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On level crossings of certain Markovian jump processes

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1. The process

We consider a real-valued piecewise-deterministic Markov process $X = (X_t)_{t \geq 0}$ driven by three characteristics:

(i) a jump intensity function $\lambda : \mathbb{R} \to [0, \infty)$,

(ii) a drift coefficient $\mu : \mathbb{R} \to \mathbb{R}$,

(iii) and a stochastic kernel $J(x, dz)$ from $\mathbb{R}$ to $\mathbb{R}$ satisfying $J(x, \{0\}) = 0$.

Interpretation of the characteristics:

(i) The function λ gives the stochastic intensity of jumps:

$$\mathbb{P}(X \text{ jumps in } [t, t+h] | \mathcal{F}_t^X) = \lambda(X_t)h + o(h).$$

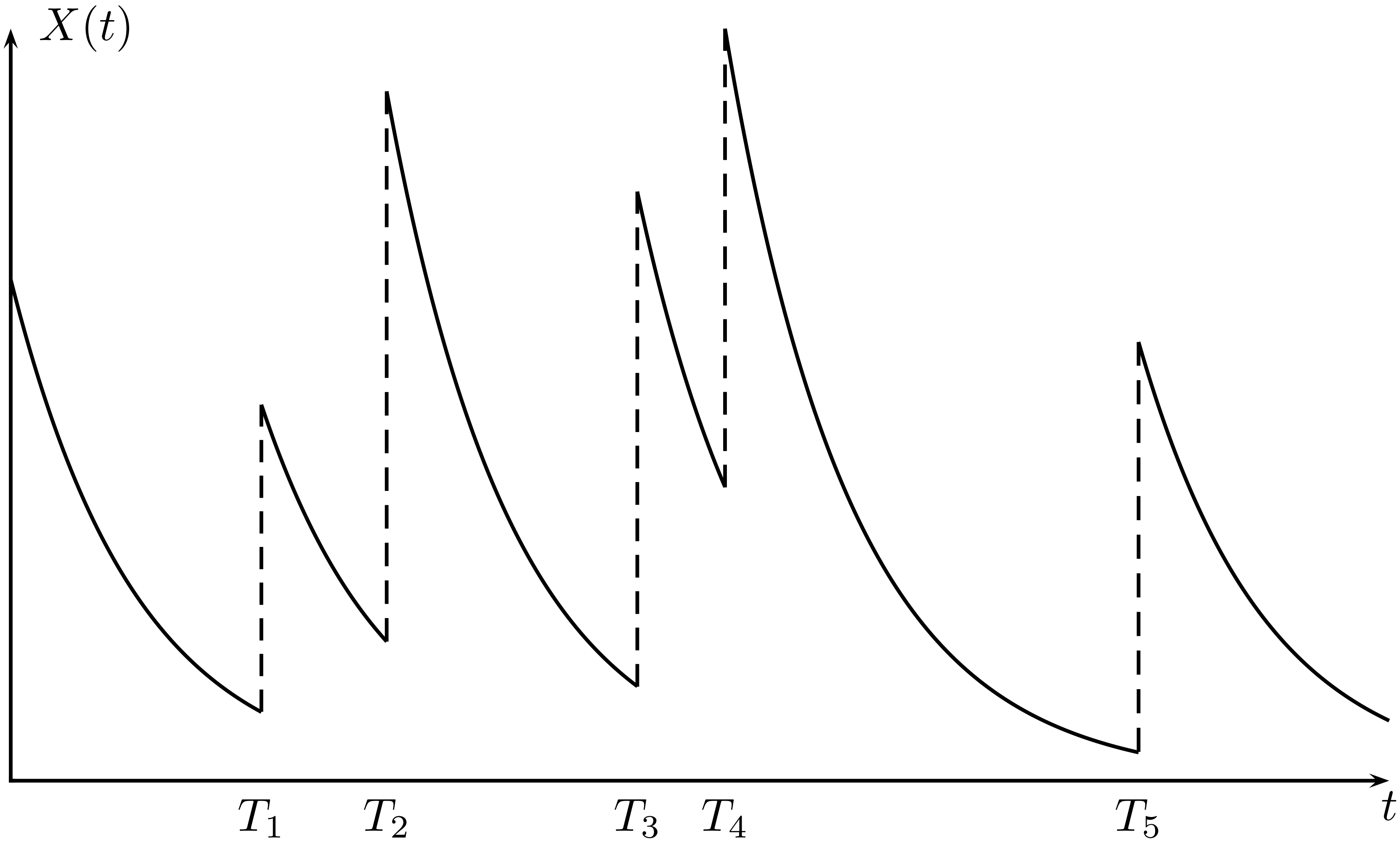
(ii) The function μ describes the deterministic evolution of X between the jumps:

$$\frac{d}{dt} X_t = \mu(X_t).$$

(iii) The kernel J is the conditional jump size distribution:

$$\mathbb{P}(X_t - X_{t-} \in \cdot | \mathcal{F}_{t-}^X, X \text{ jumps at } t) = J(X_{t-}, \cdot).$$

A possible trajectory in case $\mu(x) = -x$



Assumptions:

(i) The function μ is right-continuous and the set

$$D_\mu := \{x \in \mathbb{R} : \mu(x) = 0\}$$

is locally finite. For any $x \in \mathbb{R}$, there exists a unique continuous function $q(x, \cdot) : [0, \infty) \to \mathbb{R}$ satisfying the integral equation

$$q(x,t) = x + \int_0^t \mu(q(x,s))\, ds, \quad t \geq 0.$$

(ii) The jump intensity λ is asumed to be measurable, locally bounded and such that

$$\int_0^\infty \lambda(q(x,s))ds = \infty, \quad x \in \mathbb{R}.$$

Applications:

- storage processes (Harrison and Resnick 1976, Meyn and Tweedie 1993)
- stress release models (Vere-Jones 1988, Zheng 1991, Borovkov and Vere-Jones 2000, Last 2004)
- queueing models (Browne and Sigman 1992, Meyn and Tweedie 1993)
- repairable systems (Last and Szekli 1998)

Definition: (X_t) is the homogeneous Markov process w.r.t. a family $\{\mathbb{P}_x : x \in \mathbb{R}\}$ of probability measures on $(\Omega, \mathcal{F})$ satisfying

$$\mathbb{E}_x f(X_{t\wedge\tau_m}) = f(x) + \mathbb{E}_x \int_0^{t\wedge\tau_m} f'(X_s)\mu(X_s)ds$$
$$+ \mathbb{E}_x \int_0^{t\wedge\tau_m} \int_{\mathbb{R}} (f(X_s+z) - f(X_s))\lambda(X_s)J(X_s, dz)ds.$$

for all $x \in \mathbb{R}$, $m \in \mathbb{N}$, and all absolutely continuous $f : \mathbb{R} \to \mathbb{R}$ such that the Radon–Nikodym derivative f' as well as the function

$$x \mapsto \lambda(x) \int (f(x+z) - f(x))J(x, dz)$$

be locally bounded. Here,

$$\tau_m := \inf\{t \geq 0 : |X(t)| \geq m\}, \quad m \in \mathbb{N}.$$

Remark: The infinitesimal generator $\mathcal{A}_m$ of $(X_{t\wedge\tau_m})$ is given by

$$\mathcal{A}_m f(x) = f'(x)\mu(x) + \lambda(x)\int_{\mathbb{R}}(f(x+z)-f(x))J(x,dz), \quad |x|<m.$$

Remark: Let $N := (T_n)_{n\geq 1}$ be the point process of jump times of X and let $(\mathcal{F}_t)$ be the filtration generated by X. Then N has stochastic intensity $\lambda(X_{t-})$. More generally, the marked point process

$$\Phi := ((T_n, X_{T_n} - X_{T_n-}))_{n\geq 1}$$

has the compensator

$$\nu(d(t,z)) := \lambda(X_t)J(X_t,dz)dt$$

Assumptions:

(i) We have $\mathbb{P}_x(\lim_{n\to\infty} T_n = \infty) = 1$ for all $x \in \mathbb{R}$.

(ii) The process (X_t) has an invariant distribution π satisfying

$$\lambda_\pi := \mathbb{E}_\pi N(1) < \infty,$$

where $N(1)$ is the number of jumps in the time interval $[0, 1]$.

Remark: Assume that there is an $\varepsilon > 0$ such that

$$\liminf_{x\to-\infty}(\mu(x) + \lambda(x)m^+(x)(1-\varepsilon) - \lambda(x)m^-(x)) > 0,$$
$$\limsup_{x\to\infty}(\mu(x) + \lambda(x)m^+(x) - \lambda(x)m^-(x)(1-\varepsilon)) < 0,$$

where

$$m^-(x) := -\int_{-\infty}^{0} zJ(x,dz), \quad m^+(x) := \int_{0}^{\infty} zJ(x,dz), \quad x \in \mathbb{R}.$$

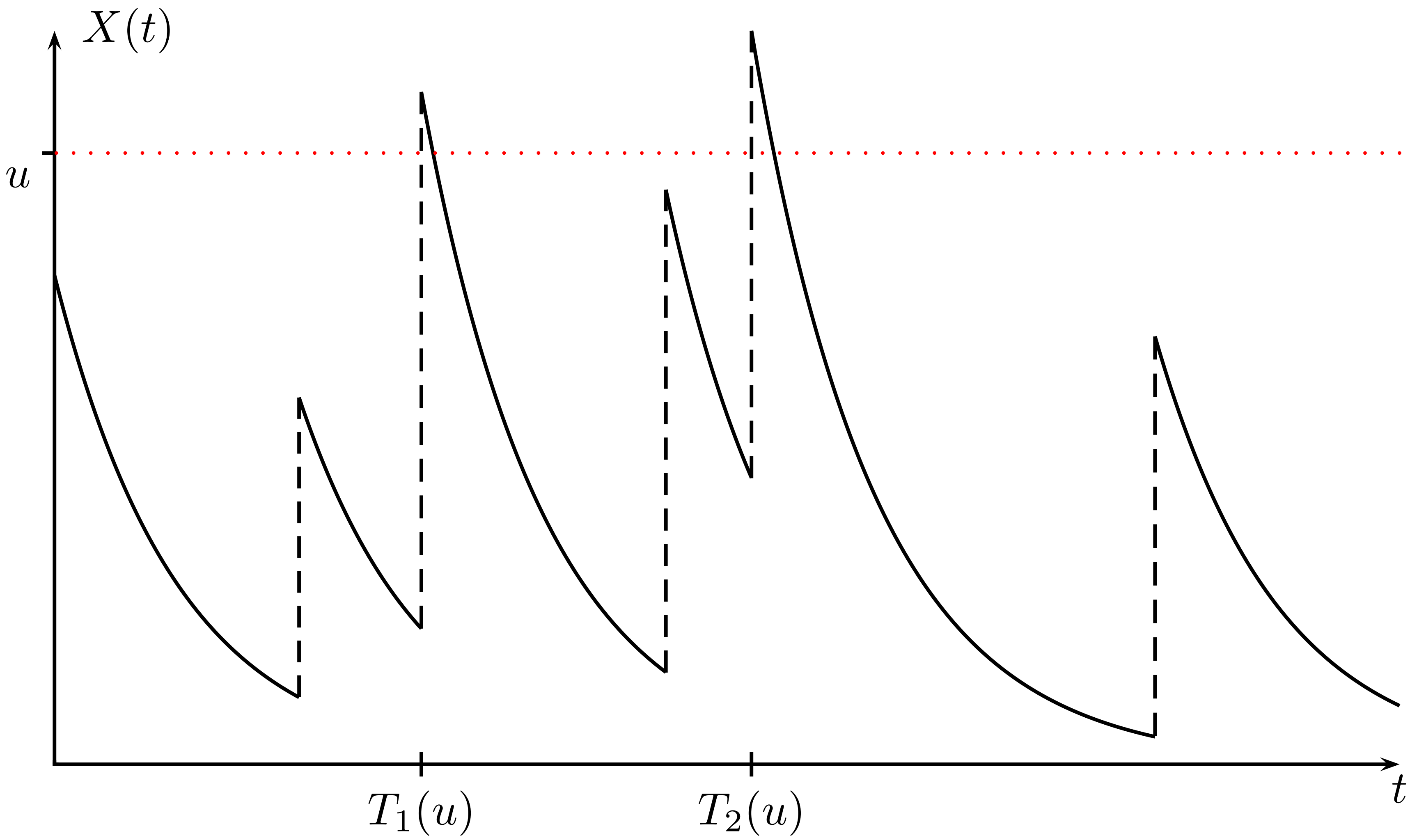
Under further technical assumptions there is a unique invariant distribution π. The condition $\lambda_\pi < \infty$ requires a slight strengthening of the above ergodicity assumption.

2. Up- and downcrossings

Definition: The process (X_t) has an *upcrossing* of level $u \in \mathbb{R}$ at time $s > 0$ if there is some $\delta > 0$ such that $X_t < u$ for $s - \delta \leq t < s$ and $X_t \geq u$ for $s < t \leq s + \delta$. If, in addition, $X_{s-} = X_s (= u)$ then the upcrossing is *continuous*. Otherwise it is *discontinuous*. *Downcrossings* are defined similarly.

Definition: The point process of all times of continuous up- and downcrossings of level u is denoted by N^u. The point process of all times of upcrossings of level u is denoted by N_+^u.

Upcrossings of level u



Remark: Under the stationary probability measure

$$\mathbb{P} := \int \mathbb{P}_x \, \pi(dx)$$

the point processes N^u and N^u_+ are (jointly) stationary. Their intensities are denoted by $\nu(u)$ and $\nu_+(u)$ respectively.

Theorem: (Rice's formula) *The stationary distribution π is absolutely continuous on $\mathbb{R} \setminus D_\mu$. The density p satisfies*

$$\nu(u) = |\mu(u)| p(u), \quad u \notin D_\mu.$$

3. Compound Poisson limit

Definition: Let $\rho \in [0,1)$. A geometrically compound Poisson process is a point process with independent increments defined as follows. Each point of a homogeneous Poisson process of intensity $(1-\rho)$ gets (independently of the other points) a mass $k \in \{1,2,\ldots\}$ with probability $(1-\rho)\rho^{k-1}$.

Remark: As the above geometric distribution has mean $1/(1-\rho)$, the intensity of Π_ρ is 1.

Remark: Compound Poisson limits occur quite frequently in extreme value theory, see e.g. Leadbetter and Hsing (1990) or the book Falk, Hüsler and Reiss (2004).

Assumption: We have $\nu_+(u) > 0$ for all sufficiently large $u \in \mathbb{R}$.

Theorem: *Consider, for $b \in \mathbb{R}$, the time-scaled level crossing process*

$$M^b(t) := N_+^b(\nu_+(b)^{-1}t), \quad t \geq 0.$$

Under the scenarios described below we have weak convergence

$$\mathbb{P}(M^b \in \cdot) \stackrel{w}{\Longrightarrow} \mathbb{P}(\Pi_\rho \in \cdot) \quad \text{as } b \to \infty.$$

The number $\rho \in [0, 1)$ is determined by the characteristics of (X_t). More generally, we have weak convergence of $\mathbb{P}_x(M^b \in \cdot)$ for any $x \in \mathbb{R}$ that is attracted *by the support of π.*

Assumptions: We assume that one of the following assumptions is satisfied.

(i) We have $J(x,(0,\infty)) = 1$ and $\mu(x) < 0$ for all $x > 0$.

(ii) We have $J(x,(-\infty,0)) = 1$ for all $x \in \mathbb{R}$ and $\mu(x) > 0$ for all $x > 0$.

Remark: It is sufficient to make this assumptions only in an asymptotic sense.

Definition: Let

$$\tau(u) := \inf\{t > 0 : N^u(t) \geq 1\}, \quad u \in \mathbb{R},$$

be the time of the first continuous crossing of level u. Define

$$\gamma(u,b) := \mathbb{P}_b(\tau(b) < \tau(u)), \quad u < b,$$

as the probability that the process, when starting at level b (continuously) crosses level b before dropping to level u.

Theorem: *Assume that*

$$\lim_{u\to\infty} \liminf_{b\to\infty} \gamma(u,b) = \lim_{u\to\infty} \limsup_{b\to\infty} \gamma(u,b) = \rho$$

for some $\rho \in [0,1)$. Then the compound Poisson limit theorem holds.

4. Scenarios for the compound Poisson limit

Assumption (A1):

(i) $\mu(y) \to \mu(\infty) < 0$ and $\lambda(y) \to \lambda(\infty) \in [0, \infty)$ as $y \to \infty$.

(ii) There are random variables $\overline{\xi}(u)$ and $\underline{\xi}(u)$, $u > 0$, such that $\mathbb{E}\overline{\xi}(u) < \infty$ and

$$\underline{\xi}(u) \overset{d}{\leq} \xi(x) \overset{d}{\leq} \overline{\xi}(u), \quad x \geq u > 0,$$

where $\xi(x)$ has distribution $J(x, \cdot)$.

(iii) As $y \to \infty$ we have that $\overline{\xi}(y), \underline{\xi}(y) \xrightarrow{d} \xi(\infty)$, where $\xi(\infty)$ is an integrable random variable satisfying

$$\mathbb{E}\xi(\infty) + \mu(\infty)/\lambda(\infty) < 0.$$

Lemma: *If assumption (A1) holds then*

$$\lim_{u\to\infty} \liminf_{b\to\infty} \gamma(u,b) = \lim_{u\to\infty} \limsup_{b\to\infty} \gamma(u,b) = \rho,$$

where $\rho \in [0,1)$ *is given by*

$$\rho = -\frac{\lambda(\infty)}{\mu(\infty)}\, \mathbb{E}\xi(\infty).$$

Remark: ρ is the probability that a stationary $M/GI/1/\infty$-queue with arrival intensity $-\lambda(\infty)/\mu(\infty)$ and generic service time $\xi(\infty)$ is non-empty.

Assumption (A2):

(i) $\mu(y) \to \mu(\infty) > 0$ and $\lambda(y) \to \lambda(\infty) \in [0, \infty)$ as $y \to \infty$.

(ii) There are random variables $\overline{\xi}(u)$ and $\underline{\xi}(u)$, $u > 0$, such that $\mathbb{E}\overline{\xi}(u) < \infty$ and

$$\underline{\xi}(u) \overset{d}{\leq} \xi(x) \overset{d}{\leq} \overline{\xi}(u), \quad x \geq u > 0,$$

where $\xi(x)$ has distribution $J(x, \cdot)$.

(iii) As $y \to \infty$ we have that $\overline{\xi}(y), \underline{\xi}(y) \overset{d}{\to} \xi(\infty)$, where $\xi(\infty)$ is an integrable random variable satisfying

$$\mathbb{E}\xi(\infty) + \mu(\infty)/\lambda(\infty) < 0.$$

Lemma: *If assumption (A2) holds then*

$$\lim_{u\to\infty} \liminf_{b\to\infty} \gamma(u,b) = \lim_{u\to\infty} \limsup_{b\to\infty} \gamma(u,b) = \rho,$$

where

$$\rho = 1 - \frac{w\mu(\infty)}{\lambda(\infty)}$$

and w is the unique solution of

$$\mathbb{E}e^{w\xi(\infty)} = 1 - w\mu(\infty)/\lambda(\infty).$$

Remark: ρ is the probability that a typical customer, arrving at a stationary $GI/M/1/\infty$-queue with service intensity $\lambda(\infty)/\mu(\infty)$ and generic interarrival time $-\xi(\infty)$, finds the system non-empty.

Assumption (A3):

(i) $\mu(y) \to -\infty$ as $y \to \infty$.

(ii) There are random variables $\overline{\xi}(u)$, $u > 0$, such that $\mathbb{E}\overline{\xi}(u) < \infty$ and

$$\xi(x) \overset{d}{\leq} \overline{\xi}(u), \quad x \geq u > 0,$$

where $\xi(x)$ has distribution $J(x, \cdot)$.

(iii) We have

$$\mathbb{E}\overline{\xi}(u_0) + \overline{\mu}(u_0)/\overline{\lambda}(u_0) < 0,$$

for some $u_0 > 0$, where

$$\overline{\mu}(u_0) := \sup_{x \geq u_0} \mu(x), \quad \overline{\lambda}(u_0) := \sup_{x \geq u_0} \lambda(x).$$

Assumption (A4):

(i) $\lambda(y) \to \infty$ as $y \to \infty$.

(ii) There are random variables $\underline{\xi}(u)$, $u > 0$, such that $\mathbb{E}\overline{\xi}(u) < \infty$ and

$$\xi(x) \overset{d}{\leq} \overline{\xi}(u), \quad x \geq u > 0,$$

where $\xi(x)$ has distribution $J(x, \cdot)$.

(iii) We have

$$\mathbb{E}\overline{\xi}(u_0) + \overline{\mu}(u_0)/\underline{\lambda}(u_0) < 0,$$

for some $u_0 > 0$, where $\underline{\lambda}(u_0) := \inf_{x \geq u_0} \lambda(x)$.

Lemma: *If assumption (A3) or assumption (A4) holds, then*

$$\lim_{b \to \infty} \gamma(u, b) = 0$$

for all sufficiently large u.

Remark: If assumption (A3) or assumption (A4) holds, then (X_t) drops very quickly from a high level.

Remark: If one of the Assumptions (A1)–(A4) holds, then $\nu_+(b) = \nu(b)$ for all $b > 0$. By Rice's formula this means that

$$\nu_+(b) = |\mu(b)| p(b), \quad b > 0.$$

5. Asymptotics for the first crossing times

Theorem: *Let $b \in \mathbb{R}$ and define the first and second upward crossing time of level b*

$$T_1(b) := \inf\{t > 0 : X_t \geq b\},$$
$$T_2(b) := \inf\{t > T_1(b) : X_t \geq b, X_{t-\delta} < b \text{ for some } \delta > 0\}.$$

If one of the Assumptions (A1)–(A4) holds, then, as $b \to \infty$,

$$\mathbb{P}_x((1-\rho)\nu_+(b)T_1(b) > s) \to e^{-s},$$
$$\mathbb{P}_x((1-\rho)\nu_+(b)T_2(b) > s) \to e^{-s}(1 + (1-\rho)s)$$

for any x that is attracted by π.