact goes back even to Minkowski. As a counterpart motivated by differential geometry, curvature measures $\Phi_j(K, \cdot)$, $j = 0, \dots, d-1$, were introduced and studied by Federer in his seminal paper [6]. The curvature measures also describe $K$ uniquely, but sit on the boundary $\operatorname{bd} K$ of $K$. The two sequences of measures can be unified in the notion of support measures $\Xi_j(K, \cdot)$, which are measures on $\mathbb{R}^d \times S^{d-1}$, concentrated on the *generalized normal bundle*

$$\operatorname{Nor} K = \{(x, u) : x \in \operatorname{bd} K, u \in S^{d-1} \text{ an outer unit normal at } x\}.$$

We refer to Schneider [27] for an excellent survey on the Brunn-Minkowski theory and for background information on most notions and results from convex geometry which we use here and in the sequel.

A fundamental relation between global and local functionals is the integral formula for the special mixed volume

$$V(K[1], M[d-1]) = \frac{2}{d} \int_{S^{d-1}} h(K, u)\, \Psi_{d-1}(M, du) \tag{1}$$

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which holds true for arbitrary convex bodies $K, M$ and involves the support function $h(K, \cdot)$ of $K$. This relation was generalized to mixed volumes

$$V(K[m], M[d-m]) = \frac{2^{d-m} m!}{d!} \int_{G(d,d-m)} V_m(K|E^\perp)\, \rho_{d-m}(M, dE), \tag{2}$$

for $m \in \{0, \dots, d-1\}$, provided $M$ is centrally symmetric and smooth (differentiable of a sufficiently high order), whereas $K$ may be arbitrary. Here, $G(d, d-m)$ is the Grassmannian of $(d-m)$-dimensional linear subspaces of $\mathbb{R}^d$, $V_m(K|E^\perp)$ is the $m$-dimensional volume of the orthogonal projection of $K$ on the orthogonal complement $E^\perp \in G(d,m)$ of $E \in G(d, d-m)$, and the measure $\rho_{d-m}(M, \cdot)$ is the $(d-m)$th projection generating measure of $M$, normalized as in [11, p. 1315]. If both bodies, $K$ and $M$, are centrally symmetric and smooth (differentiable of a sufficiently high order), then the mixed volume $V(K[m], M[d-m])$ can be expressed in a symmetric way as

$$V(K[m], M[d-m]) = \frac{2^d}{d!} \int_{G(d,d-m)} \int_{G(d,m)} |\langle E, F^\perp \rangle|\, \rho_m(K, \mathrm{d}F)\, \rho_{d-m}(M, \mathrm{d}E), \tag{3}$$

where $|\langle E, F^\perp \rangle| = |\langle E^\perp, F \rangle|$ denotes the absolute value of the determinant of the orthogonal projection of $E$ on $F^\perp$.

It is known that (2) and (3) remain true without smoothness assumptions on $M$ (resp. $K$ and $M$), if distributions are used. See the introduction of [15], for remarks on the literature. However, the central symmetry of $M$ (or $K$ and $M$) seems to be essential, as long as integrals over Grassmannians are considered. In fact, a generalization of (3) to arbitrary bodies $K, M$ was recently obtained in [15], which uses measures on certain flag manifolds. This is strong evidence that flag measures and other flag-type notions can play an important role in the further development of modern Brunn-Minkowski theory.

The first appearance of a flag-type result in convex geometry seems to be in a paper by Firey [8]. He showed that the area measure $\Psi_{d-1-j}(K, A)$ of a convex body $K$ evaluated at a Borel set $A \subset S^{d-1}$ can be interpreted as the natural measure of $j$-flats touching $K$ in boundary points $x$ with normal directions $u \in A$. Schneider [24] provided a shorter proof of this result and in [25] he gave a corresponding interpretation of the curvature measure $\Phi_{d-1-j}(K, B)$ of a convex body $K$ at a Borel set $B \subset \mathbb{R}^d$. It measures the set of $j$-flats touching $K$ in boundary points $x \in B$. For these integral-geometric interpretations of area and curvature measures, a measure theoretic foundation was given in [32] by providing, for each $j \in \{0, \dots, d-1\}$, a finite Borel measure on the space of $j$-flats touching a convex body $K$, the measure being obtained by a disintegration of the Haar measure on the affine Grassmannian $A(d, j)$ of affine $j$-flats in $\mathbb{R}^d$. The results can be reformulated in terms of flats which touch a convex body randomly and are also strongly related to local formulas in integral geometry. The study of collision probabilities (touching probabilities) was continued by Firey and others in a variety of similar situations, for example to a moving convex body $M$ randomly touching a fixed convex body $K$. Surveys on the subsequent development can be found in [33], [30] and [29, Section 8.5]. The measure constructed in [32] can be seen as a measure $\tilde{\Phi}^{(j)}_{d-j-1}(K, \cdot)$ on pairs $(x, L)$, where $x$ is a boundary point of $K$ and $L \in G(d, j)$ such that the affine $j$-flat $x + L$ touches $K$. In a diploma thesis, Kropp [17] introduced and studied corresponding measures $\tilde{\Psi}^{(j)}_{d-j-1}(K, \cdot)$ on the flag manifold of pairs $(u, L) \in S^{d-1} \times G(d, j)$ such that $u \perp L$ (that is, $u$ is orthogonal to $L$). For bodies $K$, which are strictly convex and have unique support planes, the measures $\tilde{\Phi}^{(j)}_{d-j-1}(K, \cdot)$ and $\tilde{\Psi}^{(j)}_{d-j-1}(K, \cdot)$ are image measures of each other under the Gauss map $x \mapsto u(x)$, where $u(x)$ denotes the (outer) unit normal in $x \in \operatorname{bd} K$, respectively its inverse. The curvature and area measures appear as projection images,

$$\Phi_{d-j-1}(K, \cdot) = \tilde{\Phi}^{(j)}_{d-j-1}(K, \cdot \times G(d, j)), \qquad \Psi_{d-j-1}(K, \cdot) = \tilde{\Psi}^{(j)}_{d-j-1}(K, \cdot \times G(d, j)),$$

for $j = 0, \dots, d-1$, and the intrinsic volumes equal the total measures,

$$V_{d-j-1}(K) = \tilde{\Phi}^{(j)}_{d-j-1}(K, S^{d-1} \times G(d,j)) = \tilde{\Psi}^{(j)}_{d-j-1}(K, \operatorname{bd} K \times G(d,j)).$$

Kropp [17] also introduced flag-type support measures $\tilde{\Xi}^{(j)}_{d-j-1}(K,\cdot)$ which are concentrated on the manifold of triples $(x, u, L)$, where $x \in \operatorname{bd} K$, $u$ is a unit normal of $K$ at $x$, $L \in G(d,j)$ and $u \perp L$, and thus he unified the measures $\tilde{\Phi}^{(j)}_{d-j-1}(K,\cdot)$ and $\tilde{\Psi}^{(j)}_{d-j-1}(K,\cdot)$. The set of all such triples $(x, u, L)$ is denoted by $\operatorname{Nor}_j K$ and can be used to parametrize $j$-flats touching $K$.

In the following, it will be convenient to use the different normalization $\Xi^{(j)}_{d-j-1}(K,\cdot)$ of the flag-type support measures and of the corresponding flag area and flag curvature measures, then denoted by $\Psi^{(j)}_{d-1-j}(K,\cdot)$ and $\Phi^{(j)}_{d-1-j}(K,\cdot)$ (see equation (25) for the reason of this normalization). The two normalizations are related by

$$\tilde{\Xi}^{(j)}_{d-j-1}(K,\cdot) = c(d,j) \cdot \Xi^{(j)}_{d-j-1}(K,\cdot) \tag{4}$$

with

$$c(d,j) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{j+2}{2}\right)\Gamma\left(\frac{d-j+1}{2}\right)} = \binom{d-1}{j} \frac{\Gamma\left(\frac{d-j}{2}\right)\Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1}{2}\right)},$$

where the second equality follows from Legendre's duplication formula. Note the relations $c(d,0) = c(d,d-1) = 1$ and $c(d,j) = c(d,d-j-1)$.

The constructions in [32] and [17] were based on projection formulas for curvature and area measures, respectively. In his PhD thesis, Hinderer [12] studied the flag measures $\Xi^{(j)}_{d-j-1}(K,\cdot)$ more systematically, starting from a local Steiner-type formula. As a main result, he showed that the projection function

$$L \mapsto V_j(P|L),$$

for $j = 1, \dots, d-2$, a polytope $P$ and $L \in G(d,j)$, such that $P$ and $L$ are in general relative position, can be expressed as an integral over flag measures (the case $j = d-1$ is a classical consequence of (1)). The results of [12] build the basis of the recent papers [13] and [10].

Flag measures were also used by Ambartzumian [1, 2] in his $\sin^2$-representation of the width function of convex bodies in $\mathbb{R}^3$. He introduced a flag measure, different from the one mentioned above, for polytopes and extended it to arbitrary bodies in $\mathbb{R}^3$ by a compactness argument. However, as Hinderer [12] showed, this extension is not continuous and thus lacks an important property which the measure $\Xi^{(j)}_{d-j-1}(K,\cdot)$ has.

From a different point of view, flag-type measures for sets of positive reach were constructed on the natural generalization $\operatorname{Nor}_j K$ of the normal bundle $\operatorname{Nor} K$, as integrals with respect to the corresponding Hausdorff measures. This development started with Zähle's [35] integral representation of (signed) support measures, then called (Lipschitz-Killing) curvature measures, for sets with positive reach. Seizing a suggestion in [6], M. Zähle introduced *absolute curvature measures*, corresponding to $\Xi^{(j)}_{d-1-j}(K, \cdot \times G(d,j))$, for sets with positive reach, as non-negative measures on $\mathbb{R}^d \times S^{d-1}$. These measures are introduced in [36], and further studied in [22], as mean projection measures with the interpretation as the measure of $j$-flats locally colliding with a set of positive reach in a given subset of $\mathbb{R}^d \times S^{d-1}$. This description admits a comparison of the total absolute curvature measures to corresponding quantities in the setting of smooth $(d-1)$-dimensional submanifolds that have been considered before by Santaló [23] and Baddeley [4]. For convex sets, the absolute curvature measures, as measures on $\mathbb{R}^d \times S^{d-1}$, coincide with the support measures, but they differ from these and from the total variation measures of support measures for sets with positive reach.

In [36, 22], absolute curvature measures are also described as integrals over $\operatorname{Nor}_j K$, for a set $K$ of positive reach, involving generalized (signed) curvature functions on $\operatorname{Nor} K$. At a given point

$(x,u,V) \in \mathrm{Nor}_j K$, the definition involves sums of products of generalized curvatures of $K$ at $(x,u)$, and these products are weighted with a quantity that depends on the relative position of the linear subspace $V$ and the directions of curvature. In these papers, some basic properties of absolute curvature measures, corresponding to $\Xi_{d-1-j}^{(j)}(K, \cdot \times G(d,j))$ in the present notation, are explored. In particular, for sets of the form $B \times S^{d-1}$, with a Borel sets $B \subset \mathbb{R}^d$, a Crofton formula is established. A more general translative and a kinematic Crofton formula, for certain unions of sets with positive reach, have been obtained by Rataj [20, 21].

In the present context, for each $j \in \{0, \dots, d-1\}$, we investigate flag support measures $\Xi_r^{(j)}(K, \cdot)$, for all $r = 0, \dots, d-1-j$, as measures on the product space $\mathbb{R}^d \times S^{d-1} \times G(d,j)$. On a more technical level, a brief measure geometric description of $\Xi_{d-1-j}^{(j)}(K, \cdot)$ for convex bodies $K$ is provided in [15]. An extension of the measure geometric approach to all measures $\Xi_r^{(j)}(K, \cdot)$ and an analogous investigation for convex functions is initiated in [5].

In the following, we study flag measures for convex bodies systematically, we compare the different approaches leading to flag measures and we collect various results which can be obtained in analogy to the well-known theory of curvature and area measures. We include proofs where it is convenient or where results are new, but refer to the literature when recent publications are available. We start with a section which collects the necessary notations and we recall some classical results for support measures. Then we introduce flag measures as coefficients in a local Steiner formula in Section 3. In Section 4 we describe an approach to flag measures via projection averages and Section 5 contains various properties and extensions of flag measures, including mixed flag measures. In Section 6, we present formulas for (mixed) flag measures of generalized zonoids, a well-known class of centrally symmetric bodies, and in the final Section 7 we mention shortly some recent applications of flag measures to projection functions, mixed volumes and translation invariant valuations of convex bodies.

## 2 Notations and classical results

We work in Euclidean space $\mathbb{R}^d$, $d \geq 2$, with (standard) scalar product $\langle \cdot, \cdot \rangle$ and Euclidean norm $\| \cdot \|$. The closed unit ball in $\mathbb{R}^d$ is denoted by $B^d$, and the unit sphere by $S^{d-1}$. We use $\lambda_d$ for the $d$-dimensional Lebesgue measure and $\mathcal{H}^j$ for the $j$-dimensional Hausdorff measure. Let $G(d,k)$ be the Grassmannian of $k$-dimensional linear subspaces of $\mathbb{R}^d$, supplied with its invariant probability measure $\nu_k$, and let $A(d,k)$ be the corresponding affine Grassmannian with invariant measure $\mu_k$. The latter is normalized as in [29]. Both spaces, $G(d,k)$ and $A(d,k)$, carry the usual Fell-Matheron topology. For a topological space $X$, we let $\mathcal{B}(X)$ be the $\sigma$-algebra of Borel sets in $X$, and we denote by $\mathrm{bd}\, A$, $\mathrm{relbd}\, A$, $\mathrm{int}\, A$ and $\mathrm{relint}\, A$ the boundary, relative boundary, interior and relative interior of a set $A$. For non-empty, closed sets $A, B \subset \mathbb{R}^d$, the projection point $p(A,B)$ is the point in $A$ closest to $B$, provided this point exists and is unique. We write $p(A,x)$ if $B = \{x\}$. The standard set class to be used is the class $\mathcal{K}$ of convex bodies (non-empty, compact convex sets in $\mathbb{R}^d$). We endow $\mathcal{K}$ with the Hausdorff metric. For a convex body $K$ and a vector $u \neq o$, the support function of $K$ at $u$ is $h(K,u)$, whereas the support set of $K$ in direction $u$ is denoted by $F(K,u)$ (independent of the length of $u$). For convex sets we frequently use the volume functional $V_d$ instead of $\lambda_d$. The constants $\kappa_d = V_d(B^d)$ and $\omega_d = d\kappa_d$ will often appear in formulas.

Coming now to the more special notions, we start with the *intrinsic volumes* $V_j, j = 0, \dots, d-1$, which are conveniently defined through the Steiner formula

$$V_d((K + \rho B^d) \setminus K) = \sum_{j=0}^{d-1} \rho^{d-j} \kappa_{d-j} V_j(K), \tag{5}$$

where $K \in \mathcal{K}$ and $\rho > 0$.

The *support measures* $\Xi_j(K, \cdot)$, $j = 0, \dots, d-1$, of a convex body $K \in \mathcal{K}$ can be introduced by a local version of (5). In the following presentation, we follow Schneider [27, Chapter 4]. Let

$$d(A, x) = \min\{\|x - y\| \; : \; y \in A\}$$

be the distance of a point $x \in \mathbb{R}^d$ to a closed set $A \subset \mathbb{R}^d$ and let $p(K, \cdot) : \mathbb{R}^d \to K$, for $K \in \mathcal{K}$ be the *metric projection* onto $K$, that is, $p(K, x)$ is the (in this case) unique projection point in $K$ closest to $x$. For $x \notin K$, the direction from $p(K, x)$ to $x$ is denoted by

$$u(K, x) = \frac{x - p(K, x)}{\|x - p(K, x)\|}.$$

The vector $u(K, x)$ is an outer normal of $K$ at $p(K, x)$. Therefore, $(p(K, x), u(K, x)) \in \operatorname{Nor} K$ is called a *support element* of $K$. For $K \in \mathcal{K}$, a Borel set $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1})$ and $\rho > 0$, we define the *local parallel set* $M_\rho(K, \eta)$ by

$$M_\rho(K, \eta) = \{x \in \mathbb{R}^d \; : \; 0 < d(K, x) \le \rho, \; (p(K, x), u(K, x)) \in \eta\}.$$

This is a Borel set and $\eta \mapsto \mu_\rho(K, \eta) = \lambda_d(M_\rho(K, \eta))$ is a finite Borel measure which satisfies the following *local Steiner formula*.

**Theorem 2.1. [27, Theorem 4.2.1]** *For $K \in \mathcal{K}$, there are finite (positive) Borel measures $\Xi_0(K, \cdot), \dots, \Xi_{d-1}(K, \cdot)$ on $\mathbb{R}^d \times S^{d-1}$ such that, for all $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1})$ and $\rho > 0$,*

$$\mu_\rho(K, \cdot) = \sum_{j=0}^{d-1} \rho^{d-j} \kappa_{d-j} \, \Xi_j(K, \cdot). \tag{6}$$

*The measure $\Xi_j(K, \cdot)$, $j \in \{0, \dots, d-1\}$, is concentrated on* $\operatorname{Nor} K$.

*For $j = 0, \dots, d-1$, the mapping $K \mapsto \Xi_j(K, \cdot)$ is weakly continuous and $K \mapsto \Xi_j(K, \eta)$ is measurable, for each $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1})$.*

For a polytope $P \in \mathcal{K}$, we have a specific representation of the support measures,

$$\Xi_j(P, \cdot) = \frac{1}{\omega_{d-j}} \sum_{F \in \mathcal{F}_j(P)} \int_F \int_{n(P,F)} \mathbf{1}\{(y, u) \in \cdot\} \, \mathcal{H}^{d-1-j}(du) \, \mathcal{H}^j(dy), \tag{7}$$

for $j = 0, \dots, d-1$ (see [27, (4.2.2)]). Here, $\mathcal{F}_j(P)$ is the set of $j$-faces of $P$ and $n(P, F) = n(P, x)$ is the set of unit vectors in the normal cone of $P$ at $x$, where the point $x \in \operatorname{relint} F$ is arbitrary.

For a general convex body $K \in \mathcal{K}$, the coefficient measures $\Xi_0(K, \cdot), \dots, \Xi_{d-1}(K, \cdot)$ in (6) are the *support measures* of $K$. If $\eta \supset \operatorname{Nor} K$ is measurable, then (6) turns into (5), hence the total measures equal the intrinsic volumes,

$$\Xi_j(K, \mathbb{R}^d \times S^{d-1}) = V_j(K),$$

for $j = 0, \dots, d-1$. The classical *area measures* $\Psi_j(K, \cdot)$ appear now as image measures of the support measures under projection onto the second component,

$$\Psi_j(K, \cdot) = \Xi_j(K, \mathbb{R}^d \times \cdot),$$

and the *curvature measures* $\Phi_j(K, \cdot)$ are the image measures under projection onto the first component,

$$\Phi_j(K, \cdot) = \Xi_j(K, \cdot \times S^{d-1}).$$

For polytopes, (7) implies corresponding representations for curvature and area measures.

An important additional property of support measures concerns the behavior under rigid motions $g \in G_d$. If $g_0$ denotes the rotational part of $g$ and if $g\eta = \{(gx, g_0u) : (x, u) \in \eta\}$, then

$$\Xi_j(gK, g\eta) = \Xi_j(K, \eta).$$

In particular, the curvature measures are translation covariant, whereas the area measures are translation invariant. Another important property of $\Xi_j(K, \cdot)$ is its *homogeneity*, which can be expressed by

$$\Xi_j(\alpha K, \alpha\eta) = \alpha^j \Xi_j(K, \eta),$$

for $\alpha > 0$, where $\alpha\eta = \{(\alpha x, u) : (x, u) \in \eta\}$.

For different normalizations of support measures, area measures and curvature measures, see [27] and [29]. We also mention that the support measures are *locally defined*; for a given Borel set $\eta \subset \mathbb{R}^d \times S^{d-1}$ they only depend on an (arbitrarily) small neighborhood of the boundary points $x$ with $(x, u) \in \eta$ (see [27], for more details).

Since support measures arise from a local examination (expansion) of the sum set $K + \rho B^d$, one can expect that support measures can also be expanded, for general combinations $\rho_1 K_1 + \cdots + \rho_k K_k$, where $K_1, \ldots, K_k \in \mathcal{K}$ and $\rho_i \geq 0$. In fact, under suitable additional assumptions, the measure $\Xi_j(\rho_1 K_1 + \cdots + \rho_k K_k, \cdot)$ allows a multilinear expansion. For the area measures this is classical, for the curvature measures it was proved in [16] (see also [9] and [14], for special cases). Since the support measure $\Xi_j(K, \cdot)$ corresponds to the mixed curvature measure $\Phi_{j,d-j}(K, B^d, \cdot)$ (cf. [16]), the following multilinear expansion of support measures is a consequence of formula (5.16) in [16]. It holds for convex bodies $K_1, \ldots, K_k$ in *general relative position*. This condition requires that, for each direction $u \in S^{d-1}$, the support set $F(K_1 + \cdots + K_k, u) = F(K_1, u) + \cdots + F(K_k, u)$ satisfies $\dim F(K_1 + \cdots + K_k, u) = \dim F(K_1, u) + \cdots + \dim F(K_k, u)$. It is fulfilled, for example, if $K_1, \ldots, K_k$ are strictly convex. Let $K_1, \ldots, K_k \in \mathcal{K}$ be in general relative position, and let $\beta_1, \ldots, \beta_k \subset \mathbb{R}^d$ be Borel sets with $\beta_i \subset K_i$ for $i = 1, \ldots, k$. In general, $\beta_1 + \ldots + \beta_k \subset \mathbb{R}^d$ need not be a Borel set. However, by [16, Lemma 3.2] it follows that $(\beta_1 + \ldots + \beta_k) \cap \text{bd}(K_1 + \ldots + K_k)$ is a Borel set. Since for a convex body $K \in \mathcal{K}$ we have $\Xi_j(K, \cdot) = \Xi_j(K, \cdot \cap (\text{bd}\, K \times S^{d-1}))$, the left-hand side of equation (8) below is well defined. For later use we also mention that the orthogonal projection of a Borel set $\beta \subset \mathbb{R}^d$ to a subspace need not be a Borel set, but in case $K \in \mathcal{K}$ and $\beta \subset K$ the intersection $(\beta|L) \cap \text{bd}_L(K|L)$ is a Borel set for $\nu_k$-almost all $L \in G(d, k)$, where $\text{bd}_L$ is the boundary with respect to $L$ as the ambient space (see the proof of Theorem 6.2.1 in [29]).

In the following expressions, the notation $K[r]$ means that the corresponding entry $K$ appears $r$ times and $(A)^r$, for a set $A$, is the $r$-fold product set.

**Theorem 2.2.** *For $k \in \mathbb{N}$, let $\rho_1, \ldots, \rho_k \geq 0$, let $K_1, \ldots, K_k \in \mathcal{K}$ be convex bodies in general relative position, let $\beta_1, \ldots, \beta_k \subset \mathbb{R}^d$ be Borel sets with $\beta_i \subset K_i$ and let $\gamma \subset S^{d-1}$ be a Borel set.*

**(a)** *For $j \in \{0, \ldots, d-1\}$ and $i_1, \ldots, i_k \in \{0, \ldots, j\}$ with $i_1 + \cdots + i_k = j$, there exist measures $\Xi_{i_1,\ldots,i_k}(K_1, \ldots, K_k, \cdot)$ on $(\mathbb{R}^d)^k \times S^{d-1}$ concentrated on $\text{bd}\, K_1 \times \cdots \times \text{bd}\, K_k \times S^{d-1}$ such that*

$$\Xi_j\left(\sum_{i=1}^k \rho_i K_i, \left(\sum_{i=1}^k \rho_i \beta_i\right) \times \gamma\right) \tag{8}$$

$$= \sum_{i_1,\ldots,i_k=0}^{j} \binom{j}{i_1, \ldots, i_k} \rho_1^{i_1} \cdots \rho_k^{i_k}\, \Xi_{i_1,\ldots,i_k}(K_1, \ldots, K_k, \beta_1 \times \cdots \times \beta_k \times \gamma).$$

*The measure $\Xi_{i_1,\ldots,i_k}(K_1, \ldots, K_k, \cdot)$ depends in a weakly continuous way on $K_1, \ldots, K_k$ in general relative position, and it is symmetric, in the sense that*

$$\Xi_{i_1,\ldots,i_k}(K_1, \ldots, K_k, \beta_1 \times \cdots \times \beta_k \times \gamma) = \Xi_{i_{\pi(1)},\ldots,i_{\pi(k)}}(K_{\pi(1)}, \ldots, K_{\pi(k)}, \beta_{\pi(1)} \times \cdots \times \beta_{\pi(k)} \times \gamma),$$

*for all permutations $\pi$ of $1,\dots,k$. If $i_1=0$, then*

$$\Xi_{i_1,\dots,i_k}(K_1,\dots,K_k,\operatorname{bd}K_1\times\cdot)=\Xi_{i_2,\dots,i_k}(K_2,\dots,K_k,\cdot).$$

*If $i_1\neq 0$, then $\Xi_{i_1,\dots,i_k}(K_1,\dots,K_k,\cdot)$ is translation covariant and homogeneous of degree $i_1$ in the first component, hence*

$$\begin{aligned}\Xi_{i_1,i_2,\dots,i_k}(\alpha K_1+x,K_2,\dots,K_k,&(\alpha\beta_1+x)\times\beta_2\times\cdots\times\beta_k\times\gamma)\\&=\alpha^{i_1}\Xi_{i_1,\dots,i_k}(K_1,\dots,K_k,\beta_1\times\cdots\times\beta_k\times\gamma),\end{aligned}$$

*for $\alpha>0$, $x\in\mathbb{R}^d$. Furthermore, $\Xi_{i_1,i_2,\dots,i_k}(\cdot,K_2,\dots,K_k,\cdot\times\beta_2\times\cdots\times\beta_k\times\gamma)$ has a polynomial expansion.*

**(b)** *Let $j\in\{1,\dots,d-1\}$, $i_1,\dots,i_k\in\{1,\dots,j\}$ be such that $i_1+\cdots+i_k=j$. If $K_1,\dots,K_k\in\mathcal{K}$ are strictly convex bodies, we have*

$$\Xi_{i_1,\dots,i_k}(K_1,\dots,K_k,\beta_1\times\cdots\times\beta_k\times\gamma)=\Xi_{1,\dots,1}(K_1[i_1],\dots,K_k[i_k],(\beta_1)^{i_1}\times\cdots\times(\beta_k)^{i_k}\times\gamma),$$

*where the lower index 1 on the right-hand side appears $j$ times.*

**(c)** *For $j\in\{0,\dots,d-1\}$, $r\in\{0,\dots,j\}$, a convex body $K\in\mathcal{K}$ and Borel sets $\beta\subset K$, $\gamma\subset S^{d-1}$, we have*

$$\Xi_r(K,\beta\times\gamma)=\frac{\binom{d}{r}}{\binom{d}{j}}\frac{\kappa_{d-j}}{\kappa_{d-r}}\,\Xi_{r,j-r}(K,B^d,\beta\times S^{d-1}\times\gamma).$$

*In addition, for convex bodies $K_1,\dots,K_s\in\mathcal{K}$ in general relative position and Borel sets $\gamma\subset S^{d-1}$, $\beta_1,\dots,\beta_s\subset\mathbb{R}^d$ with $\beta_i\subset K_i$, we have*

$$\begin{aligned}&\Xi_{i_1,\dots,i_s}(K_1,\dots,K_s,\beta_1\times\cdots\times\beta_s\times\gamma)\\&=\frac{\binom{d}{r}}{\binom{d}{j}}\frac{\kappa_{d-j}}{\kappa_{d-r}}\,\Xi_{i_1,\dots,i_s,j-r}(K_1,\dots,K_s,B^d,\beta_1\times\cdots\times\beta_s\times S^{d-1}\times\gamma),\end{aligned}$$

*where $i_1,\dots,i_s\in\{0,\dots,r\}$ with $i_1+\cdots+i_s=r$.*

The strict convexity is needed, since the $i_1$ copies of $K_1$ etc. have to be in general relative position. The condition on the general relative position can be neglected if the projections of the mixed measures onto the last (spherical) component are considered. The resulting expansion then reduces to the classical multilinearity of the mixed area measures.

*Proof of Theorem 2.2.* **(a)** By [16, (5.16)] and since $\Xi_j(L,\cdot)=\frac{\binom{d}{j}}{d\kappa_{d-j}}\Theta_j(L,\cdot)$ we obtain the polynomial expansion (8) with coefficient measures $\Xi_{i_1,\dots,i_k}$ which are determined by this expansion. The asserted properties of these mixed measures then follow from [16, displayed formula on page 328] and from the properties stated in [16, Theorem 5.6 and Corollary 3.6].

**(b)** Let $j\in\{1,\dots,d-1\}$ and $i_1,\dots,i_k\in\{1,\dots,j\}$ with $j=i_1+\cdots+i_k$. Let $K_1,\dots,K_k\in\mathcal{K}$ be strictly convex. We consider $\rho_{11},\dots,\rho_{1i_1},\dots,\rho_{k1},\dots,\rho_{ki_k}\geq 0$, $\beta_i\subset K_i$ for $i=1,\dots,k$, $\gamma\subset S^{d-1}$. Then, by (8), in the expansion of

$$\Xi_j\Big(\sum_{r=1}^{k}\sum_{s_r=1}^{i_r}\rho_{rs_r}K_r,\Big(\sum_{r=1}^{k}\sum_{s_r=1}^{i_r}\rho_{rs_r}\beta_r\Big)\times\gamma\Big)\tag{9}$$

the coefficient of $\rho_{11}\cdots\rho_{ki_k}$ is

$$j!\,\Xi_{1,\ldots,1}(K_1[i_1],\ldots,K_k[i_k],(\beta_1)^{i_1}\times\cdots\times(\beta_k)^{i_k}\times\gamma).$$

Here, the index 1 in the measure $\Xi_{1,\ldots,1}$ appears $j$ times. On the other hand, using that $K_r$ is strictly convex and $\beta_r\subset K_r$, we get

$$\left(\sum_{r=1}^{k}\sum_{s_r=1}^{i_r}\rho_{rs_r}\beta_r\right)\cap\mathrm{bd}\left(\sum_{r=1}^{k}\sum_{s_r=1}^{i_r}\rho_{rs_r}K_r\right)=\left[\sum_{r=1}^{k}\left(\sum_{s_r=1}^{i_r}\rho_{rs_r}\right)\beta_r\right]\cap\mathrm{bd}\left[\sum_{r=1}^{k}\left(\sum_{s_r=1}^{i_r}\rho_{rs_r}\right)K_r\right].$$

Hence, it follows that (9) is equal to

$$\begin{aligned}\Xi_j(&(\rho_{11}+\cdots+\rho_{1i_1})K_1+\cdots+(\rho_{k1}+\cdots+\rho_{ki_k})K_k,\\ &[(\rho_{11}+\cdots+\rho_{1i_1})\beta_1+\cdots+(\rho_{k1}+\cdots+\rho_{ki_k})\beta_k]\times\gamma).\end{aligned}$$

Expanding this according to (8), we get

$$\begin{aligned}\sum_{r_1,\ldots,r_k=0}^{j}\binom{j}{r_1,\ldots,r_k}&(\rho_{11}+\cdots+\rho_{1i_1})^{r_1}\cdots(\rho_{k1}+\cdots+\rho_{ki_k})^{r_k}\\ &\times\ \Xi_{r_1,\ldots,r_k}(K_1,\ldots,K_k,\beta_1\times\cdots\times\beta_k\times\gamma).\end{aligned}$$

Then expanding each of the $k$ expressions $(\ldots)^{r_j}$, $j=1,\ldots,k$, we arrive at a polynomial in $\rho_{11},\ldots,\rho_{ki_k}$ which is homogeneous of degree $j$. The monomial $\rho_{11}\cdots\rho_{1i_1}\cdots\rho_{k1}\cdots\rho_{ki_k}$ arises only in the expansion of $r_1=i_1,\ldots,r_k=i_k$ and occurs with multiplicity $\binom{i_1}{1,\ldots,1}\cdots\binom{i_k}{1,\ldots,1}$. Hence the coefficient of this monomial is

$$\begin{aligned}\binom{j}{i_1,\ldots,i_k}\binom{i_1}{1,\ldots,1}\cdots\binom{i_k}{1,\ldots,1}&\,\Xi_{i_1,\ldots,i_k}(K_1,\ldots,K_k,\beta_1\times\cdots\times\beta_k\times\gamma)\\ &=j!\,\Xi_{i_1,\ldots,i_k}(K_1,\ldots,K_k,\beta_1\times\cdots\times\beta_k\times\gamma).\end{aligned}$$

A comparison of coefficients now shows that

$$\Xi_{1,\ldots,1}(K_1[i_1],\ldots,K_k[i_k],(\beta_1)^{i_1}\times\cdots\times(\beta_k)^{i_k}\times\gamma)=\Xi_{i_1,\ldots,i_k}(K_1,\ldots,K_k,\beta_1\times\cdots\times\beta_k\times\gamma).$$

**(c)** If $K\in\mathcal{K}$, we obtain from **(a)** that

$$\Xi_j(K+\rho B^d,(\beta+\rho B^d)\times\gamma)=\sum_{r=0}^{j}\rho^{j-r}\binom{j}{r}\,\Xi_{r,j-r}(K,B^d,\beta\times S^{d-1}\times\gamma).$$

Let $T_\rho(x,u)=(x+\rho u,u)$ for $(x,u)\in\mathbb{R}^d\times S^{d-1}$. Since

$$\begin{aligned}\Xi_j(K+\rho B^d,(\beta+\rho B^d)\times\gamma)&=\Xi_j(K+\rho B^d,[(\beta+\rho B^d)\times\gamma]\cap\mathrm{Nor}(K+\rho B^d))\\ &=\Xi_j(K+\rho B^d,T_\rho(\beta\times\gamma))\end{aligned}$$

and using the polynomial expansion of the support measures ([27, Theorem 4.2.2] or [29, Theorem 14.2.4])

$$\Xi_j(K+\rho B^d,T_\rho(\beta\times\gamma))=\sum_{r=0}^{j}\rho^{j-r}\binom{d-r}{j-r}\frac{\kappa_{d-r}}{\kappa_{d-j}}\,\Xi_r(K,\beta\times\gamma),\tag{10}$$

a comparison of coefficients yields

$$\Xi_r(K, \beta \times \gamma) = \frac{\binom{d}{r}}{\binom{d}{j}} \frac{\kappa_{d-j}}{\kappa_{d-r}} \Xi_{r,j-r}(K, B^d, \beta \times S^{d-1} \times \gamma).$$

Replacing $K$ by a Minkowski combination $\alpha_1 K_1 + \cdots + \alpha_s K_s$, for $K_1, \ldots, K_s \in \mathcal{K}$ in general relative position, expanding both sides and comparing coefficients, we get

$$\begin{aligned}&\Xi_{i_1,\ldots,i_s}(K_1, \ldots, K_s, \beta_1 \times \cdots \times \beta_s \times \gamma)\\ &= \frac{\binom{d}{r}}{\binom{d}{j}} \frac{\kappa_{d-j}}{\kappa_{d-r}} \Xi_{i_1,\ldots,i_s,j-r}(K_1, \ldots, K_s, B^d, \beta_1 \times \cdots \times \beta_s \times S^{d-1} \times \gamma),\end{aligned}$$

where $i_1 + \cdots + i_s = r$. □

**Remark 2.3.** *In the following, we simply write $\Xi(K_1, \ldots, K_j, \cdot)$ for $\Xi_{1,\ldots,1}(K_1, \ldots, K_j, \cdot)$. Here it is clear from the context that the lower index 1 appears $j$ times.*

We finish this section with a short description of the proof of Theorem 2.1, since the structure of this proof also underlies the more general construction of flag measures which we discuss in the next section. Formula (6) is first proved for polytopes $K$, by discussing the contributions to the measure $\mu_\rho(K, \eta)$ coming from the different faces of $K$. Here, an important aspect is to see that the summand which is contributed by the faces of dimension $j$, is homogeneous of degree $j$. The formula for polytopes then also implies the representation (7) of the support measures. Having proved this polynomial expansion for polytopes $K$, one considers $\mu_\rho(K, \eta)$, for $\rho = 1, \ldots, d$. This yields a system of linear equations for the coefficients $\Xi_0(K, \eta), \ldots, \Xi_{d-1}(K, \eta)$ which is invertible (the corresponding determinant is a Vandermonde determinant), hence we obtain $\Xi_j(K, \eta)$ as a linear combination of the values $\mu_\rho(K, \eta)$, $\rho = 1, \ldots, d$. Now $K \mapsto \mu_\rho(K, \cdot)$, $K \in \mathcal{K}$, is weakly continuous (this is shown directly by an application of the Portmanteau theorem). Thus, the linear combination of the values $\mu_\rho(K, \eta)$, $\rho = 1, \ldots, d$, which was obtained for polytopes, extends to arbitrary $K \in \mathcal{K}$ by continuity and defines $\Xi_j(K, \eta)$ for $K \in \mathcal{K}$ (in a weakly continuous way). Therefore, the local Steiner formula (6) also extends from polytopes to arbitrary $K \in \mathcal{K}$, by continuity. It is a convenient strategy to establish properties of support measures of general convex bodies first for polytopes in a direct way, and then to deduce the corresponding property for arbitrary bodies by approximation with polytopes.

## 3 Flag measures as coefficients of a local Steiner formula

In this section, we introduce flag measures as natural generalizations of support measures by a similar procedure, namely a further variant of a local Steiner formula. The new aspect is that points in the neighborhood of a convex body $K$ in $\mathbb{R}^d$ are replaced by $k$-flats, $k \in \{0, \ldots, d-1\}$. We mainly follow Hinderer [12, Chapter 4] and mostly skip the proofs since a detailed exposition with proofs is available in the recent paper [13].

Let $K$ be a convex body, $\rho > 0$ and $k \in \{0, \ldots, d-1\}$. For a flat $E \in A(d, k)$, we let $p(K, E)$ and $l(K, E) = p(E, K)$ be the points in $K$ and $E$ closest to each other, provided this pair of closest points is unique. The distance between $K$ and $E$ is then given by $d(K, E) = \|p(K, E) - l(K, E)\|$ and the direction from $K$ to $E$ is

$$u(K, E) = \frac{l(K, E) - p(K, E)}{d(K, E)}$$

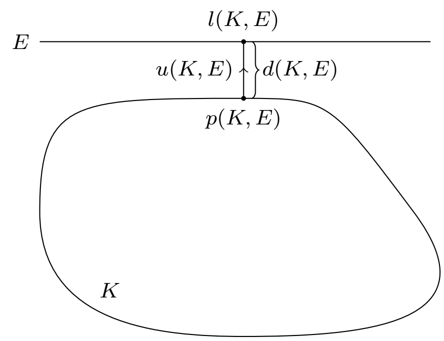


Figure 1: Points realizing the distance between $K$ and $E$

provided that $d(K,E) > 0$ (see Figure 1). Note that this direction is always uniquely defined even if a pair of closest points is not unique. It follows from [27, Corollary 2.3.11] that, for $\mu_k$-almost all flats $E \in A(d,k)$ with $E \cap K = \emptyset$, the distance $d(K,E)$ is realized by a unique pair $(x,y) \in K \times E$, and thus $x = p(K,E)$ and $y = l(K,E)$.

Let $A(d,k,K)$ denote the Borel set of all flats $E \in A(d,k)$ with $E \cap K = \emptyset$ and for which the pair of nearest points is unique. Then, the mappings $d(\cdot,\cdot)$, $p(\cdot,\cdot)$, $l(\cdot,\cdot)$ and $u(\cdot,\cdot)$ are continuous on $\{(K,E) : K \in \mathcal{K}, E \in A(d,k,K)\}$ (see [12, Lemma 23]). We also define a continuous mapping $E \mapsto L(E)$ from $A(d,k)$ to $G(d,k)$, which maps each flat to the parallel linear subspace.

For $K \in \mathcal{K}$, $\rho > 0$ and $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1} \times G(d,k))$, we now consider the *local parallel set*

$$M_\rho^{(k)}(K,\eta) = \{E \in A(d,k,K) \ : \ 0 < d(K,E) \leq \rho, (p(K,E), u(K,E), L(E)) \in \eta\}.$$

Then $M_\rho^{(k)}(K,\eta)$ is a Borel set in $A(d,k)$ and

$$\mu_\rho^{(k)}(K,\cdot) = \mu_k(M_\rho^{(k)}(K,\cdot)) \tag{11}$$

defines a finite Borel measure on $\mathcal{B}(\mathbb{R}^d \times S^{d-1} \times G(d,k))$.

**Theorem 3.1.** *For $K \in \mathcal{K}$ and $k \in \{0,\dots,d-1\}$, there are finite (positive) Borel measures $\Xi_0^{(k)}(K,\cdot),\dots,\Xi_{d-k-1}^{(k)}(K,\cdot)$ on $\mathbb{R}^d \times S^{d-1} \times G(d,k)$ such that, for all $\rho > 0$,*

$$\mu_\rho^{(k)}(K,\cdot) = \sum_{m=0}^{d-k-1} \rho^{d-k-m} \kappa_{d-k-m}\, \Xi_m^{(k)}(K,\cdot). \tag{12}$$

*The measure $\Xi_j^{(k)}(K,\cdot)$, $j \in \{0,\dots,d-k-1\}$, is concentrated on*

$$\mathrm{Nor}_k\, K = \{(x,u,L) : (x,u) \in \mathrm{Nor}\, K, L \in G(d,k), L \perp u\}.$$

*For $j \in \{0,\dots,d-k-1\}$, the mapping $K \mapsto \Xi_j^{(k)}(K,\cdot)$ is weakly continuous. Moreover, $K \mapsto \Xi_j^{(k)}(K,\eta)$ is measurable, for each $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1} \times G(d,k))$.*

The proof of the theorem is based on the following two lemmas.

**Lemma 3.2.** *The mapping $K \mapsto \mu_\rho^{(k)}(K,\cdot)$ is weakly continuous.*

This implies, in particular, that $K \mapsto \mu_\rho^{(k)}(K,\eta)$ is measurable, for $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1} \times G(d,k))$.

**Lemma 3.3.** *For $m \in \{0, \dots, d-k-1\}$ and polytopes $K \subset \mathbb{R}^d$, Theorem 3.1 holds with*

$$\Xi_m^{(k)}(K, \cdot) = \frac{1}{\omega_{d-k-m}} \int_{G(d,k)} \sum_{F \in \mathcal{F}_m(K)} \int_{F|L^\perp} \int_{L^\perp \cap\, n(K,F)} \times \mathbf{1}\{(p(F, L+x), u, L) \in \cdot\}\, \mathcal{H}^{d-k-m-1}(du)\, \mathcal{H}^m(dx)\, \nu_k(dL).$$

Note that for a given $F \in \mathcal{F}_m(K)$, with $m \in \{0, \dots, d-k-1\}$, and $\nu_k$-almost all $L \in G(d,k)$, $F$ and $L$ are in general relative position, that is, $L(F) \cap L = \{o\}$, where $L(F)$ is the linear subspace parallel to $F$. This fact can be used to show that all expressions involved in the preceding integral are well defined. In particular, for $\nu_k$-almost all $L \in G(d,k)$ we have $\dim(L^\perp \cap \operatorname{lin} n(K,F)) = d-k-m$, where $\operatorname{lin}$ denotes the linear hull of a set, and $p(F, L+x)$ is the unique intersection point of $F$ and $L+x$ for a given point $x \in F|L^\perp$.

Using (12) with $\rho = 1, \dots, d-k$, we obtain a system of linear equations for the values $\Xi_0^{(k)}(K, \eta), \dots, \Xi_{d-k-1}^{(k)}(K, \eta)$, where $K$ is a polytope and $\eta$ is a fixed Borel set. This system is invertible (again we have a Vandermonde determinant), hence there is a representation

$$\Xi_m^{(k)}(K, \eta) = \sum_{i=1}^{d-k} a(d,k,m,i)\, \mu_i^{(k)}(K, \eta) \tag{13}$$

with real numbers $a(d,k,m,i)$, which do not depend on $\eta$ or $K$.

For an arbitrary body $K \in \mathcal{K}$, we choose a sequence of polytopes $K_r, r \in \mathbb{N}$, converging to $K$. Then (13) holds, for each polytope $K_r$. Due to Lemma 3.2, the right-hand side of (13) converges weakly to $\sum_{i=1}^{d-k} a(d,k,m,i)\mu_i^{(k)}(K, \cdot)$. Therefore, also the left-hand side converges and defines the limit measure $\Xi_m^{(k)}(K, \cdot)$. The measures thus obtained could be signed, but the explicit representation in Lemma 3.3 shows that they are non-negative (first for polytopes but then for arbitrary bodies by approximation). Lemma 3.2 also shows that the measures $\Xi_m^{(k)}(K, \cdot)$ depend continuously on $K \in \mathcal{K}$. Since (12) holds for the polytopes $K_r$, $r \in \mathbb{N}$, a continuity argument shows that it also holds for $K$, and the proof of Theorem 3.1 is complete.

The measures $\Xi_m^{(k)}(K, \cdot)$, $m \in \{0, \dots, d-k-1\}$, are called *flag measures* of $K$. More precisely, $\Xi_m^{(k)}(K, \cdot)$ is called flag support measure of type $(k, m)$.

For each $k \in \{0, \dots, d-1\}$, we thus obtain a sequence $\Xi_0^{(k)}(K, \cdot), \dots, \Xi_{d-k-1}^{(k)}(K, \cdot)$ of flag measures. In particular, for $k = 0$ we get back the classical support measures. More generally, the support measures appear as image measures of the flag measures under a projection map. This fact is expressed by Proposition 4.5 below.

We also mention that the flag measures $\Xi_m^{(k)}(K, \cdot)$ induce flag area measures $\Psi_m^{(k)}(K, \cdot)$ and flag curvature measures $\Phi_m^{(k)}(K, \cdot)$ as image measures under the projection map $(x, u, L) \mapsto (u, L)$, respectively $(x, u, L) \mapsto (x, L)$. The general results on flag measures, which will be discussed in the sequel, always include corresponding assertions on flag area measures and flag curvature measures as special cases, even if we will not point this out explicitly in each case.

# 4 Flag measures as projection averages

In Lemma 3.3 we have seen an explicit representation for the flag measure $\Xi_m^{(k)}(K, \cdot)$ of a polytope $K$ as a sum over the $m$-dimensional faces of $K$. As an alternative approach, we could take this formula

$$\Xi_m^{(k)}(K, \cdot) = \frac{1}{\omega_{d-k-m}} \int_{G(d,k)} \sum_{F \in \mathcal{F}_m(K)} \int_{F|L^\perp} \int_{L^\perp \cap\, n(K,F)} \times \mathbf{1}\{(p(F, L+x), u, L) \in \cdot\}\, \mathcal{H}^{d-k-m-1}(du)\, \mathcal{H}^m(dx)\, \nu_k(dL) \tag{14}$$

as the definition of $\Xi_m^{(k)}(K,\cdot)$ and ask whether this measure has a (weakly) continuous extension to arbitrary bodies $K \in \mathcal{K}$ and whether this extension, if it exists, satisfies a local Steiner formula. In this section, we will provide answers to these questions.

In view of an extension to general bodies $K \in \mathcal{K}$, our first goal is to reformulate the right-hand side of (14) appropriately. For this purpose, we replace $p(F, L+x)$ on the right-hand side of (14) by the value of a more generally defined function which has suitable continuity properties (see Lemma 4.4). Let $c(M)$ denote the midpoint of the circumsphere of a body $M \in \mathcal{K}$ and define

$$g : \mathbb{R}^d \times G(d,k) \times \mathcal{K} \to \mathbb{R}^d, \qquad (z, L, K) \mapsto c\Big((p(K|L^\perp, z) + L) \cap K\Big).$$

Then, for $F \in \mathcal{F}_m(K)$, we have $p(F, L+x) = g(x, L, K)$ for $\nu_k$-almost all $L \in G(d,k)$ such that $L^\perp \cap \operatorname{relint} n(K,F) \neq \emptyset$ and all $x \in F|L^\perp$, and therefore

$$\Xi_m^{(k)}(K,\cdot) = \frac{1}{\omega_{d-k-m}} \int_{G(d,k)} \sum_{F \in \mathcal{F}_m(K)} \int_{F|L^\perp} \int_{L^\perp \cap n(K,F)} \times \mathbf{1}\{(g(x,L,K), u, L) \in \cdot\}\, \mathcal{H}^{d-k-m-1}(du)\, \mathcal{H}^m(dx)\, \nu_k(dL). \tag{15}$$

As we shall show now, the inner part of the integral can be expressed in terms of the support measure $\Xi_m^{L^\perp}(K|L^\perp,\cdot)$ of $K|L^\perp$ in $L^\perp$. Namely, from (7) we have

$$\Xi_m^{L^\perp}(K|L^\perp,\cdot) = \frac{1}{\omega_{d-k-m}} \sum_{G \in \mathcal{F}_m(K|L^\perp)} \int_G \int_{n_{L^\perp}(K|L^\perp, G)} \times \mathbf{1}\{(x,u) \in \cdot\}\, \mathcal{H}^{d-k-m-1}(du)\, \mathcal{H}^m(dx), \tag{16}$$

where $n_{L^\perp}(K|L^\perp, G)$ denotes the set of exterior unit normals of $K|L^\perp$ at $G$ with respect to $L^\perp$ as the ambient space. In order to see the connection between (15) and (16), the following lemmas are useful.

**Lemma 4.1.** *Let $K \in \mathcal{K}$ be a polytope, $k \in \{0,\dots,d-1\}$, $m \in \{0,\dots,d-k-1\}$, $F \in \mathcal{F}_m(K)$ and $L \in G(d,k)$. If $F$ and $L$ are in general relative position and $u \in L^\perp \cap \operatorname{relint} n(K,F)$, then*

$$F|L^\perp \in \mathcal{F}_m(K|L^\perp)$$

*and*

$$u \in \operatorname{relint} n_{L^\perp}(K|L^\perp, F|L^\perp).$$

*Proof.* Let $F, L$ and $u$ be given as in the statement of the lemma. Since $u \in \operatorname{relint} n(K,F)$, [27, (2.4.3)] implies that $F = F(K,u) = H(K,u) \cap K$, where $H(K,u)$ is the supporting hyperplane of $K$ with exterior unit normal $u$. Since $u \in L^\perp$, $H(K,u) \cap L^\perp$ is a supporting hyperplane of $K|L^\perp$ with outer normal $u$ and $F|L^\perp = (K|L^\perp) \cap H(K,u) \cap L^\perp$. This implies that $F|L^\perp = F_{L^\perp}(K|L^\perp, u)$ is a face of $K|L^\perp$. Since $L$ and $F$ are in general relative position, we have $L \cap L(F) = \{o\}$ and therefore $\dim(F|L^\perp) = m$, which shows that $F|L^\perp \in \mathcal{F}_m(K|L^\perp)$. Using [27, (2.4.3)] again, we see that $u \in \operatorname{relint} n_{L^\perp}(K|L^\perp, F|L^\perp)$. □

For a convex polytope $K \in \mathcal{K}$ and a subspace $L \in G(d,k)$ we say that $K$ and $L$ are in general relative position if, for all faces $F$ of $K$, $F$ and $L$ are in general relative position.

**Lemma 4.2.** *Let $K \in \mathcal{K}$ be a polytope, $k \in \{0,\dots,d-1\}$, and let $L \in G(d,k)$ be such that $K$ and $L$ are in general relative position. If $G \in \mathcal{F}_m(K|L^\perp)$, $m \in \{0,\dots,d-k-1\}$, and if $u \in \operatorname{relint} n_{L^\perp}(K|L^\perp, G)$, then there is a unique face $F \in \mathcal{F}_m(K)$ with $F|L^\perp = G$ and $u \in \operatorname{relint} n(P,F)$.*

*Proof.* Let $G, L, u$ be given as in the statement of the lemma. We use [27, (2.4.3)] again to get $F_{L^\perp}(K|L^\perp, u) = G$. Moreover, $F = F(K, u)$ is a face of $K$ with $F|L^\perp = G$. Since $K$ and $L$ are in general relative position, we have $\dim(F) = \dim(G) = m$, hence $F \in \mathcal{F}_m(K)$ with $G = F|L^\perp$ and $u \in \operatorname{relint} n(K, F)$ by another application of [27, (2.4.3)]. The uniqueness assertion is clear. □

Now we formulate the main result in this section. It describes the flag support measures of a convex body as mixtures of support measures of projections of the given convex body. Intuitively, it can be interpreted as the measure of $k$-flats touching a convex body in a given set of support elements and such that the linear subspaces parallel to the $k$-flats also lie in a prescribed Borel set.

**Theorem 4.3.** *Let $K \in \mathcal{K}$ be a polytope and let the measure $\Xi_m^{(k)}(K, \cdot)$ be defined by* (15)*, for $k \in \{0, \dots, d-1\}$ and $m \in \{0, \dots, d-k-1\}$. Then*

$$\Xi_m^{(k)}(K, \cdot) = \int_{G(d,k)} \int \mathbf{1}\{(g(x, L, K), u, L) \in \cdot\} \, \Xi_m^{L^\perp}(K|L^\perp, d(x, u)) \, \nu_k(dL). \tag{17}$$

*If $\Xi_m^{(k)}(K, \cdot)$ is defined for arbitrary $K \in \mathcal{K}$ by* (17)*, then the extended mapping $K \mapsto \Xi_m^{(k)}(K, \cdot)$ is weakly continuous on $\mathcal{K}$.*

*Proof.* We start with a polytope $K$ and a subspace $L \in G(d, k)$ which are in general relative position. Since $n_{L^\perp}(K|L^\perp, G)$ and $\operatorname{relint} n_{L^\perp}(K|L^\perp, G)$ only differ by a set of $\mathcal{H}^{d-k-m-1}$-measure 0, the representation (16) turns into

$$\Xi_m^{L^\perp}(K|L^\perp, \cdot) = \frac{1}{\omega_{d-k-m}} \sum_{G \in \mathcal{F}_m(K|L^\perp)} \int_G \int_{\operatorname{relint} n_{L^\perp}(K|L^\perp, G)} \times \mathbf{1}\{(x, u) \in \cdot\} \, \mathcal{H}^{d-k-m-1}(du) \, \mathcal{H}^m(dx).$$

Using Lemma 4.1 and Lemma 4.2, we thus obtain

$$\Xi_m^{L^\perp}(K|L^\perp, \cdot) = \frac{1}{\omega_{d-k-m}} \sum_{F \in \mathcal{F}_m(K)} \int_{F|L^\perp} \int_{L^\perp \cap \operatorname{relint} n(K,F)} \times \mathbf{1}\{(x, u) \in \cdot\} \, \mathcal{H}^{d-k-m-1}(du) \, \mathcal{H}^m(dx).$$

The spherical set $n(K, F)$ is the intersection of the unit sphere with a $(d-m)$-dimensional polyhedral cone (the normal cone $N(K, F)$) and the relative boundary of $N(K, F)$ is contained in a finite union of $(d-m-1)$-dimensional subspaces. Excluding a set in $G(d, k)$ of $\nu_k$-measure 0, we may assume that $L^\perp$ and each of these subspaces bounding $N(K, F)$ are in general relative position. This implies

$$\mathcal{H}^{d-k-m-1}(L^\perp \cap \operatorname{relbd} n(K, F)) = 0,$$

and hence

$$\Xi_m^{L^\perp}(K|L^\perp, \cdot) = \frac{1}{\omega_{d-k-m}} \sum_{F \in \mathcal{F}_m(K)} \int_{F|L^\perp} \int_{L^\perp \cap n(K,F)} \times \mathbf{1}\{(x, u) \in \cdot\} \, \mathcal{H}^{d-k-m-1}(du) \, \mathcal{H}^m(dx). \tag{18}$$

Combining (18) with (15) yields (17).

Now let $K \in \mathcal{K}$ be arbitrary and let $\Xi_m^{(k)}(K, \cdot)$ be defined by (17). We show that $K \mapsto \Xi_m^{(k)}(K, \cdot)$ is weakly continuous.

Let $\mathrm{USP}_k(K)$ be the set of all subspaces $L \in G(d,k)$ which have the *unique support property* for $K$. This means that every $k$-flat parallel to $L$, which supports $K$, meets $K$ only in one point. If the subspace $L$ is fixed, we write $g_L(x,K)$ instead of $g(x,L,K)$, in the following. We first discuss the continuity properties of $g_L$.

**Lemma 4.4.** *Let $K_i \in \mathcal{K}$ and $x_i \in \mathbb{R}^d$, $i \in \mathbb{N}$, be converging sequences, $K_i \to K_0$, $x_i \to x_0$ (as $i \to \infty$). If $L \in \mathrm{USP}_k(K_0)$, then*

$$\lim_{i\to\infty} g_L(x_i, K_i) = g_L(x_0, K_0).$$

*In particular, $x \mapsto g_L(x, K_0)$ is continuous and the convergence $g_L(\cdot, K_i) \to g_L(\cdot, K_0)$, as $i \to \infty$, is uniform on compact subsets of $\mathbb{R}^d$.*

*Proof.* The continuity of the metric projection ([27, Lemma 1.8.9]) and [29, Theorem 12.3.5] yield that $p(K_i|L^\perp, x_i) + L \to p(K_0|L^\perp, x_0) + L$, as $i \to \infty$.

*Case* 1: $p(K_0|L^\perp, x_0) + L$ and $K_0$ cannot be separated by a hyperplane.

Then the proof of Theorem 1.8.8 in [27] (with Theorem 1.8.7 replaced by Theorem 12.2.2 in [29]) shows that $(p(K_i|L^\perp, x_i) + L) \cap K_i \to (p(K_0|L^\perp, x_0) + L) \cap K_0$, as $i \to \infty$, in $\mathcal{K}$. From the continuity of the map $K \mapsto c(K)$ (see [27, Lemma 4.1.1]) we thus get the assertion.

*Case* 2: $p(K_0|L^\perp, x_0) + L$ and $K_0$ can be separated by a hyperplane.

Since $L \in \mathrm{USP}(K_0)$, we get $(p(K_0|L^\perp, x_0) + L) \cap K_0 = \{z_0\}$, for some $z_0 \in \mathbb{R}^d$. We show that

$$(p(K_i|L^\perp, x_i) + L) \cap K_i \to \{z_0\}, \tag{19}$$

which proves the assertion.

For this, we choose points $y_i \in (p(K_i|L^\perp, x_i) + L) \cap K_i$, $i \in \mathbb{N}$, which converge to some $y_0 \in K_0$ (as $i \to \infty$ and possibly after selecting a subsequence). Since $y_i \in p(K_i|L^\perp, x_i) + L \to p(K_0|L^\perp, x_0) + L$, we also have $y_0 \in p(K_0|L^\perp, x_0) + L$ and thus $y_0 \in (p(K_0|L^\perp, x_0) + L) \cap K_0$. This implies $y_0 = z_0$. Hence every sequence $y_i \in (p(K_i|L^\perp, x_i) + L) \cap K_i \neq \emptyset$, $i \in \mathbb{N}$, has a subsequence which converges to $z_0$. Thus $y_i$ converges to $z_0$ and so (19) follows from [29, Theorem 12.2.2].

The remaining assertions are clear. □

Now we continue the proof of Theorem 4.3. Let $F$ be a (non-negative) continuous function on $\mathbb{R}^d \times S^{d-1} \times G(d,k)$. We have to show that

$$K \mapsto \int_{G(d,k)} \int F(g(x,L,K), u, L)\, \Xi_m^{L^\perp}(K|L^\perp, d(x,u))\, \nu_k(dL)$$

is continuous on $\mathcal{K}$. For this purpose, we consider $K_i, K_0 \in \mathcal{K}$, $i \in \mathbb{N}$, with $K_i \to K_0$, as $i \to \infty$, and show that

$$\begin{aligned} \lim_{i\to\infty} \int F(g_L(x,K_i), u, L)\, \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u)) \\ = \int F(g_L(x,K_0), u, L)\, \Xi_m^{L^\perp}(K_0|L^\perp, d(x,u)) \end{aligned} \tag{20}$$

for $\nu_k$-almost all $L \in G(d,k)$. The assertion then follows by the dominated convergence theorem.

From [27, Corollary 2.3.11] we get $L \in \bigcap_{i\in\mathbb{N}_0} \mathrm{USP}_k(K_i)$ for $\nu_k$-almost all linear subspaces $L \in G(d,k)$. For such an $L$, we have

$$
\begin{aligned}
&\left| \int F(g_L(x,K_i),u,L)\, \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u)) - \int F(g_L(x,K_0),u,L)\, \Xi_m^{L^\perp}(K_0|L^\perp, d(x,u)) \right| \\
&= \left| \int F(g_L(x,K_i),u,L) - F(g_L(x,K_0),u,L)\, \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u)) \right. \\
&\qquad + \int F(g_L(x,K_0),u,L)\, \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u)) \\
&\qquad\qquad \left. - \int F(g_L(x,K_0),u,L)\, \Xi_m^{L^\perp}(K_0|L^\perp, d(x,u)) \right| \\
&\le \int \left| F(g_L(x,K_i),u,L) - F(g_L(x,K_0),u,L) \right| \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u)) \\
&\qquad + \left| \int F(g_L(x,K_0),u,L)\, \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u)) \right. \\
&\qquad\qquad \left. - \int F(g_L(x,K_0),u,L)\, \Xi_m^{L^\perp}(K_0|L^\perp, d(x,u)) \right| .
\end{aligned} \tag{21}
$$

Since $K_i \to K_0$ and so $K_i|L^\perp \to K_0|L^\perp$, the support of the measures $\Xi_m^{L^\perp}(K_i|L^\perp, \cdot)$, $i \in \mathbb{N}_0$, is contained in $RB^d \times S^{d-1}$ for some sufficiently large $R > 0$. Therefore, Lemma 4.4 implies that, for $\epsilon > 0$, we find $i_0 = i_0(\epsilon)$ such that, for $i \ge i_0$,

$$\left| F(g_L(x,K_i),u,L) - F(g_L(x,K_0),u,L) \right| \le \epsilon$$

for all $(x,u) \in RB^d \times S^{d-1}$. Hence, the first summand in (21) can be made smaller than $c(d,k,R)\cdot\epsilon$.

Since, by Lemma 4.4, the mapping $(x,u) \mapsto F(g_L(x,K_0),u,L)$ is continuous and bounded on $RB^d \times S^{d-1}$ and since $\Xi_m^{L^\perp}(K_i|L^\perp, \cdot) \to \Xi_m^{L^\perp}(K_0|L^\perp, \cdot)$, as $i \to \infty$, in the weak topology, by [27, Theorem 4.2.1], we can also make the second summand in (21) smaller than $\epsilon$, for $i \ge i_1(\epsilon)$.

This proves (20) and finishes the proof of Theorem 4.3. □

We now show that the measures $\Xi_m^{(k)}(K,\cdot)$ defined by (17) satisfy a local Steiner formula. Let $\mu_\rho^{(k)}(K,\eta) = \mu_k(M_\rho^{(k)}(K,\eta))$ be the measure of the outer parallel set which we considered in (11), evaluated at some Borel set $\eta \subset \mathbb{R}^d \times S^{d-1} \times G(d,k)$. By definition of $\mu_k$, we get

$$\mu_\rho^{(k)}(K,\eta) = \int_{G(d,k)} \mathcal{H}^{d-k}(T(L))\, \nu_k(dL) \tag{22}$$

with

$$T(L) = \{x \in L^\perp : L + x \in M_\rho^{(k)}(K,\eta)\}.$$

For $\nu_k$-almost all $L \in G(d,k)$ and for $x \in L^\perp$, the condition $L + x \in M_\rho^{(k)}(K,\eta)$ is equivalent to

$$0 < d(K|L^\perp, x) \le \rho, \qquad (g(x,L,K), u(K|L^\perp,x), L) \in \eta.$$

Hence, if $\eta(L)$ denotes the set of pairs $(x|L^\perp, u) \in L^\perp \times (L^\perp \cap S^{d-1})$ with $(x, u, L) \in \eta$, we have

$$\begin{aligned}
&\mu_\rho^{(k)}(K,\eta)\\
&= \int_{G(d,k)} \int_{L^\perp} \mathbf{1}\{0 < d(K|L^\perp, x) \le \rho, (p(K|L^\perp, x), u(K|L^\perp, x)) \in \eta(L)\}\, \mathcal{H}^{d-k}(dx)\, \nu_k(dL)\\
&= \int_{G(d,k)} \mathcal{H}^{d-k}(M_\rho^{L^\perp}(K|L^\perp, \eta(L)))\, \nu_k(dL), \qquad (23)
\end{aligned}$$

where $M_\rho^{L^\perp}(K|L^\perp, \eta(L))$ denotes an (ordinary) local outer parallel set of $K|L^\perp$ in $L^\perp$. Applying the classical local Steiner formula (6) to $K|L^\perp$ in $L^\perp$, we get

$$\mu_\rho^{(k)}(K,\eta) = \sum_{m=0}^{d-k-1} \rho^{d-k-m} \kappa_{d-k-m} \int_{G(d,k)} \Xi_m^{L^\perp}(K|L^\perp, \eta(L))\, \nu_k(dL). \qquad (24)$$

For $\nu_k$-almost all $L \in G(d,k)$, for $x \in L^\perp$ and $u \in L^\perp \cap S^{d-1}$, we have $(g(x, L, K), u, L) \in \eta$ if and only if $(x, u) \in \eta(L)$. Therefore,

$$\int_{G(d,k)} \Xi_m^{L^\perp}(K|L^\perp, \eta(L))\, \nu_k(dL) = \Xi_m^{(k)}(K, \eta), \qquad (25)$$

which shows that (12) holds.

Using [27, Theorem 4.5.10] (see also Note 2 for Section 6.2 in [29, p. 223]), we obtain from (25) the following relationship.

**Proposition 4.5.** *For $K \in \mathcal{K}$, $k \in \{0, \dots, d-1\}$ and $m \in \{0, \dots, d-k-1\}$, the measure $\Xi_m(K, \cdot)$ is the image of $\alpha(d,k,m) \cdot \Xi_m^{(k)}(K, \cdot)$ under the projection $(x, u, L) \mapsto (x, u)$, where*

$$\alpha(d,k,m) = \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-k-m+1}{2}\right)}{\Gamma\left(\frac{d-k+1}{2}\right)\Gamma\left(\frac{d-m+1}{2}\right)}.$$

Note that $\alpha(d,k,d-k-1) = c(d,k)$ and thus we arrive at the normalization (4) from the introduction.

The arguments just given also lead to a possible introduction of the flag measures $\Xi_m^{(k)}(K, \cdot)$. This is the approach chosen in Kropp [17]. Namely one starts with (22), shows that this implies (23), uses the Steiner formula (6) to obtain formula (24) and then defines the measure $\Xi_m^{(k)}(K, \cdot)$ by (25).

# 5 Further properties of flag measures

We now collect various additional properties of flag measures and study representations and extensions.

Four major properties are homogeneity, motion covariance, local definedness and additivity. Homogeneity means that $\Xi_m^{(k)}(K, \cdot)$ is *homogeneous of degree* $m$, in the sense that

$$\Xi_m^{(k)}(\alpha K, \alpha\eta) = \alpha^m\, \Xi_m^{(k)}(K, \eta),$$

for $K \in \mathcal{K}$, $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1} \times G(d,k))$ and $\alpha > 0$, with

$$\alpha\eta = \{(\alpha x, u, L) : (x, u, L) \in \eta\}.$$

Furthermore, we say that $\Xi_m^{(k)}(K,\cdot)$ is *motion covariant*, if

$$\Xi_m^{(k)}(gK,g\eta)=\Xi_m^{(k)}(K,\eta)$$

holds for $K\in\mathcal{K}$, each Borel set $\eta\in\mathcal{B}(\mathbb{R}^d\times S^{d-1}\times G(d,k))$ and each rigid motion $g\in G_d$, where $g\eta=\{(gx,g_0u,g_0L):(x,u,L)\in\eta\}$ and $g_0$ is the rotational part of $g$. We call $\Xi_m^{(k)}(K,\cdot)$ *locally defined*, if

$$\Xi_m^{(k)}(K,\eta)=\Xi_m^{(k)}(M,\eta)$$

for all bodies $K,M\in\mathcal{K}$ such that there is an open set $A\subset\mathbb{R}^d$ with $K\cap A=M\cap A$ and for all Borel sets $\eta$, for which the projection $\eta_1$ onto the first component lies in $A$. Finally, additivity means that the mapping $K\mapsto\Xi_m^{(k)}(K,\cdot)$ is *additive* in the sense that

$$\Xi_m^{(k)}(K\cup M,\cdot)+\Xi_m^{(k)}(K\cap M,\cdot)=\Xi_m^{(k)}(K,\cdot)+\Xi_m^{(k)}(M,\cdot)$$

for all $K,M\in\mathcal{K}$ for which $K\cup M\in\mathcal{K}$.

**Theorem 5.1.** *For $k\in\{0,\dots,d-1\}$ and $m\in\{0,\dots,d-k-1\}$, the flag measure $\Xi_m^{(k)}(K,\cdot)$ is homogeneous of degree $m$, it is motion covariant and locally defined and, as a function of $K$, it is additive.*

*Proof.* Concerning the homogeneity property, we notice that

$$\begin{aligned}\Xi_m^{(k)}(\alpha K,\alpha\eta)&=\int_{G(d,k)}\int\mathbf{1}\{(g(x,L,\alpha K),u,L)\in\alpha\eta\}\,\Xi_m^{L^\perp}(\alpha K|L^\perp,d(x,u))\,\nu_k(dL)\\&=\int_{G(d,k)}\int\mathbf{1}\{(g(x,L,\alpha K),u,L)\in\alpha\eta\}\,\Xi_m^{L^\perp}(\alpha(K|L^\perp),d(x,u))\,\nu_k(dL)\\&=\int_{G(d,k)}\int\mathbf{1}\{(g(\alpha x,L,\alpha K),u,L)\in\alpha\eta\}\,\alpha^m\,\Xi_m^{L^\perp}(K|L^\perp,d(x,u))\,\nu_k(dL)\\&=\alpha^m\int_{G(d,k)}\int\mathbf{1}\{(\alpha g(x,L,K),u,L)\in\alpha\eta\}\,\Xi_m^{L^\perp}(K|L^\perp,d(x,u))\,\nu_k(dL)\\&=\alpha^m\,\Xi_m^{(k)}(K,\eta).\end{aligned}$$

Here, we used $\alpha K|L^\perp=\alpha(K|L^\perp)$, the homogeneity of $\Xi_m^{L^\perp}$ and $g(\alpha x,L,\alpha K)=\alpha g(x,L,K)$.

The motion covariance follows from the fact that

$$\begin{aligned}&M_\rho^{(k)}(gK,g\eta)\\&=\{E\in A(d,k,gK):0<d(gK,E)\le\rho,(p(gK,E),u(gK,E),L(E))\in g\eta\}\\&=\{gE:E\in A(d,k,K),0<d(gK,gE)\le\rho,(p(gK,gE),u(gK,gE),L(gE))\in g\eta\}\\&=\{gE:E\in A(d,k,K),0<d(K,E)\le\rho,(gp(K,E),g_0u(K,E),g_0L(E))\in g\eta\}\\&=gM_\rho^{(k)}(K,\eta).\end{aligned}$$

The motion invariance of $\mu_k$ shows that $\mu_\rho^{(k)}(gK,g\eta)=\mu_\rho^{(k)}(K,\eta)$, and so the motion covariance transfers to the coefficient measures in (12), by the procedure described in detail in Section 3.

To show that $\Xi_m^{(k)}(K,\cdot)$ is locally defined, let $K,M$ and $A$ be such that $K\cap A=M\cap A$, where $A\subset\mathbb{R}^d$ is open, and let $\eta$ be a Borel set such that $\{x\in\mathbb{R}^d:(x,u,L)\in\eta\}\subset A$. By definition,

$$M_\rho^{(k)}(K,\eta)=\{E\in A(d,k,K)\;:\;0<d(K,E)\le\rho,(p(K,E),u(K,E),L(E))\in\eta\}.$$

Hence, for $E \in M_\rho^{(k)}(K,\eta)$, we have $p(K,E) \in A$. Since $A$ is open, the boundaries of $K$ and $M$ coincide in a neighborhood of $p(K,E)$. Therefore, excluding possibly a set of $E \in A(d,k)$ of $\mu_k$-measure zero, we have $E \in A(d,k,M)$ and $p(K,E) = p(M,E)$. This implies $u(K,E) = u(M,E)$ and thus $E \in M_\rho^{(k)}(K,\eta)$. Interchanging the role of $K$ and $M$, we conclude that $M_\rho^{(k)}(K,\eta) = M_\rho^{(k)}(M,\eta)$, up to a set of $E \in A(d,k)$ of $\mu_k$-measure zero. With the same argument as above, this implies $\Xi_m^{(k)}(K,\eta) = \Xi_m^{(k)}(M,\eta)$.

Concerning the additivity, assume $K, M, K \cup M \in \mathcal{K}$. It is sufficient, by the same principle, to show that

$$\begin{aligned} &\mathbf{1}\{E \in M_\rho^{(k)}(K \cup M,\eta)\} + \mathbf{1}\{E \in M_\rho^{(k)}(K \cap M,\eta)\} \\ &\quad = \mathbf{1}\{E \in M_\rho^{(k)}(K,\eta)\} + \mathbf{1}\{E \in M_\rho^{(k)}(M,\eta)\} \end{aligned} \tag{26}$$

holds for $\mu_k$-almost all $E$. In proving this, we follow [12, Lemma 28]. We assume $E \in A(d,k,K) \cap A(d,k,M)$ and put $y = p(K,E), z = p(M,E)$.

We first consider the case $d(K \cup M,E) = d(K,E)$, hence $d(K \cup M,E) = d(y,E)$. If $d(M,E) < d(K,E)$, then $p(K \cup M,E) = y$. If $d(M,E) = d(K,E)$, then $[y,z] \subset K \cup M$. Since $E \in A(d,k,K) \cap A(d,k,M)$, it follows that $y = z$. Then, we have again $p(K \cup M,E) = y$. Since $K \cup M$ is convex, $[y,z]$ is a subset of $K \cup M$, and so a point $a \in [y,z] \cap K \cap M$ exists. The mapping

$$t \mapsto d(tz + (1-t)y, E)$$

is convex on $[0,1]$ and has a minimum at $t = 0$. Therefore, $d(y,E) \le d(a,E) \le d(z,E)$. As $z = p(M,E)$, we have $d(a,E) \ge d(z,E)$ and thus $d(a,E) = d(z,E)$ (and $a, z \in M$). The uniqueness of the nearest point map now implies $z = a \in K \cap M$. We get

$$d(K \cup M,E) = d(K,E), \quad d(K \cap M,E) = d(M,E)$$

and

$$u(K \cup M,E) = u(K,E), \quad u(K \cap M,E) = u(M,E).$$

Therefore,

$$\begin{aligned} \mathbf{1}\{E \in M_\rho^{(k)}(K \cup M,\eta)\} &= \mathbf{1}\{E \in M_\rho^{(k)}(K,\eta)\}, \\ \mathbf{1}\{E \in M_\rho^{(k)}(K \cap M,\eta)\} &= \mathbf{1}\{E \in M_\rho^{(k)}(M,\eta)\}, \end{aligned}$$

which implies (26).

In the other case, that is $d(K \cup M,E) = d(M,E)$, we first get $p(K \cup M,E) = z$, and then we conclude in a similar way that

$$\begin{aligned} \mathbf{1}\{E \in M_\rho^{(k)}(K \cup M,\eta)\} &= \mathbf{1}\{E \in M_\rho^{(k)}(M,\eta)\}, \\ \mathbf{1}\{E \in M_\rho^{(k)}(K \cap M,\eta)\} &= \mathbf{1}\{E \in M_\rho^{(k)}(K,\eta)\}, \end{aligned}$$

which again implies (26). □

Since the flag measures are the coefficients in a local Steiner formula, it is a natural question whether they themselves admit a polynomial expansion, if $\Xi_m^{(k)}(K + \rho B^d, \cdot)$ is considered. For the classical support measures, a corresponding result is formula (10), which we used earlier, and the following proposition shows that this carries over to flag measures.

**Proposition 5.2.** *Let $K \in \mathcal{K}$, $\rho > 0$, $k \in \{0, \dots, d-1\}$ and $m \in \{0, \dots, d-k-1\}$. Then we have*

$$\Xi_m^{(k)}(K + \rho B^d, t_\rho \eta) = \sum_{j=0}^{m} \rho^j \binom{d-k+j-m}{j} \frac{\kappa_{d-k+j-m}}{\kappa_{d-k-m}} \Xi_{m-j}^{(k)}(K, \eta),$$

*where $t_\rho(x, u, L) = (x + \rho u, u, L)$.*

*Proof.* Using (10) and $(K + \rho B^d)|L^\perp = K|L^\perp + \rho B^d|L^\perp$, we get

$$\begin{aligned}
&\Xi_m^{(k)}(K + \rho B^d, t_\rho \eta)\\
&\quad = \int_{G(d,k)} \int \mathbf{1}\{(g(x, L, K + \rho B^d), u, L) \in t_\rho \eta\} \, \Xi_m^{L^\perp}((K + \rho B^d)|L^\perp, d(x, u)) \, \nu_k(dL)\\
&\quad = \int_{G(d,k)} \int \mathbf{1}\{(g(x - \rho u, L, K) + \rho u, u, L) \in t_\rho \eta\} \, \Xi_m^{L^\perp}(K|L^\perp + \rho B^d|L^\perp, d(x, u)) \, \nu_k(dL)\\
&\quad = \int_{G(d,k)} \int \mathbf{1}\{(g(x - \rho u, L, K), u, L) \in \eta\} \, \Xi_m^{L^\perp}(K|L^\perp + \rho B^d|L^\perp, d(x, u)) \, \nu_k(dL)\\
&\quad = \sum_{j=0}^{m} \rho^j \binom{d-k+j-m}{j} \frac{\kappa_{d-k+j-m}}{\kappa_{d-k-m}} \int_{G(d,k)} \int\\
&\qquad\qquad \times \mathbf{1}\{(g(z, L, K), u, L) \in \eta\} \, \Xi_{m-j}^{L^\perp}(K|L^\perp, d(z, u)) \, \nu_k(dL)\\
&\quad = \sum_{j=0}^{m} \rho^j \binom{d-k+j-m}{j} \frac{\kappa_{d-k+j-m}}{\kappa_{d-k-m}} \Xi_{m-j}^{(k)}(K, \eta).
\end{aligned}$$

□

Proposition 5.2 is actually a special case of a multilinear expansion, which is in analogy to (8) and follows from this formula by using the projection average approach. For the latter, we need to know that the condition of general relative position is compatible with projections. Such a result is provided by the following lemma.

**Lemma 5.3.** *Let $K_1, K_2 \in \mathcal{K}$ be in general relative position in $\mathbb{R}^d$. Let $L \in G(d, k)$, $k \in \{0, \dots, d-1\}$, be such that there is no supporting $k$-flat of $K_1 + K_2$ parallel to $L$ that contains a $1$-dimensional convex subset of $K_1 + K_2$. Then $K_1|L^\perp$ and $K_2|L^\perp$ are in general relative position in $L^\perp$.*

*Proof.* For $k \in \{0, d-1\}$ there is nothing to prove. Hence we assume that $L$ is as in the assumptions of the lemma with $1 \le k \le d-2$.

Assume that $K_1|L^\perp$ and $K_2|L^\perp$ are not in general relative position in $L^\perp$. Then there is a unit vector $v \in L^\perp$ and there are parallel segments $\tilde{S}_i \subset F(K_i|L^\perp, v)$, $i = 1, 2$. Hence there are segments $S_i \subset F(K_i, v)$ with $\tilde{S}_i = S_i|L^\perp$, $i = 1, 2$. Since $K_1, K_2$ are in general relative position in $\mathbb{R}^d$, the segments $S_1, S_2$ are not parallel, and therefore $\dim(S_1 + S_2) = 2$. But then $S_1 + S_2$ contains a segment $S$ parallel to $L$ and $S + L \subset H(K_1 + K_2, v)$ is a supporting $k$-flat of $K_1 + K_2$ parallel to $L$ which contains a 1-dimensional subset of $K_1 + K_2$. This is a contradiction to the choice of $L$. □

**Theorem 5.4.** *Let $l \in \mathbb{N}$, $K_1, \dots, K_l \in \mathcal{K}$ be convex bodies in general relative position, $\rho_1, \dots, \rho_l \ge 0$ and $\beta_1, \dots, \beta_l$ be Borel sets with $\beta_i \subset K_i$. Let $k \in \{0, \dots, d-1\}$, and let $\gamma \subset S^{d-1} \times G(d, k)$ be a Borel set.*

**(a)** *For $m \in \{0, \dots, d-k-1\}$ and $i_1, \dots, i_l \in \{0, \dots, m\}$ with $i_1 + \dots + i_l = m$, there exist measures $\Xi^{(k)}_{i_1,\dots,i_l}(K_1, \dots, K_l, \cdot)$ concentrated on $\operatorname{bd} K_1 \times \dots \times \operatorname{bd} K_l \times S^{d-1} \times G(d,k)$ such that*

$$\Xi^{(k)}_m \left( \sum_{i=1}^{l} \rho_i K_i, \Big( \sum_{i=1}^{l} \rho_i \beta_i \Big) \times \gamma \right) \tag{27}$$

$$= \sum_{i_1,\dots,i_l=0}^{m} \binom{m}{i_1, \dots, i_l} \rho_1^{i_1} \cdots \rho_l^{i_l} \, \Xi^{(k)}_{i_1,\dots,i_l}(K_1, \dots, K_l, \beta_1 \times \dots \times \beta_l \times \gamma).$$

*The measure $\Xi^{(k)}_{i_1,\dots,i_l}(K_1, \dots, K_l, \cdot)$ depends in a weakly continuous way on $K_1, \dots, K_l$ in general relative position, and it is symmetric, in the sense that*

$$\Xi^{(k)}_{i_1,\dots,i_l}(K_1, \dots, K_l, \beta_1 \times \dots \times \beta_l \times \gamma) = \Xi^{(k)}_{i_{\pi(1)},\dots,i_{\pi(l)}}(K_{\pi(1)}, \dots, K_{\pi(l)}, \beta_{\pi(1)} \times \dots \times \beta_{\pi(l)} \times \gamma),$$

*for all permutations $\pi$ of $1, \dots, l$. If $i_1 = 0$, then*

$$\Xi^{(k)}_{i_1,\dots,i_l}(K_1, \dots, K_l, \operatorname{bd} K_1 \times \cdot) = \Xi^{(k)}_{i_2,\dots,i_l}(K_2, \dots, K_l, \cdot).$$

*If $i_1 \neq 0$, then $\Xi^{(k)}_{i_1,\dots,i_l}(K_1, \dots, K_l, \cdot)$ is translation covariant and homogeneous of degree $i_1$ in the first component, hence*

$$\Xi^{(k)}_{i_1,i_2,\dots,i_l}(\alpha K_1 + x, K_2, \dots, K_l, (\alpha \beta_1 + x) \times \beta_2 \times \dots \times \beta_l \times \gamma)$$

$$= \alpha^{i_1} \Xi^{(k)}_{i_1,\dots,i_l}(K_1, \dots, K_l, \beta_1 \times \dots \times \beta_l \times \gamma),$$

*for $\alpha > 0$, $x \in \mathbb{R}^d$. Moreover, $\Xi^{(k)}_{i_1,i_2,\dots,i_l}(\cdot, K_2, \dots, K_l, \cdot \times \beta_2 \times \dots \times \beta_l \times \gamma)$ has a polynomial expansion.*

**(b)** *Let $m \in \{1, \dots, d-1-k\}$ and $i_1, \dots, i_l \in \{1, \dots, m\}$ be such that $i_1 + \ldots + i_l = m$. Then, for strictly convex bodies $K_1, \dots, K_l$, we have*

$$\Xi^{(k)}_{i_1,\dots,i_l}(K_1, \dots, K_l, \beta_1 \times \dots \times \beta_l \times \gamma) = \Xi^{(k)}_{1,\dots,1}(K_1[i_1], \dots, K_l[i_l], (\beta_1)^{i_1} \times \dots \times (\beta_l)^{i_l} \times \gamma),$$

*where the lower index 1 on the right-hand side appears $m$ times.*

**(c)** *For $m \in \{0, \dots, d-1-k\}$, $r \in \{0, \dots, m\}$, $K \in \mathcal{K}$ and Borel sets $\beta \subset K$, $\gamma \subset S^{d-1} \times G(d,k)$, we have*

$$\Xi^{(k)}_r(K, \beta \times \gamma) = \frac{\binom{d-k}{r}}{\binom{d-k}{m}} \frac{\kappa_{d-k-m}}{\kappa_{d-k-r}} \, \Xi^{(k)}_{r,m-r}(K, B^d, \beta \times S^{d-1} \times \gamma).$$

*In addition, for $i_1, \dots, i_s \in \{0, \dots, r\}$ with $i_1 + \ldots + i_s = r$, convex bodies $K_1, \dots, K_s \in \mathcal{K}$ in general relative position and Borel sets $\beta_1, \dots, \beta_s \subset \mathbb{R}^d$ with $\beta_i \subset K_i$ and $\gamma \subset S^{d-1} \times G(d,k)$, we have*

$$\Xi^{(k)}_{i_1,\dots,i_s}(K_1, \dots, K_s, \beta_1 \times \dots \times \beta_s \times \gamma)$$

$$= \frac{\binom{d-k}{r}}{\binom{d-k}{m}} \frac{\kappa_{d-k-m}}{\kappa_{d-k-r}} \, \Xi^{(k)}_{i_1,\dots,i_s,m-r}(K_1, \dots, K_s, B^d, \beta_1 \times \dots \times \beta_s \times S^{d-1} \times \gamma).$$

*Proof.* **(a)** We may assume $\gamma = \omega \times A$ with Borel sets $\omega \subset S^{d-1}$ and $A \subset G(d,k)$.

In order to use (17) together with Theorem 2.2, we first need to show that $K_1|L^\perp, \dots, K_l|L^\perp$ are in general relative position, for $\nu_k$-almost all $L \in G(d,k)$. For this purpose, we combine Lemma 5.3 with a result due to Zalgaller (see the case $s = 1$ and $r = k$ of the Theorem in Notes for Section 2.3 in [27, p. 93]) to obtain that if $K_1$ and $K_2$ are in general relative position in $\mathbb{R}^d$, then $K_1|L^\perp$ and $K_2|L^\perp$ are in general relative position in $L^\perp$ for $\nu_k$-almost all $L \in G(d,k)$. A straightforward induction argument yields the extension of this result to finitely many convex bodies in general relative position.

By formulas (17) and (8), we now have

$$\begin{aligned}
&\Xi_m^{(k)}\left(\sum_{i=1}^l \rho_i K_i, \Big(\sum_{i=1}^l \rho_i \beta_i\Big) \times \gamma\right)\\
&\quad = \int_{G(d,k)} \int \mathbf{1}\Big\{(g\Big(x, L, \sum_{i=1}^l \rho_i K_i\Big), u, L) \in \Big(\sum_{i=1}^l \rho_i \beta_i\Big) \times \gamma\Big\}\\
&\qquad\qquad \times \Xi_m^{L^\perp}\Big(\Big(\sum_{i=1}^l \rho_i K_i\Big)|L^\perp, d(x,u)\Big)\, \nu_k(dL)\\
&\quad = \int_{G(d,k)} \int \mathbf{1}\Big\{(g\Big(x, L, \sum_{i=1}^l \rho_i K_i\Big), u, L) \in \Big(\sum_{i=1}^l \rho_i \beta_i\Big) \times \gamma\Big\}\\
&\qquad\qquad \times \Xi_m^{L^\perp}\Big(\sum_{i=1}^l \rho_i (K_i|L^\perp), d(x,u)\Big)\, \nu_k(dL)\\
&\quad = \int_{G(d,k)} \mathbf{1}\{L \in A\}\, \Xi_m^{L^\perp}\left(\sum_{i=1}^l \rho_i (K_i|L^\perp), \Big(\sum_{i=1}^l \rho_i (\beta_i|L^\perp)\Big) \times (\omega \cap L^\perp)\right) \nu_k(dL)\\
&\quad = \sum_{i_1,\dots,i_l=0}^m \binom{m}{i_1,\dots,i_l} \rho_1^{i_1} \cdots \rho_l^{i_l} \int_{G(d,k)} \mathbf{1}\{L \in A\}\\
&\qquad\qquad \times \Xi_{i_1,\dots,i_l}^{L^\perp}(K_1|L^\perp, \dots, K_l|L^\perp, (\beta_1|L^\perp) \times \cdots \times (\beta_l|L^\perp) \times (\omega \cap L^\perp))\, \nu_k(dL)\\
&\quad = \sum_{i_1,\dots,i_l=0}^m \binom{m}{i_1,\dots,i_l} \rho_1^{i_1} \cdots \rho_l^{i_l}\, \Xi_{i_1,\dots,i_l}^{(k)}(K_1, \dots, K_l, \beta_1 \times \cdots \times \beta_l \times \gamma).
\end{aligned}$$

Here we denote the mixed support measure of the convex bodies $K_1|L^\perp, \dots, K_l|L^\perp$ in $L^\perp$ by $\Xi_{i_1,\dots,i_l}^{L^\perp}(K_1|L^\perp, \dots, K_l|L^\perp, \cdot)$ and define the mixed flag measure $\Xi_{i_1,\dots,i_l}^{(k)}(K_1, \dots, K_l, \cdot)$ by

$$\begin{aligned}
&\Xi_{i_1,\dots,i_l}^{(k)}(K_1, \dots, K_l, \beta_1 \times \cdots \times \beta_l \times \omega \times A) \qquad (28)\\
&\quad = \int_{G(d,k)} \mathbf{1}\{L \in A\}\, \Xi_{i_1,\dots,i_l}^{L^\perp}(K_1|L^\perp, \dots, K_l|L^\perp, (\beta_1|L^\perp) \times \cdots \times (\beta_l|L^\perp) \times (\omega \cap L^\perp))\, \nu_k(dL).
\end{aligned}$$

The remaining assertions of **(a)** follow from (27), (28) and the corresponding properties of mixed support measures in Theorem 2.2(a).

**(b)** and **(c)** follow from (28) and Theorem 2.2(b) and (c). □

**Remark 5.5.** *In the following, we simply write* $\Xi^{(k)}(K_1,\dots,K_m,\cdot)$ *for* $\Xi^{(k)}_{1,\dots,1}(K_1,\dots,K_m,\cdot)$, *where* $m\in\{1,\dots,d-1-k\}$ *and the index* 1 *in the measure* $\Xi^{(k)}_{1,\dots,1}$ *appears* $m$ *times.*

We emphasize two special cases of the preceding theorem.

**(1)** Let $m=d-1-k$ and $r\in\{0,\dots,d-1-k\}$ in Theorem 5.4(c). Then, for $K\in\mathcal{K}$ we have

$$\Xi^{(k)}_r(K,\beta\times\gamma)=\frac{2\binom{d-k}{r}}{(d-k)\kappa_{d-k-r}}\,\Xi^{(k)}_{r,d-1-k-r}(K,B^d,\beta\times S^{d-1}\times\gamma).$$

**(2)** Let $m=d-1-k\geq 1$ and $r\in\{0,\dots,d-1-k\}$. If $K\in\mathcal{K}$ is strictly convex, Theorem 5.4(b) and (c) imply that

$$\Xi^{(k)}_r(K,\beta\times\gamma)=\frac{2\binom{d-k}{r}}{(d-k)\kappa_{d-k-r}}\,\Xi^{(k)}_{1,\dots,1}(K[r],B^d[d-1-k-r],(\beta)^r\times(S^{d-1})^{d-1-k-r}\times\gamma).$$

The condition of general relative position is not necessary for the multilinear expansion of the area measures. This carries over to the flag area measures. We list some of the resulting formulas (using the obvious notation for the mixed measures), which are obtained from the preceding results by passing to image measures with respect to the map $(x,u,L)\mapsto(u,L)$ and using the weak continuity of these image measures without the restriction to convex bodies in general relative position. Thus we obtain

$$\begin{aligned}\Psi^{(k)}_m(\rho_1K_1+\dots+\rho_lK_l,\cdot)&=\sum_{i_1,\dots,i_l=0}^{m}\binom{m}{i_1,\dots,i_l}\rho_1^{i_1}\cdots\rho_l^{i_l}\,\Psi^{(k)}(K_1[i_1],\dots,K_l[i_l],\cdot)\\&=\sum_{j_1,\dots,j_m=1}^{l}\rho_{j_1}\cdots\rho_{j_m}\Psi^{(k)}(K_{j_1},\dots,K_{j_m},\cdot),\end{aligned}$$

$$\Psi^{(k)}_m(K,\cdot)=\frac{2\binom{d-k}{m}}{(d-k)\kappa_{d-k-m}}\,\Psi^{(k)}(K[m],B^d[d-1-k-m],\cdot)\tag{29}$$

and

$$\begin{aligned}\Psi^{(k)}_m(K+\rho B^d,\cdot)&=\sum_{r=0}^{m}\rho^r\binom{m}{r}\,\Psi^{(k)}(K[m-r],B^d[r],\cdot)\\&=\sum_{r=0}^{m}\rho^r\binom{m}{r}\,\Psi^{(k)}_{m-r,r}(K,B^d,\cdot)\\&=\sum_{r=0}^{m}\rho^{m-r}\binom{d-k-r}{d-k-m}\frac{\kappa_{d-k-r}}{\kappa_{d-k-m}}\,\Psi^{(k)}_r(K,\cdot).\end{aligned}$$

The latter also follows directly from Proposition 5.2.

Finally, in this section, we consider relations between the flag measures $\Xi^{(k)}_m(K,\cdot)$, for fixed $m$ but different $k$. In Theorem 5.6 we show that these measures are connected by an integral relation. For $k\in\{0,\dots,d-1\}$, $r\in\{k,\dots,d-1\}$, $m\in\{0,\dots,d-r-1\}$ and $L\in G(d,k)$, we denote by $\Xi^{L^\perp,(r-k)}_m(K|L^\perp,\cdot)$ the flag measure of type $(r-k,m)$ of the convex body $K|L^\perp$, computed in $L^\perp$. The same convention applies to $g^{L^\perp}$. Note that the special case $r=k$ of the following theorem boils down to relation (17).

**Theorem 5.6.** *For $K \in \mathcal{K}$, $k \in \{0, \dots, d-1\}$, $r \in \{k, \dots, d-1\}$, $m \in \{0, \dots, d-r-1\}$, we have*

$$\Xi_m^{(r)}(K, \cdot) = \int_{G(d,k)} \int \mathbf{1}\{(g(x, L, K), u, V + L) \in \cdot\} \, \Xi_m^{L^\perp,(r-k)}(K|L^\perp, d(x, u, V)) \, \nu_k(dL).$$

*Proof.* In the following, if $V \in G(d, k)$ is a subspace of $\mathbb{R}^d$ and $l \in \{0, \dots, k\}$, we write $G(V, l)$ for the set of $W \in G(d, l)$ with $W \subset V$. However, if $l \in \{k, \dots, d\}$, then $G(V, l)$ is the set of all $W \in G(d, l)$ with $V \subset W$. In any case, the Haar probability measure on $G(V, l)$ is denoted by $\nu_l^V$. Let $f : \mathbb{R}^d \times S^{d-1} \times G(d, k) \to [0, \infty)$ be measurable. Then, applying (17) in $L^\perp$, we get

$$\begin{aligned}
&\int_{G(d,k)} \int f(g(x, L, K), u, V + L) \, \Xi_m^{L^\perp,(r-k)}(K|L^\perp, d(x, u, V)) \, \nu_k(dL) \\
&= \int_{G(d,k)} \int_{G(L^\perp,r-k)} \int f\left(g\left(g^{L^\perp}(z, W, K|L^\perp), L, K\right), w, W + L\right) \\
&\qquad \times \Xi_m^{L^\perp \cap W^\perp}\left((K|L^\perp)|(L^\perp \cap W^\perp), d(z, w)\right) \nu_{r-k}^{L^\perp}(dW) \, \nu_k(dL) \\
&= \int_{G(d,k)} \int_{G(L^\perp,r-k)} \int f(g(z, W + L, K), w, W + L) \\
&\qquad \times \Xi_m^{(W+L)^\perp}\left(K|(W + L)^\perp, d(z, w)\right) \nu_{r-k}^{L^\perp}(dW) \, \nu_k(dL) \\
&= \int_{G(d,k)} \int_{G(L,r)} \int f(g(z, U, K), w, U) \, \Xi_m^{U^\perp}\left(K|U^\perp, d(z, w)\right) \nu_r^L(dU) \, \nu_k(dL),
\end{aligned}$$

where we put $U = W + L$. Now [29, Theorem 7.1.1] and another application of (17) yield that

$$\begin{aligned}
&\int_{G(d,k)} \int f(g(x, L, K), u, V + L) \, \Xi_m^{L^\perp,(r-k)}(K|L^\perp, d(x, u, V)) \, \nu_k(dL) \\
&= \int_{G(d,r)} \int f(g(z, U, K), u, U) \, \Xi_m^{U^\perp}(K|U^\perp, d(z, u)) \, \nu_r(dU) \\
&= \int f(z, u, U) \, \Xi_m^{(r)}(K, d(z, u, U)).
\end{aligned}$$

□

From the preceding result, extensions for mixed measures can be obtained by the procedure described before.

# 6 Flag measures for generalized zonoids

A *generalized zonoid* is a centrally symmetric convex body $Z \in \mathcal{K}$, the support function of which is of the form

$$h(Z, \cdot) = \int_{S^{d-1}} |\langle \cdot, v \rangle| \, \rho(Z, dv)$$

(up to a linear function). Here, $\rho(Z, \cdot)$ is an even finite signed Borel measure on $S^{d-1}$, the *generating measure* of $Z$. For simplicity (and since flag area measures are invariant under translations of the bodies), we assume that the center of $Z$ is at the origin $o$. Then, the mapping $Z \mapsto \rho(Z, \cdot)$ is injective and the measure $\rho(Z, \cdot)$ is uniquely determined by $Z$. If $\rho(Z, \cdot) \geq 0$, then $Z$ is a *zonoid*, and if $\rho(Z, \cdot)$

is moreover discrete (a finite combination of Dirac measures), $Z$ is a *zonotope* (a finite sum of line segments).

For generalized zonoids, formulas are known which express mixed area measures (and thus also mixed volumes) in terms of the generating measures (see [31]). For the curvature measures, such representations are not available. In this section, we discuss corresponding results for flag area measures. We first give a general result for the (ordinary) mixed area measure. For vectors $u_1, \dots, u_k \neq 0$, we denote by $E(u_1, \dots, u_k)$ their linear hull, if this space is $k$-dimensional, and we put $E(u_1, \dots, u_k) = E_0$, for a fixed subspace $E_0 \in G(d, k)$, otherwise. Let $D_k(u_1, \dots, u_k)$ denote the $k$-dimensional volume of the parallelepiped spanned by $u_1, \dots, u_k$. If $D_k(u_1, \dots, u_k) > 0$, this is also the absolute value of the determinant of $u_1, \dots, u_k$ (calculated in $E(u_1, \dots, u_k)$). For a subspace $M \in G(d, m)$ and $k + m \leq d$, we define $D_{k+m}(u_1, \dots, u_k, M)$ by $D_{k+m}(u_1, \dots, u_k, v_1, \dots, v_m)$, where $v_1, \dots, v_m$ is an orthonormal basis in $M$. This notion does not depend on the choice of the basis.

In order to simplify the comparison with the literature, we remark that the normalization of the mixed surface area measures $S(K_1, \dots, K_{d-1}, \cdot)$ and area measures $S_j(K, \cdot)$, as used in [27], is related to the present normalization by

$$2\Psi(K_1, \dots, K_{d-1}, \cdot) = S(K_1, \dots, K_{d-1}, \cdot)$$

and

$$\Psi_j(K, \cdot) = \frac{\binom{d}{j}}{d\kappa_{d-j}} \, S_j(K, \cdot).$$

Since $S_j(K) = S(K[j], B^d[d-1-j], \cdot)$, we get (cf. (29))

$$\Psi_j(K, \cdot) = \frac{2\binom{d}{j}}{d\kappa_{d-j}} \, \Psi(K[j], B^d[d-1-j], \cdot).$$

**Theorem 6.1.** *For $k \in \{1, \dots, d-2\}$, let $Z_1, \dots, Z_k$ be generalized zonoids with generating measures $\rho(Z_1, \cdot), \dots, \rho(Z_k, \cdot)$ and let $K_{k+1}, \dots, K_{d-1} \in \mathcal{K}$ be arbitrary bodies. Then, for a Borel set $\omega \subset S^{d-1}$, we have*

$$\begin{aligned}\Psi(Z_1, \dots, Z_k, K_{k+1}, \dots, K_{d-1}, \omega) = \frac{2^k}{k!\binom{d-1}{k}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} D_k(u_1, \dots, u_k) \\ \times \Psi^{E^\perp}(K_{k+1}|E^\perp, \dots, K_{d-1}|E^\perp, \omega \cap E^\perp) \, \rho(Z_1, du_1) \dots \rho(Z_k, du_k)\end{aligned}$$

*with $E = E(u_1, \dots, u_k)$.*

If in the statement of the theorem, the dimension of the subspace $E$ is smaller than $k$, then $\Psi^{E^\perp}(\cdots)$ is not properly defined. In this case, however, $D_k(u_1, \dots, u_k) = 0$ and therefore it is consistent to define the integrand as zero. Similar conventions will be adopted in the following.

*Proof.* The proof follows the lines in [26, Proposition 3.7].

We define a measure $\mu$ by

$$\begin{aligned}\mu(\omega) = &\frac{2^k}{k!\binom{d-1}{k}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} D_k(u_1, \dots, u_k) \\ &\times \Psi^{E^\perp}(K_{k+1}|E^\perp, \dots, K_{d-1}|E^\perp, \omega \cap E^\perp) \, \rho(Z_1, du_1) \dots \rho(Z_k, du_k),\end{aligned}$$

where $\omega \subset S^{d-1}$ is a Borel set.

For $M \in \mathcal{K}$, we use [27, Theorem 5.1.6] and [28, formula (31)] and get

$$\begin{aligned}
&\int_{S^{d-1}} h(M,v)\,\mu(dv)\\
&= \frac{2^k}{k!\binom{d-1}{k}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{S^{d-1}\cap E^\perp} D_k(u_1,\dots,u_k)\, h(M,v)\\
&\qquad \times \Psi^{E^\perp}(K_{k+1}|E^\perp,\dots,K_{d-1}|E^\perp,dv)\,\rho(Z_1,du_1)\dots\rho(Z_k,du_k)\\
&= \frac{2^k(d-k)}{2k!\binom{d-1}{k}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} D_k(u_1,\dots,u_k)\\
&\qquad \times V^{E^\perp}(K_{k+1}|E^\perp,\dots,K_{d-1}|E^\perp,M|E^\perp)\,\rho(Z_1,du_1)\dots\rho(Z_k,du_k)\\
&= \frac{2^k(d-k)}{2k!\binom{d-1}{k}} \frac{k!\binom{d}{k}}{2^k} V(K_{k+1},\dots,K_{d-1},M,Z_1,\dots,Z_k)\\
&= \frac{d}{2} V(M,K_{k+1},\dots,K_{d-1},Z_1,\dots,Z_k)\\
&= \int_{S^{d-1}} h(M,v)\,\Psi(Z_1,\dots,Z_k,K_{k+1},\dots,K_{d-1},dv).
\end{aligned}$$

Since $M$ was arbitrary and differences of support functions are dense in the Banach space of continuous functions on $S^{d-1}$, we deduce $\mu = \Psi(Z_1,\dots,Z_k,K_{k+1},\dots,K_{d-1},\cdot)$. □

For $k = d-1$, there would be no bodies $K_{k+1},\dots,K_{d-1}$ in the above theorem. If we interpret $\Psi^{E^\perp}(K_{k+1}|E^\perp,\dots,K_{d-1}|E^\perp,\cdot)$, in this case, as $\frac{1}{2}\mathcal{H}^0 \llcorner (E^\perp \cap S^{d-1})$ ($\llcorner$ denotes the restriction of a measure), then the proof goes through and yields

$$\begin{aligned}
\Psi(Z_1,\dots,Z_{d-1},\omega) = \frac{2^{d-1}}{(d-1)!} \int_{S^{d-1}} \cdots \int_{S^{d-1}} D_{d-1}(u_1,\dots,u_{d-1})\\
\times \frac{1}{2}\mathcal{H}^0(\omega \cap E^\perp)\,\rho(Z_1,du_1)\dots\rho(Z_{d-1},du_{d-1})
\end{aligned} \tag{30}$$

with $E = E(u_1,\dots,u_{d-1})$. We can modify this formula slightly by introducing a (measurable) mapping $T : (S^{d-1})^{d-1} \to S^{d-1}$. For linearly independent $u_1,\dots,u_{d-1}$, we put $T(u_1,\dots,u_{d-1}) = u$, where $u \in E(u_1,\dots,u_{d-1})^\perp$ is such that $u_1,\dots,u_{d-1},u$ are positively oriented (with respect to a fixed reference basis). For linearly dependent $u_1,\dots,u_{d-1}$, we put $T(u_1,\dots,u_{d-1}) = u_0$ for a fixed vector $u_0 \in S^{d-1}$. Let $T\rho$ denote the image measure (on $S^{d-1}$) of a (signed) measure $\rho$ on $(S^{d-1})^{d-1}$ under $T$. If $L \in G(d,k)$, we use an upper index $T^L$ to denote the corresponding mapping $T^L : (S^{d-1}\cap L)^{k-1} \to S^{d-1}\cap L$. In this case, we associate with each $L \in G(d,k)$ an orientation in a measurable way. We also use the mappings $T$ and $T^L$ in connection with a subspace $M \in G(d,m)$, with which we associate an orientation in a measurable way, and in case of $T^L$, we assume $M \subset L, M \neq L$. Namely, $T(u_1,\dots,u_{d-m-1},M)$, for linearly independent unit vectors $u_1,\dots,u_{d-m-1}$, is defined as $T(u_1,\dots,u_{d-m-1},v_1,\dots,v_m)$ where $v_1,\dots,v_m$ is a (positively oriented) orthonormal basis of $M$. In a similar way, $T^L(u_1,\dots,u_{k-m-1},M)$ is defined. Since we are integrating with respect to even measures, the particular choice of the orientations is irrelevant.

The following corollary arises as a special case of Theorem 6.1. The results stated in Corollary 6.2 were previously obtained in [31].

**Corollary 6.2.** *Let $Z, Z_1, \dots, Z_d$ be generalized zonoids with corresponding generating measures $\rho(Z,\cdot), \rho(Z_1,\cdot), \dots, \rho(Z_{d-1},\cdot)$. Then we have*

$$\Psi(Z_1,\dots,Z_{d-1},\cdot) = \frac{2^{d-1}}{(d-1)!}\, T\left[\int_{(\cdot)} D_{d-1}\, d(\rho(Z_1,\cdot)\times\cdots\times\rho(Z_{d-1},\cdot))\right]$$

*and, for $j \in \{0,\dots,d-1\}$,*

$$\Psi_j(Z,\cdot) = \frac{2^{j+1}\binom{d}{j}}{d!\kappa_{d-1}^{d-j-1}\kappa_{d-j}}\, T\left[\int_{(\cdot)} D_{d-1}\, d(\rho(Z,\cdot)^j \times (\mathcal{H}^{d-1})^{d-j-1})\right].$$

The first formula is a reformulation of (30) and the second follows from the first since $B^d$ is a zonoid with generating measure (see [31, Satz 3 and its proof])

$$\rho(B^d,\cdot) = \frac{1}{2\kappa_{d-1}}\, \mathcal{H}^{d-1} \llcorner S^{d-1}.$$

We remark that the formulas in Theorem 6.1 and Corollary 6.2 imply corresponding integral representations for mixed volumes involving generalized zonoids, since

$$V(K_1,\dots,K_d) = \frac{2}{d}\int_{S^{d-1}} h(K_d,u)\,\Psi(K_1,\dots,K_{d-1},du)$$

for $K_1,\dots,K_d \in \mathcal{K}$.

In order to obtain similar results for mixed flag area measures, we apply a special case of the projection formula (28), namely

$$\begin{aligned}&\Psi^{(k)}(K_1,\dots,K_{d-k-1},\cdot) \qquad (31)\\ &= \int_{G(d,k)}\int_{S^{d-1}\cap L^\perp} \mathbf{1}\{(u,L)\in\cdot\}\,\Psi^{L^\perp}(K_1|L^\perp,\dots,K_{d-k-1}|L^\perp,du)\,\nu_k(dL),\end{aligned}$$

where $k \in \{0,\dots,d-2\}$ and $K_1,\dots,K_{d-k-1} \in \mathcal{K}$ are arbitrary.

It is therefore important that orthogonal projections $Z|L^\perp$ of generalized zonoids $Z$ to a subspace $L^\perp$, for $L \in G(d,k)$, are generalized zonoids again (in $L^\perp$) (see [34]). In order to describe the generating measure $\rho(Z|L^\perp,\cdot)$ of $Z|L^\perp$, we introduce the *spherical projection*

$$\Pi_{L^\perp} : S^{d-1}\backslash L \to S^{d-1}\cap L^\perp, \qquad u \mapsto \frac{u|L^\perp}{\|u|L^\perp\|}.$$

Then [34, Theorem 4.1] shows that

$$\rho(Z|L^\perp,\cdot) = \int_{S^{d-1}\backslash L} \mathbf{1}\{\Pi_{L^\perp}u \in\cdot\}\,\|u|L^\perp\|\,\rho(Z,du). \qquad (32)$$

In the integration domain we can replace $S^{d-1}\setminus L$ by $S^{d-1}$, since $\|u|L^\perp\| = 0$ if $u \in L$, and the integrand is taken as zero in this case. We now extend Theorem 6.1 to flag measures.

**Theorem 6.3.** *For $k \in \{0,\dots,d-3\}$ and $l \in \{1,\dots,d-k-2\}$, let $Z_1,\dots,Z_l$ be generalized zonoids with generating measures $\rho(Z_1,\cdot),\dots,\rho(Z_l,\cdot)$ and let $K_{l+1},\dots,K_{d-k-1} \in \mathcal{K}$ be arbitrary. Then we have*

$$\begin{aligned}&\Psi^{(k)}(Z_1,\dots,Z_l,K_{l+1},\dots,K_{d-k-1},\cdot)\\ &= \frac{2^l}{l!\binom{d-k-1}{l}}\int_{S^{d-1}}\cdots\int_{S^{d-1}}\int_{G(d,k)}\int_{S^{d-1}\cap L^\perp\cap E^\perp} D_l(v_1|L^\perp,\dots,v_l|L^\perp)\,\mathbf{1}\{(u,L)\in\cdot\}\\ &\quad\times \Psi^{L^\perp\cap E^\perp}(K_{l+1}|(L^\perp\cap E^\perp),\dots,K_{d-k-1}|(L^\perp\cap E^\perp),du)\,\nu_k(dL)\,\rho(Z_1,dv_1)\dots\rho(Z_l,dv_l),\end{aligned}$$

*with $E = E(v_1|L^\perp,\dots,v_l|L^\perp)$.*

*Proof.* From (31) and from Theorem 6.1 applied in $L^\perp$, we get

$$\begin{aligned}
&\Psi^{(k)}(Z_1,\dots,Z_l,K_{l+1},\dots,K_{d-k-1},\cdot)\\
&=\int_{G(d,k)}\int_{S^{d-1}\cap L^\perp}\mathbf{1}\{(u,L)\in\cdot\}\\
&\quad\times\ \Psi^{L^\perp}(Z_1|L^\perp,\dots,Z_l|L^\perp,K_{l+1}|L^\perp,\dots,K_{d-k-1}|L^\perp,du)\,\nu_k(dL)\\
&=\int_{G(d,k)}\frac{2^l}{l!\binom{d-k-1}{l}}\int_{S^{d-1}\cap L^\perp}\cdots\int_{S^{d-1}\cap L^\perp}\int_{S^{d-1}\cap L^\perp\cap E^\perp}D_l(v_1,\dots,v_l)\,\mathbf{1}\{(u,L)\in\cdot\}\\
&\quad\times\Psi^{L^\perp\cap E^\perp}(K_{l+1}|(L^\perp\cap E^\perp),\dots,K_{d-k-1}|(L^\perp\cap E^\perp),du)\\
&\quad\times\rho(Z_1|L^\perp,dv_1)\dots\rho(Z_l|L^\perp,dv_l)\,\nu_k(dL),
\end{aligned}$$

where $E=E(v_1,\dots,v_l)$. Using (32), we obtain

$$\begin{aligned}
&\Psi^{(k)}(Z_1,\dots,Z_l,K_{l+1},\dots,K_{d-k-1},\cdot)\\
&=\frac{2^l}{l!\binom{d-k-1}{l}}\int_{G(d,k)}\int_{S^{d-1}\setminus L}\cdots\int_{S^{d-1}\setminus L}\int_{S^{d-1}\cap L^\perp\cap E^\perp}D_l(\Pi_{L^\perp}v_1,\dots,\Pi_{L^\perp}v_l)\,\mathbf{1}\{(u,L)\in\cdot\}\\
&\quad\times\ \|v_1|L^\perp\|\cdots\|v_l|L^\perp\|\,\Psi^{L^\perp\cap E^\perp}(K_{l+1}|(L^\perp\cap E^\perp),\dots,K_{d-k-1}|(L^\perp\cap E^\perp),du)\\
&\quad\times\rho(Z_1,dv_1)\dots\rho(Z_l,dv_l)\,\nu_k(dL)\\
&=\frac{2^l}{l!\binom{d-k-1}{l}}\int_{G(d,k)}\int_{S^{d-1}\setminus L}\cdots\int_{S^{d-1}\setminus L}\int_{S^{d-1}\cap L^\perp\cap E^\perp}D_l(v_1|L^\perp,\dots,v_l|L^\perp)\,\mathbf{1}\{(u,L)\in\cdot\}\\
&\quad\times\Psi^{L^\perp\cap E^\perp}(K_{l+1}|(L^\perp\cap E^\perp),\dots,K_{d-k-1}|(L^\perp\cap E^\perp),du)\\
&\quad\times\ \rho(Z_1,dv_1)\dots\rho(Z_l,dv_l)\,\nu_k(dL)\\
&=\frac{2^l}{l!\binom{d-k-1}{l}}\int_{S^{d-1}}\cdots\int_{S^{d-1}}\int_{G(d,k)}\int_{S^{d-1}\cap L^\perp\cap E^\perp}D_l(v_1|L^\perp,\dots,v_l|L^\perp)\,\mathbf{1}\{(u,L)\in\cdot\}\\
&\quad\times\Psi^{L^\perp\cap E^\perp}(K_{l+1}|(L^\perp\cap E^\perp),\dots,K_{d-k-1}|(L^\perp\cap E^\perp),du)\\
&\quad\times\ \nu_k(dL)\,\rho(Z_1,dv_1)\dots\rho(Z_l,dv_l),
\end{aligned}$$

where $E=E(v_1|L^\perp,\dots,v_l|L^\perp)$.

In the last step, we first replaced $S^{d-1}\setminus L$ by $S^{d-1}$, since the integrand $D_l(v_1|L^\perp,\dots,v_l|L^\perp)=0$ if $v_i\in L$ for some $i\in\{1,\dots,l\}$, and then we applied Fubini's theorem. □

Again, the case $l=d-k-1$, where there are no additional bodies $K_{l+1},\dots,K_{d-k-1}$ and also $k=d-2$ is admitted, can be treated with the same proof, if Corollary 6.2 instead of Theorem 6.1 is used. This gives the following result.

**Corollary 6.4.** *For $k\in\{0,\dots,d-2\}$, let $Z_1,\dots,Z_{d-k-1}$ be generalized zonoids with generating* *measures* $\rho(Z_1,\cdot),\dots,\rho(Z_{d-k-1},\cdot)$. *Then we have*

$$\begin{aligned}
&\Psi^{(k)}(Z_1,\dots,Z_{d-k-1},\cdot)\\
&= \frac{2^{d-k-1}}{(d-k-1)!}\int_{S^{d-1}}\dots\int_{S^{d-1}}\int_{G(d,k)} D_{d-k-1}(u_1|L^\perp,\dots,u_{d-k-1}|L^\perp)\\
&\quad\times \mathbf{1}\{(T^{L^\perp}(\Pi_{L^\perp}u_1,\dots,\Pi_{L^\perp}u_{d-k-1}),L)\in\cdot\}\,\nu_k(dL)\,\rho(Z_1,du_1)\dots\rho(Z_{d-k-1},du_{d-k-1})\\
&= \frac{2^{d-k-1}}{(d-k-1)!}\int_{S^{d-1}}\dots\int_{S^{d-1}}\int_{G(d,d-k)} D_{d-k-1}(u_1|L,\dots,u_{d-k-1}|L)\\
&\quad\times \mathbf{1}\{(T^{L}(\Pi_{L}u_1,\dots,\Pi_{L}u_{d-k-1}),L^\perp)\in\cdot\})\,\nu_{d-k}(dL)\,\rho(Z_1,du_1)\dots\rho(Z_{d-k-1},du_{d-k-1}).
\end{aligned}$$

On the other hand, we can also consider the case of Theorem 6.3, where the bodies $K_i$ are generalized zonoids as well. Then we get intermediate formulas for $\Psi^{(k)}(Z_1,\dots,Z_{d-k-1},\cdot)$.

**Remark 6.5.** *For generalized zonoids* $Z_1,\dots,Z_{d-k-1}$, $k\in\{0,\dots,d-3\}$, *and* $l\in\{1,\dots,d-k-2\}$, *we have*

$$\begin{aligned}
&\Psi^{(k)}(Z_1,\dots,Z_{d-k-1},\cdot)\\
&= \frac{2^l}{l!\binom{d-k-1}{l}}\int_{S^{d-1}}\dots\int_{S^{d-1}}\int_{G(d,k)}\int_{S^{d-1}\cap L^\perp\cap E^\perp} D_l(v_1|L^\perp,\dots,v_l|L^\perp)\,\mathbf{1}\{(u,L)\in\cdot\}\\
&\quad\times \Psi^{L^\perp\cap E^\perp}(Z_{l+1}|(L^\perp\cap E^\perp),\dots,Z_{d-k-1}|(L^\perp\cap E^\perp),du)\,\nu_k(dL)\,\rho(Z_1,dv_1)\dots\rho(Z_l,dv_l),
\end{aligned}$$

*where* $E=E(v_1|L^\perp,\dots,v_l|L^\perp)$.

A case of special interest is given if $Z_{l+1},\dots,Z_{d-k-1}=B^d$. Here, we obtain the following result.

**Theorem 6.6.** *For* $k\in\{1,\dots,d-2\}$ *and* $l\in\{1,\dots,d-k-1\}$, *let* $Z_1,\dots,Z_l$ *be generalized zonoids with generating measures* $\rho(Z_1,\cdot),\dots,\rho(Z_l,\cdot)$. *Then we have*

$$\begin{aligned}
&\Psi^{(k)}(Z_1,\dots,Z_l,B^d[d-k-l-1],\cdot)\\
&= c(d,k,l)\int_{S^{d-1}}\dots\int_{S^{d-1}}\int_{G(d,d-k)}\int_{G(L,d-k-l-1)} D_{d-k-1}(u_1|L,\dots,u_l|L,U)\\
&\quad\times \mathbf{1}\{(T^L(\Pi_L u_1,\dots,\Pi_L u_l,U),L^\perp)\in\cdot\}\,\nu^L_{d-k-l-1}(dU)\,\nu_{d-k}(dL)\,\rho(Z_1,du_1)\dots\rho(Z_l,du_l)
\end{aligned}$$

*with*

$$c(d,k,l)=\frac{2^{d-k-1}}{(d-k-1)!}\,\pi^{d-k-l-1}\frac{\omega_{l+2}}{\omega_{d-k+1}}.$$

*Proof.* For $l=d-k-1$ the assertion is implied by Corollary 6.4. Therefore, we assume that $k\le d-3$ and $l<d-k-1$, in the following. Let $L\in G(d,d-k)$. Since

$$\rho(B^d|L,\cdot)=\frac{1}{2\kappa_{d-k-1}}\mathcal{H}^{d-k-1}\llcorner(S^{d-1}\cap L),$$

we obtain for a measurable function $f\ge 0$ on $S^{d-1}\cap L$ that

$$\int_{S^{d-1}\cap L} f(u)\,\rho(B^d|L,du)=\frac{1}{2\kappa_{d-k-1}}\int_{S^{d-1}\cap L} f(u)\,\mathcal{H}^{d-k-1}(du).$$

On the other hand,

$$\int_{S^{d-1}\cap L} f(u)\,\rho(B^d|L,du) = \int_{S^{d-1}} f(\Pi_L u)\,\|u|L\|\,\rho(B^d,du)$$

and thus

$$\int_{S^{d-1}} f(\Pi_L u)\,\|u|L\|\,\rho(B^d,du) = \frac{1}{2\kappa_{d-k-1}}\int_{S^{d-1}\cap L} f(u)\,\mathcal{H}^{d-k-1}(du).$$

From Corollary 6.4, we therefore obtain

$$\begin{aligned}
&\Psi^{(k)}(Z_1,\dots,Z_l,B^d[d-k-l-1],\cdot)\\
&\quad= \frac{2^{d-k-1}}{(d-k-1)!}\int_{G(d,d-k)}\int_{S^{d-1}}\dots\int_{S^{d-1}} D_{d-k-1}(u_1|L,\dots,u_{d-k-1}|L)\\
&\qquad\times \mathbf{1}\{(T^L(\Pi_L u_1,\dots,\Pi_L u_{d-k-1}),L^\perp)\in\cdot\}\\
&\qquad\times \rho(Z_1,du_1)\dots\rho(Z_l,du_l)\,\rho(B^d,du_{l+1})\dots\rho(B^d,du_{d-k-1})\,\nu_{d-k}(dL)\\
&\quad= a(d,k,l)\int_{G(d,d-k)}\int_{S^{d-1}\cap L}\dots\int_{S^{d-1}\cap L}\int_{S^{d-1}}\dots\int_{S^{d-1}}\\
&\qquad\times D_{d-k-1}(u_1|L,\dots,u_l|L,u_{l+1},\dots,u_{d-k-1})\\
&\qquad\times \mathbf{1}\{(T^L(\Pi_L u_1,\dots,\Pi_L u_l,u_{l+1},\dots,u_{d-k-1}),L^\perp)\in\cdot\}\\
&\qquad\times \rho(Z_1,du_1)\dots\rho(Z_l,du_l)\,\mathcal{H}^{d-k-1}(du_{l+1})\dots\mathcal{H}^{d-k-1}(du_{d-k-1})\,\nu_{d-k}(dL)
\end{aligned}$$

with $a(d,k,l)=\frac{2^{d-k-1}}{(d-k-1)!}\left(\frac{1}{2\kappa_{d-k-1}}\right)^{d-k-l-1}$.

If $u_{l+1},\dots,u_{d-k-1}$ are linearly independent, they span a $(d-k-l-1)$-dimensional linear subspace $U\subset L$ and

$$\begin{aligned}
&D_{d-k-1}(u_1|L,\dots,u_l|L,u_{l+1},\dots,u_{d-k-1})\\
&\quad= D_{d-k-1}(u_1|L,\dots,u_l|L,U)\,D_{d-k-l-1}(u_{l+1},\dots,u_{d-k-1}).
\end{aligned}$$

Applying a special case of [3, Theorem 1] to the $(d-k-l-1)$-fold integral over $S^{d-1}\cap L$, we obtain

$$\begin{aligned}
&\Psi^{(k)}(Z_1,\dots,Z_l,B^d[d-k-l-1],\cdot)\\
&\quad= a(d,k,l)\mathcal{H}^{(l+1)(d-k-l-1)}(G(L,d-k-l-1))\\
&\qquad\times\int_{G(d,d-k)}\int_{G(L,d-k-l-1)}\int_{S^{d-1}\cap U}\dots\int_{S^{d-1}\cap U}\int_{S^{d-1}}\dots\int_{S^{d-1}}\\
&\qquad\times D_{d-k-1}(u_1|L,\dots,u_l|L,U)\,D_{d-k-l-1}(v_1,\dots,v_{d-k-l-1})^{l+2}\\
&\qquad\times \mathbf{1}\{(T^L(\Pi_L u_1,\dots,\Pi_L u_l,U),L^\perp)\in\cdot\}\,\rho(Z_1,du_1)\dots\rho(Z_l,du_l)\\
&\qquad\times \mathcal{H}^{d-k-l-2}(dv_1)\dots\mathcal{H}^{d-k-l-2}(dv_{d-k-l-1})\,\nu^L_{d-k-l-1}(dU)\,\nu_{d-k}(dL).
\end{aligned}$$

The integration

$$b(d,k,l) = \int_{S^{d-1}\cap U} \dots \int_{S^{d-1}\cap U} D_{d-k-l-1}(v_1,\dots,v_{d-k-l-1})^{l+2} \times \mathcal{H}^{d-k-l-2}(dv_1)\dots\mathcal{H}^{d-k-l-2}(dv_{d-k-l-1}) = \omega_{d-k+1}^{d-k-l-1} \prod_{j=0}^{d-k-l-2} \frac{\omega_{d-k-l-1-j}}{\omega_{d-k+1-j}}$$

follows from [29, Theorem 8.2.2] by introducing polar coordinates. Therefore we obtain

$$\Psi^{(k)}(Z_1,\dots,Z_l,B^d[d-k-l-1],\cdot) = c(d,k,l) \int_{G(d,d-k)} \int_{G(L,d-k-l-1)} \int_{S^{d-1}} \dots \int_{S^{d-1}} D_{d-k-1}(u_1|L,\dots,u_l|L,U) \times \mathbf{1}\{(T^L(\Pi_L u_1,\dots,\Pi_L u_l,U),L^\perp)\in\cdot\}\,\rho(Z_1,du_1)\dots\rho(Z_l,du_l)\,\nu^L_{d-k-l-1}(dU)\,\nu_{d-k}(dL),$$

where

$$c(d,k,l) = a(d,k,l)\cdot\mathcal{H}^{(l+1)(d-k-l-1)}(G(L,d-k-l-1))\cdot b(d,k,l) = \frac{2^{d-k-1}}{(d-k-1)!}\pi^{d-k-l-1}\frac{\omega_{d-k}\cdots\omega_{d-k-l}}{\omega_1\cdots\omega_{l+1}} \prod_{j=0}^{d-k-l-2}\frac{\omega_{d-k-l-1-j}}{\omega_{d-k+1-j}} = \frac{2^{d-k-1}}{(d-k-1)!}\pi^{d-k-l-1}\frac{\omega_{l+2}}{\omega_{d-k+1}}.$$

The Hausdorff measure of the Grassmannian is determined, for instance, in [7, p. 267]. □

Choosing $Z_1 = \dots = Z_l = Z$ and using (29) and Theorem 6.6, we finally get a formula for the flag area measure $\Psi_l^{(k)}(Z,\cdot)$ of a generalized zonoid $Z$.

**Corollary 6.7.** *For $k\in\{1,\dots,d-2\}$, $l\in\{1,\dots,d-k-1\}$, and a generalized zonoid $Z$, we have*

$$\Psi_l^{(k)}(Z,\cdot) = \frac{2\binom{d-k}{l}}{(d-k)\kappa_{d-k-l}}\cdot c(d,k,l)\int_{S^{d-1}}\dots\int_{S^{d-1}}\int_{G(d,d-k)}\int_{G(L,d-k-l-1)} \times D_{d-k-1}(u_1|L,\dots,u_l|L,U)\mathbf{1}\{(T^L(\Pi_L u_1,\dots,\Pi_L u_l,U),L^\perp)\in\cdot\} \times \nu^L_{d-k-l-1}(dU)\,\nu_{d-k}(dL)\,\rho(Z,du_1)\dots\rho(Z,du_l).$$

**Remark 6.8.** *First choosing $l = d-k-1$ and then replacing $k$ by $d-k-1$ in Corollary 6.7, we obtain*

$$\Psi_k^{(d-k-1)}(Z,\cdot) = \frac{2^k}{k!}\int_{S^{d-1}}\dots\int_{S^{d-1}}\int_{G(d,k+1)} D_k(u_1|L,\dots,u_k|L) \times \mathbf{1}\{(T^L(\Pi_L u_1,\dots,\Pi_L u_k),L^\perp)\in\cdot\}\,\nu_{k+1}(dL)\,\rho(Z,du_1)\dots\rho(Z,du_k).$$

From Proposition 4.5 (see also (4)) we obtain $\Psi_k(Z,\cdot) = c(d,k)\cdot\Psi_k^{(d-k-1)}(Z,\cdot\times G(d,d-k-1))$. Then a special case of Remark 6.8 yields that

$$\Psi_k(Z,\cdot) = c(d,k)\frac{2^k}{k!}\int_{S^{d-1}}\cdots\int_{S^{d-1}}\int_{G(d,k+1)} D_k(u_1|L,\dots,u_k|L)$$
$$\times\, \mathbf{1}\{T^L(\Pi_L u_1,\dots,\Pi_L u_k)\in\cdot\}\,\nu_{k+1}(dL)\,\rho(Z,du_1)\dots\rho(Z,du_k).$$

This formula does not seem to be available in the literature, but (of course) it could be obtained more directly. Calculating the total measure, we recover a known formula (see [31, Satz 6]) for the quermassintegrals of generalized zonoids, that is

$$W_{d-k}(Z) = \frac{2^k(d-k)!\kappa_{d-k}}{d!}\cdot\int_{S^{d-1}}\cdots\int_{S^{d-1}} D_k(u_1,\dots,u_k)\,\rho(Z,du_1)\dots\rho(Z,du_k).$$

To derive this from the preceding result, we use that

$$D_k(u_1|L,\dots,u_k|L) = D_k(u_1,\dots,u_k)\cdot D_k(\xi_1|L,\dots,\xi_k|L),$$

where $\xi_1,\dots,\xi_k$ is an arbitrary orthonormal system with the same span as $u_1,\dots,u_k$ (assumed to be linearly independent). Then we apply [7, middle of page 139 and Section 3.2.13] to obtain

$$\int_{G(d,k+1)} D_k(\xi_1|L,\dots,\xi_k|L)\,\nu_{k+1}(dL) = \frac{\beta_1(d,k)}{\beta_1(k+1,k)} = \frac{1}{c(d,k)},$$

where the constant $\beta_1(d,k)$ is taken from [7, Section 3.2.13].

# 7 Concluding remarks

As an additive and continuous functional on $\mathcal{K}$, the flag measure $\Xi_m^{(k)}(K,\cdot)$ has an additive extension to the *convex ring* $\mathcal{R}$, the class of finite unions of convex bodies (polyconvex sets). This extension is given by the inclusion-exclusion formula and is a signed measure, in general. Since $\Xi_m^{(k)}(K,\cdot)$ is also locally defined, it can even be extended to countable, locally finite unions of convex bodies, the elements of the *extended convex ring* $\mathcal{S}$. For $K\in\mathcal{S}$, the flag measure $\Xi_m^{(k)}(K,\cdot)$ is then a locally finite, signed Radon measure, that is, it is only defined on bounded Borel sets $\eta$. The extended measures are no longer continuous, in general. Also, one has to be cautious with most of the formulas which we discussed so far; they are no longer valid for polyconvex sets. The reason is that the projection approach which we used and the Minkowski addition which underlies the construction of the mixed measures are both not compatible with the additive extension.

As we remarked in the introduction, one major goal in convex geometry is to express global geometric functionals of convex bodies as integrals over locally defined quantities in order to extract local information about the bodies. With respect to flag measures, there is still a variety of open problems, but there are two first results in this direction which deserve to be mentioned.

One result concerns the *projection function* $v_{ij}(K,\cdot)$ of $K\in\mathcal{K}$, where $i\in\{0,\dots,j\}$ and $j\in\{0,\dots,d-1\}$. This is a continuous function on $G(d,j)$, defined by

$$v_{ij}(K,L) = V_i(K|L),\quad L\in G(d,j).$$

The case $i=j=0$ being trivial, let us assume $1\le j\le d-1$. The case $j=d-1$ is well-known, since for $x\ne o$ we have

$$v_{i,d-1}(K,x^\perp) = c_i\int_{S^{d-1}} |\langle x,u\rangle|\,\Psi_i(K,du),\quad i=1,\dots,d-1,$$

with some constant $c_i$. Using flag area measures, this formula generalizes as follows (see [12, 10]). In the following, for $k \in \{0, \dots, d-1\}$ we put $k^* = d - 1 - k$ to emphasize the symmetry of our statements. Then there is a measurable function $g$ on $G(d, j) \times S^{d-1} \times G(d, j^*)$ such that

$$v_{ij}(K, L) = \int_{S^{d-1} \times G(d,j^*)} g(L, u, U) \Psi_i^{(j^*)}(K, d(u, U)), \quad 1 \le i \le j \le d - 1. \tag{33}$$

For the validity of (33), some assumptions on the relative position of $K$ and $L$ are needed. For example, (33) holds, if $K$ is a polytope and $K$ and $L$ are in general relative position. Also, (33) holds for arbitrary $K \in \mathcal{K}$ and $\nu_j$-almost all $L$ (see [12, 10], for more details).

A second result concerns the mixed volume $V(K[m], M[d-m])$, $m = 0, \dots, d$, of two convex bodies $K, M \in \mathcal{K}$, which we already discussed in the introduction. In [15], it is shown that there is a measurable function $f_{m,d-m}$ on $S^{d-1} \times G(d, m^*) \times S^{d-1} \times G(d, (d-m)^*)$ such that

$$V(K[m], -M[d-m]) = \int_{S^{d-1} \times G(d,m^*)} \int_{S^{d-1} \times G(d,(d-m)^*)} f_{m,d-m}(u, U, v, V) \tag{34}$$
$$\times \Psi_{d-m}^{((d-m)^*)}(M, d(v, V)) \Psi_m^{(m^*)}(K, d(u, U)),$$

for $m = 1, \dots, d-1$. Equation (34) holds for bodies $K, M \in \mathcal{K}$ in suitable relative position, e.g. if $K$ and $M$ are polytopes in general relative position or if one of the bodies has a support function of class $C^{1,1}$ (see [15], for details). We remark that (34) can be used to provide formulas for general mixed volumes $V(K_1, \dots, K_d)$ of bodies $K_1, \dots, K_d \in \mathcal{K}$, based on the concept of mixed flag area measures presented above.

A further field of application for flag measures is the theory of valuations. A *valuation* $\varphi$ is an additive functional $\varphi : \mathcal{K} \to \mathbb{R}$. If we require $\varphi$ to be also continuous and motion invariant, then Hadwiger's famous characterization theorem shows that $\varphi$ is a linear combination of the intrinsic volumes $V_j$, $j = 0, \dots, d$. There is an ongoing program to understand the corresponding structure of valuations which are only invariant under translations. McMullen [18, 19] has shown that a valuation $\varphi$ on the class $\mathcal{P}$ of polytopes is weakly continuous (that is, continuous under parallel displacements of the facets of a polytope) and translation invariant if and only if

$$\varphi(P) = \sum_{j=0}^{d-1} \sum_{F \in \mathcal{F}_j(P)} f_j(n(P, F)) V_j(F) + c_d V_d(P), \tag{35}$$

where $c_d$ is a constant and $f_j$ is a simple additive functional on the class $\wp_{d-j-1}^{d-1}$ of at most $(d-j-1)$-dimensional spherical polytopes, for $j = 0, \dots, d-1$. Notice that (35) includes a decomposition of $\varphi$ into homogeneous parts

$$\varphi = \sum_{j=0}^{d-1} \varphi_j,$$

where

$$\varphi_j(P) = \sum_{F \in \mathcal{F}_j(P)} f_j(n(P, F)) V_j(F),$$

is homogeneous of degree $j$, $j = 0, \dots, d-1$ and $\varphi_d = c_d V_d$. Thus, the structure of (weakly continuous, translation invariant) valuations on polytopes is understood to some extent. For the general picture, still many questions remain open. Which $j$-homogeneous valuations $\varphi_j$ on $\mathcal{P}$ admit a continuous extension to $\mathcal{K}$? In [13], this question is studied for a number of different continuity properties of $f_j$. In particular, if $f_j$ is *strongly flag-continuous*, in the sense that

$$f_j(p) = \int_p \int_{G([u],d-j)} \langle [u], L \rangle^2 h_j(u, L)\, \nu_{d-j}^{[u]}(dL)\, \mathcal{H}^{d-j-1}(du), \quad p \in \wp_{d-j-1}^{d-1},$$

for some continuous function $h_j$ on $S^{d-1}\times G(d,d-j)$, where $[u]$ denotes the linear hull of $u \in S^{n-1}$, then

$$\varphi_j(P) = \int_{S^{d-1}\times G(d,d-j)} h_j(u,L)\, \Psi_j^{(d-j-1)}(P, d(u,L)),$$

and a continuous extension to $\mathcal{K}$ is given by

$$\varphi_j(K) = \int_{S^{d-1}\times G(d,d-j)} h_j(u,L)\, \Psi_j^{(d-j-1)}(K, d(u,L)), \quad K \in \mathcal{K}.$$

A special case arises, if $h_j(u,L) = h_j(u)$ does not depend on the subspace $L$. For such *strongly continuous* valuations $\varphi_j$, we get

$$\varphi_j(K) = \int_{S^{d-1}} h_j(u)\, \Psi_j(K, du), \quad K \in \mathcal{K}.$$

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**Authors' address:**

Karlsruhe Institute of Technology, Department of Mathematics, Institute of Stochastics,
D-76128 Karlsruhe, Germany.

**Email addresses:**

daniel.hug@kit.edu, ines.tuerk@kit.edu, wolfgang.weil@kit.edu