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On the chaos expansion of Poisson functionals

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1. Poincaré's inequality

Theorem: Chernoff (81) *Let X be a standard normal random vector and let $g \in L^2(\mathbb{P}_X)$ be differentiable. Then*

$$\mathbb{V}\mathrm{ar}[g(X)] \leq \mathbb{E}\|\nabla g(X)\|^2.$$

Equality holds iff g is a linear function.

Theorem: Wu (00) *Let η be a Poisson process on some measurable space $(\mathbb{Y}, \mathcal{Y})$ with intensity measure λ. Then for any $f \in L^2(\mathbb{P}_\eta)$,*

$$\mathbb{V}\mathrm{ar}[f(\eta)] \leq \mathbb{E} \int (f(\eta + \delta_y) - f(\eta))^2 \lambda(dy).$$

Remark: Equality holds iff $f(\eta)$ is a linear function of η. This will be made more precise later.

Corollary: Chen (85) *Let X be an infinitely purely non-Gaussian infinitely divisible random vector with Lévy measure λ and let $g \in L^2(\mathbb{P}_X)$. Then*

$$\mathbb{V}\mathrm{ar}[g(X)] \le \mathbb{E} \int (g(X+y) - g(X))^2 \lambda(dy).$$

Equality holds iff $\Delta^2_{y_1,y_2} g(X) = 0$ $\mathbb{P}$-a.s. and for λ^2-a.e. (y_1, y_2). Here $\Delta_y g(x) := g(x+y) - g(x)$ and $\Delta^2_{y_1,y_2} := \Delta_{y_2} \circ \Delta_{y_1}$.

Proof: Let η be a Poisson process with intensity measure λ. Let

$$h(\eta) \equiv c + \int_{|x|\le 1} x(\eta - \lambda)(dx) + \int_{|x|>1} x\eta(dx),$$

where $c \in \mathbb{R}^d$ is chosen such that $h(\eta) \overset{d}{=} X$. Apply Poincaré's inequality with

$$f(\eta) = g(h(\eta)).$$

2. Clark-Ocone type representation

Setting: η is a Poisson process on the product $\mathbb{R}_+ \times \mathbb{X}$ of $\mathbb{R}_+$ and a Borel space $\mathbb{X}$.

Assumption: The intensity measure λ of η satisfies

$$\lambda(\{t\} \times \mathbb{X}) = 0, \quad t \geq 0.$$

Consequently we have a.s. that

$$\eta(\{t\} \times \mathbb{X}) \in \{0, 1\}, \quad t \geq 0.$$

Notation: Let $\hat{\eta} := \eta - \lambda$ be the *compensated Poisson process*. Integrals with respect to $\hat{\eta}$ are to be understood in a stochastic sense.

Example: Let $(X_t)_{t\geq 0}$ be a pure jump Lévy process in $\mathbb{R}^d$. Then

$$X_t = \int_{|x|\leq 1}\int_0^t x\hat{\eta}(ds, dx) + \int_{|x|>1}\int_0^t x\eta(ds, dx), \quad t \geq 0,$$

for a Poisson process η on $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$. The intensity measure λ of η is the product of Lebesgue measure and the Lévy measure ν of (X_t) on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\int (|x|^2 \wedge 1)\nu(dx) < \infty.$$

Notation: For $t \in [0, \infty]$ let η_t (resp. η_{t-}) be the restriction of η to $[0, t] \times \mathbb{X}$ (resp. $[0, t) \times \mathbb{X}$).

Theorem: Picard (96), Wu (00), L. and Penrose (09) *Let $f \in L^2(\mathbb{P}_\eta)$. Then*

$$\mathbb{E} \int \mathbb{E}[f(\eta + \delta_{(s,x)}) - f(\eta) | \eta_{s-}]^2 \lambda(d(s, x)) < \infty$$

and we have for any $t \in [0, \infty]$ that $\mathbb{P}$-a.s.

$$\mathbb{E}[f(\eta)|\eta_t] = \mathbb{E}f(\eta) + \int_{[0,t]\times\mathbb{X}} \mathbb{E}[f(\eta + \delta_{(s,x)}) - f(\eta)|\eta_{s-}]\hat{\eta}(d(s, x)).$$

Theorem: Wu (00), L. and Penrose (09) *For any* $f, g \in L^2(\mathbb{P}_\eta)$,

$$\mathbb{C}\mathrm{ov}[f(\eta), g(\eta)]$$

$$= \mathbb{E} \int \mathbb{E}[f(\eta + \delta_{(s,x)}) - f(\eta) | \eta_s] \mathbb{E}[g(\eta + \delta_{(s,x)}) - g(\eta) | \eta_s] \lambda(d(s,x)).$$

Idea of the Proof: The martingales

$$M_t := \mathbb{E}[f(\eta)|\eta_t], \qquad N_t := \mathbb{E}[g(\eta)|\eta_t]$$

satisfy

$$M_t N_t = M_0 N_0 + \int_0^t N_{s-} dM_s + \int_0^t M_{s-} dN_s + \sum_{s \le t} \Delta M_s \Delta N_s.$$

Corollary: For any $f \in L^2(\mathbb{P}_\eta)$,

$$\mathbb{V}\mathrm{ar}[f(\eta)] = \mathbb{E} \int \mathbb{E}[f(\eta + \delta_{(s,x)}) - f(\eta) | \eta_s]^2 \lambda(d(s,x)).$$

In particular,

$$\mathbb{V}\mathrm{ar}[f(\eta)] \leq \mathbb{E} \int (f(\eta + \delta_{(s,x)}) - f(\eta))^2 \lambda(d(s,x)).$$

Proof of the general Poincaré inequality: Randomization!

3. Fock space representation

Setting: η is a Poisson process on some measurable space $(\mathbb{Y}, \mathcal{Y})$ with intensity measure λ. This process can be interpreted as a random element in the space $\mathbf{N}$ of all integer-valued σ-finite measures on $\mathbb{Y}$, equipped with the usual (product) σ-field.

Definition: For $n \in \mathbb{N}$ let $\mathbf{H}_n$ be the space of symmetric functions in $L^2(\lambda^n)$, and let $\mathbf{H}_0 := \mathbb{R}$. The *Fock space* $\mathbf{H}$ associated with η (or λ) is the direct sum of the spaces $\mathbf{H}_n$, $n \geq 0$, equipped with the scalar product

$$\langle f, g\rangle_{\mathbf{H}} := \sum_{n=0}^{\infty} \frac{1}{n!} \langle f_n, g_n\rangle_n, \quad f = (f_n), g = (g_n) \in \mathbf{H},$$

where $\langle \cdot, \cdot \rangle_n$ is the scalar product in $L^2(\lambda^n)$. This is a Hilbert space.

Definition: For a measurable functions $f : \mathbf{N} \to \mathbb{R}$ and $y \in \mathbb{Y}$ we define a function $D_y f : \mathbf{N} \to \mathbb{R}$ by

$$D_y f(\mu) := f(\mu + \delta_y) - f(\mu).$$

For $y_1, \dots, y_n \in \mathbb{Y}$ we define $D^n_{y_1,\dots,y_n} f : \mathbf{N} \to \mathbb{R}$ inductively by

$$D^n_{y_1,\dots,y_n} f := D^1_{y_1} D^{n-1}_{y_2,\dots,y_n} f,$$

where $D^1 := D$ and $D^0 f = f$.

Lemma: *For any* $f \in L^2(\mathbb{P}_\eta)$

$$T_n f(y_1, \dots, y_n) := \mathbb{E} D^n_{y_1,\dots,y_n} f(\eta),$$

defines a function $T_n f \in \mathbf{H}_n$.

Theorem: L. and Penrose (09) *Let* $f, g \in L^2(\mathbb{P}_\eta)$. *Then*

$$\mathbb{E}f(\eta)g(\eta) = (\mathbb{E}f(\eta))(\mathbb{E}g(\eta)) + \sum_{n=1}^{\infty} \frac{1}{n!} \langle T_n f, T_n g \rangle_n.$$

That is,

$$\mathbb{E}f(\eta)g(\eta) = \langle Tf, Tg \rangle_{\mathbf{H}},$$

where $Tf := (T_n f)$ *and* $Tg := (T_n g)$.

Idea of the Proof:

1. Check the result for functions of the form

$$\mu \mapsto \exp\Big[-\int v(y)\mu(dy)\Big],$$

   where $v : \mathbb{Y} \to \mathbb{R}_+$ vanishes outside a set of finite λ-measure.

2. The set $\mathbf{G}$ of all linear combinations of functions of the above type is dense in $L^2(\mathbb{P}_\eta)$.

3. Use Hilbert space and completeness arguments to extend the result from $\mathbf{G}$ to $L^2(\mathbb{P}_\eta)$.

4. Back to the Poincaré inequality

It follows from the Fock space representation that any $f \in L^2(\mathbb{P}_\eta)$ satisfies

$$\mathbb{V}\mathrm{ar}\, f(\eta) = \sum_{n=1}^{\infty} \frac{1}{n!} \langle T_n f, T_n f \rangle_n$$

$$= \int (\mathbb{E} D_y f(\eta))^2 \lambda(dy)$$

$$+ \sum_{n=2}^{\infty} \frac{1}{n!} \iint (\mathbb{E} D^{n-1}_{y_2,\dots,y_n} D_y f(\eta))^2 \lambda^{n-1}(d(y_2,\dots,y_n))\lambda(dy)$$

$$\le \int (\mathbb{E} D_y f(\eta))^2 \lambda(dy)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n!} \iint (\mathbb{E} D^{n}_{y_1,\dots,y_n} D_y f(\eta))^2 \lambda^{n}(d(y_1,\dots,y_n))\lambda(dy).$$

Assuming w.l.o.g. that

$$\mathbb{E}\int (f(\eta+\delta_y)-f(\eta))^2\lambda(dy)<\infty$$

and applying the Fock representation to $D_y f$ for λ-a.e. $y\in\mathbb{Y}$ gives

$$\mathbb{V}\mathrm{ar}\, f(\eta)\le \int \mathbb{E}(D_y f(\eta))^2\lambda(dy).$$

Proposition: *We have equality in the Poincaré inequality iff*

$$D_y f(\eta)=\mathbb{E}D_y f(\eta)\quad \mathbb{P}\text{-}a.s.,\ \lambda\text{-}a.e.\ y.$$

This is equivalent to

$$D_{y_1,y_2} f(\eta)=0\quad \mathbb{P}\text{-}a.s.,\ \lambda^2\text{-}a.e.\ (y_1,y_2).$$

5. Chaos expansion

Definition: Let $n \in \mathbb{N}$ and $g \in \mathbf{H}_n$. Then

$$I_n(g) \equiv \int g \, d\hat{\eta}^n$$

denotes the *multiple Wiener-Itô integral* of g w.r.t. the compensated Poisson process $\hat{\eta} = \eta - \lambda$. For $c \in \mathbb{R}$ let $I_0(c) := c$. These integrals have the properties

$$\mathbb{E} I_n(g) I_n(h) = n! \int g h d\lambda^n, \quad n \in \mathbb{N}_0,$$

and

$$\mathbb{E} I_m(g) I_n(h) = 0, \quad m \neq n.$$

Remarks: (i) Let $g_1, \dots, g_n \in L^1(\lambda) \cap L^2(\lambda)$ have disjoint supports and let f be the symmetrization of $g_1 \otimes \dots \otimes g_n$, where

$$g_1 \otimes \dots \otimes g_n(y_1, \dots, y_n) := g_1(y_1) \cdot \ldots \cdot g_n(y_n), \quad y_1, \dots, y_n \in \mathbb{Y}.$$

Then

$$I_n(f) = \prod_{i=1}^{n} (\eta(g_i) - \lambda(g_i)).$$

(ii) Let $g \in L^1(\lambda) \cap L^2(\lambda)$ and $f := g^{\otimes n}$. Then

$$I_n(f) = \sum_{k=0}^{n} \binom{n}{k} \eta^{(k)}(h^{\otimes k})(-\lambda(h))^{n-k},$$

where $\eta^{(k)}$ is the kth factorial moment measure associated with η.

Theorem: Wiener (38), Itô (56) Assume that λ is diffuse and let $f \in L^2(\mathbb{P}_\eta)$. Then there are functions $f_n \in \mathbf{H}_n$, $n \in \mathbb{N}_0$, such that

$$f(\eta) = \sum_{n=0}^{\infty} I_n(f_n),$$

where the series converges in $L^2(\mathbb{P})$.

Theorem: Y. Ito (88), L. and Penrose (09) *For any* $f \in L^2(\mathbb{P}_\eta)$,

$$f(\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(T_n f).$$

6. Variance inequalities

Theorem: Houdré and Perez-Abreu (95), L. and Penrose (09) *Let $f \in L^2(\mathbb{P}_\eta)$ and $k \in \mathbb{N}$ be such that*

$$\mathbb{E}\|D^n f(\eta)\|_n^2 < \infty, \quad n = 1, \ldots, 2k.$$

Then

$$\sum_{n=1}^{2k} \frac{(-1)^{n+1}}{n!} \mathbb{E}\|D^n f(\eta)\|_n^2 \le \mathbb{V}\mathrm{ar}[f(\eta)] \le \sum_{n=1}^{2k-1} \frac{(-1)^{n+1}}{n!} \mathbb{E}\|D^n f(\eta)\|_n^2.$$

The equality cases can be characterized as before.

6. References

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