Gamma distributions in Poisson Voronoi and hyperplane tessellatio

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1. The Gamma distribution

Definition: A non-negative random variable X has a *gam* *bution* with *shape parameter* $\alpha > 0$ and *scale parameter* λ the probability density function

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}.$$

Fact: If X is as above, then

$$\mathbb{E}X = \frac{\alpha}{\lambda}, \quad \mathbb{E}X^2 = \frac{\alpha(\alpha+1)}{\lambda^2}, \quad \operatorname{Var} X = \frac{\alpha}{\lambda^2}.$$

Fact: If X and Y are independent and Gamma distrib
shape parameters α_1 and α_2 and the same scale param
$X + Y$ has a Gamma distribution with shape parameter

2. Gamma distributions in stochastic geometry

Complementary theorem:

Miles, R.E. (1971). Poisson flats in Euclidean spaces. Par
Homogeneous Poisson flats and complementary theorem.
Appl. Prob. **3**, 1–43.

Distribution of the typical Poisson-Delaunay simplex:

Miles, R.E. (1974). A synopsis of 'Poisson flats in Euclide
spaces'. In *Stochastic Geometry.* ed. E. F. Harding and D
Kendall, Wiley, New York.

Poisson-Voronoi flower:

Miles, R.E. and Maillardet, R.J. (1982). The basic structu
Voronoi and generalized Voronoi polygons. *J. Appl. Prob.*
97–111.

Complementary theorem and subprocesses:

Møller, J. and Zuyev, S. (1996). Gamma-type results and related properties of Poisson processes. *Adv. Appl. Prob.* 662–673.

Stopping sets:

Zuyev, S. (1999). Stopping sets: gamma-type results and properties. *Adv. Appl. Prob.* **31**, 355–366.

Typical edge of a Poisson Voronoi tessellation:

Muche, L. (2005). The Poisson-Voronoi tessellation: relat for edges. *Adv. Appl. Prob.* **37**, 279–296.

Baumstark, V. and Last, G. (2007). Some distributional r Poisson Voronoi tessellations. *Adv. Appl. Prob.* **39**, 16–40

General results:

Baumstark, V. and Last, G. (2009). Gamma distributions
stationary Poisson flat processes *Adv. Appl. Prob.* **41**, 911

3. First results

Setting: Φ is a stationary Poisson process on $\mathbb{R}^d$ with
$\lambda > 0$.

Notation: $B(x, r)$ denotes a ball with radius $r \geq 0$ centred
Its volume $|B(x, r)|$ is given by $\kappa_d r^d$.

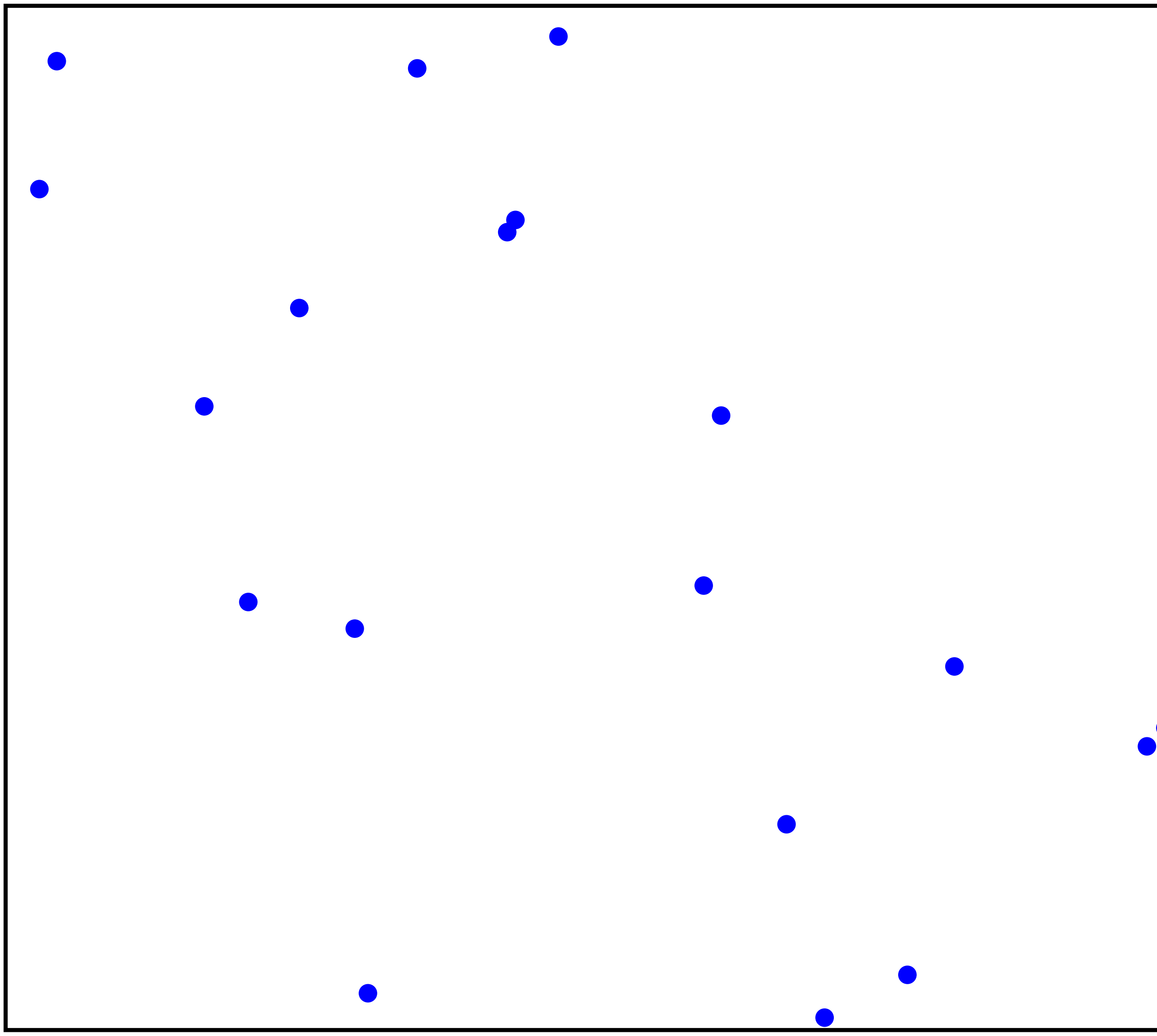
Example: Let $x \in \mathbb{R}^d$ and define

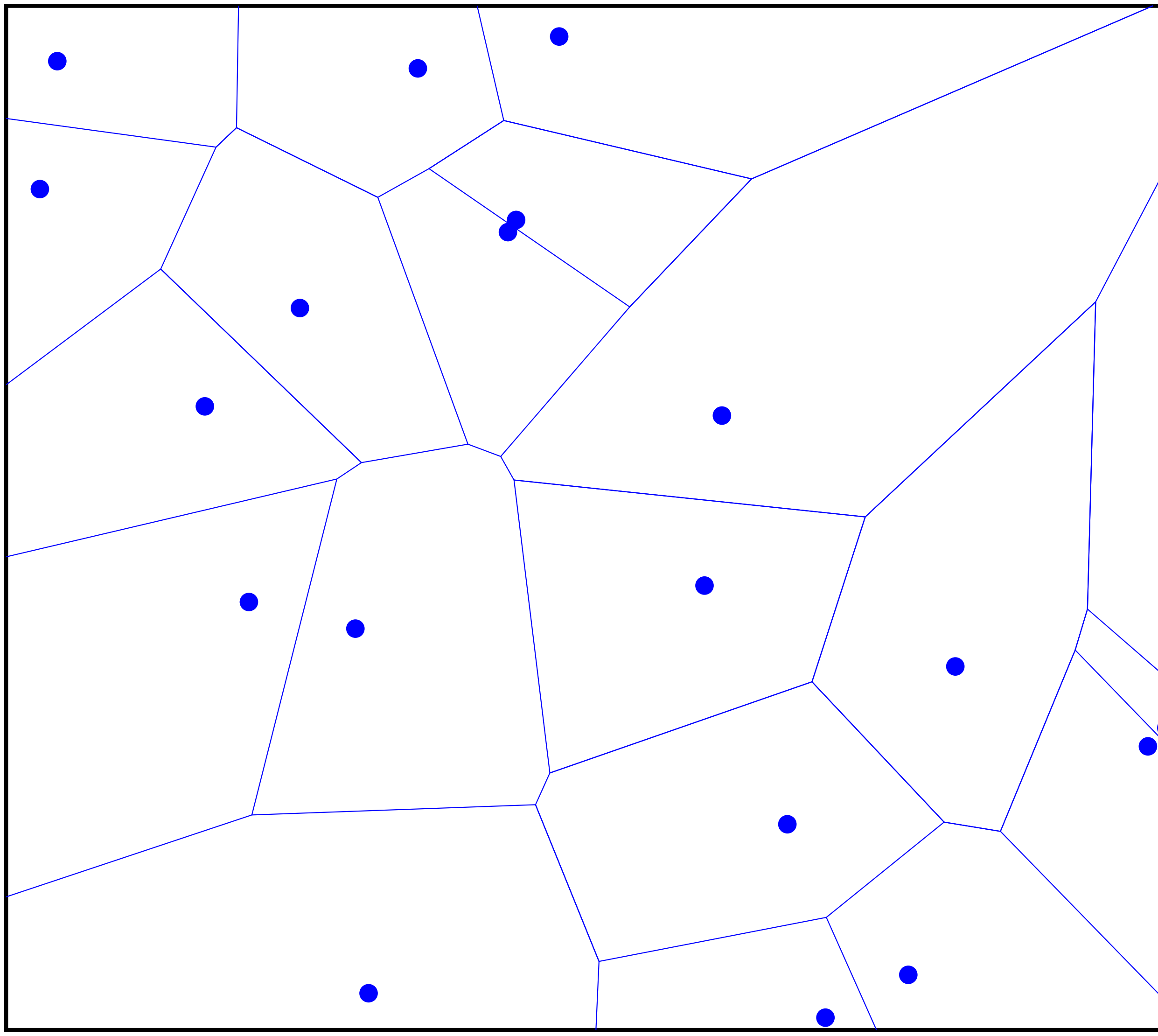
$$\tau := \inf\{t \geq 0 : \Phi(B(x, t)) \geq 1\}.$$

Then

$$|B(x, \tau)| \equiv \kappa_d \tau^d \sim \Gamma(1, \lambda).$$

$$
\begin{aligned}
\mathbb{P}(|B(x,\tau)| > t) &= \mathbb{P}(\kappa_d \tau^d > t) = \mathbb{P}(\tau > (\kappa_d^{-1} t)^{1/d} \\
&= \mathbb{P}(\Phi(B(x,(\kappa_d^{-1} t)^{1/d}) = 0) \\
&= e^{-\lambda t}.
\end{aligned}
$$





Example: (Miles 1974) Consider the Voronoi tessellation
Φ and pick one of the vertices x at random. Consider t
centred at x that has $d+1$ Poisson points in its bounda
point in its interior. Then

$$|S_0| \sim \Gamma(d, \lambda).$$

Example: (Muche 2005, Baumstark and L. 2007) Let $j \in$
Consider the Voronoi tessellation based on Φ and pick a
the j-faces at random (according to j-dimensional Lebe
sure). Consider the ball S_j centred at x that has $d - j +$
points in its boundary and no point in its interior. Then

$$|S_j| \sim \Gamma(d - j + j/d, \lambda).$$

Remark: The shape parameter

$$d - j + j/d = d - j(1 - 1/d)$$

decreases in j. The volumes of S_j are stochastically decre

Ingredients of a general gamma result

- $m, n \geq 0$ and $j \in \{0, \ldots, d\}$ are integers with $m + n - (d - j)/d > 0$;
- $S \equiv S(x_1, \ldots, x_n, x) \subset \mathbb{R}^d$ = closed set depending on $x_1, \ldots, x_n, x \in \mathbb{R}^d$ in an *equivariant* way (w.r.t. scalin translations);
- $R(x_1, \ldots, x_n, x, \Phi)$ = real-valued function depending arguments in an *invariant* way;
- $\mu(x_1, \ldots, x_n, \cdot)$ = measure on $\mathbb{R}^d$ depending on the ar in a translation equivariant way and satisfying the sca relation

$$\mu(cx_1, \ldots, cx_n, cB) = c^j \mu(x_1, \ldots, x_n, B),$$

for all $c > 0$. In the case $n = 0$, μ is Lebesgue measur $j = d$.

Definition: Define a stationary random measure M on $\mathbb{R}$
as follows. $M(B \times C)$ is given as the sum of

$$\int_B \mathbf{1}\{|S(x_1,\dots,x_n,x)| \in C\} R_m(x_1,\dots,x_n,x,\Phi)\mu(x_1,\dots$$

over all pairwise different Poisson points $x_1,\dots,x_n \in$
$R_m(x_1,\dots,x_n,x,\Phi) := R_m(x_1,\dots,x_n,\Phi)$ if $\Phi \setminus \{x_1,\dots$
m points in $S(x_1,\dots,x_n,x)$ and $R_m(x_1,\dots,x_n,x) := 0,$

Theorem: *If M has a positive and finite intensity, then*
distribution of M is $\Gamma(m+n-(d-j)/d,\lambda)$*, where* $j = d$
$n = 0$.

Proof: Assume that $R(x_1, \ldots, x_n, x, \Phi)$ does not depend c
intensity of M is given by

$$\gamma_M := \mathbb{E}M([0,1]^d \times [0,\infty)).$$

By stationarity,

$$\mathbb{E}M(d(x,t)) = \gamma_M dx \mathbb{V}(dt)$$

for a probability measure $\mathbb{V}$ on $[0,\infty)$. We need to show t

$$\mathbb{V} = \Gamma(m + n - (d-j)/d, \lambda).$$

This is done via invariance and scaling arguments.

By the iterated *Mecke formula* for the Poisson process Φ,

$$\mathbb{V}(C) = \gamma_M^{-1}\mathbb{E}M([0,1]^d \times C)$$
$$= \frac{\lambda^n}{\gamma_M m!}\int \mathbf{1}\{x \in [0,1]^d, t \in C\} \exp[-\lambda t]\nu_m(d(x$$

where, for any $b \in \mathbb{R}$,

$$\nu_b(B \times C) :=$$
$$\int \cdots \int \mathbf{1}\{x \in B, |S(x_1, \ldots, x_n, x)| \in C\}|S(x_1, \ldots, x$$
$$R(x_1, \ldots, x_n, x)\mu(x_1, \ldots, x_n, dx)dx_1 \ldots dx_n.$$

By translation invariance

$$\nu_b(d(x,t)) = \nu_b([0,1]^d \times dt)dx.$$

By the scaling properties of Lebesgue measure and the ke

$$\nu_b([0,1]^d \times aC) = a^n a^b a^{j/d} \nu_b(a^{-1/d}[0,1]^d \times C)$$
$$= a^n a^b a^{j/d} a^{-1} \nu_b([0,1]^d \times C)$$

for all $a > 0$. Choosing $b := 2 - n - j/d$ yields $a^n a^b a^{j/d} a^{-}$

$$\nu_b([0,1]^d \times dt) = cdt$$

for some $c > 0$, and hence

$$\nu_b(d(x,t)) = cdxdt.$$

Hence the mark distribution $\mathbb{V}$ is proportional to

$$\int \mathbf{1}\{x \in [0,1]^d, t \in \cdot\} \exp[-\lambda t] \nu_m(d(x,t))$$

$$\int \mathbf{1}\{x \in [0,1]^d, t \in \cdot\} t^{m-b} \exp[-\lambda t] \nu_b(d(x,t))$$

$$= \int \mathbf{1}\{t \in \cdot\} t^{m-b} \exp[-\lambda t] dt.$$

4. More examples

Example: Let $j \in \{0, \ldots, d\}$ and $m \geq 0$. Consider th
tessellation based on Φ and pick a point x on the j-fac
dom (according to j-dimensional Lebesgue measure). Co
smallest ball $S_{j,m}$ centred at x that contains $m + d - j +$
points point in its interior. Then

$$|S_{j,m}| \sim \Gamma(m + d - j + j/d, \lambda).$$

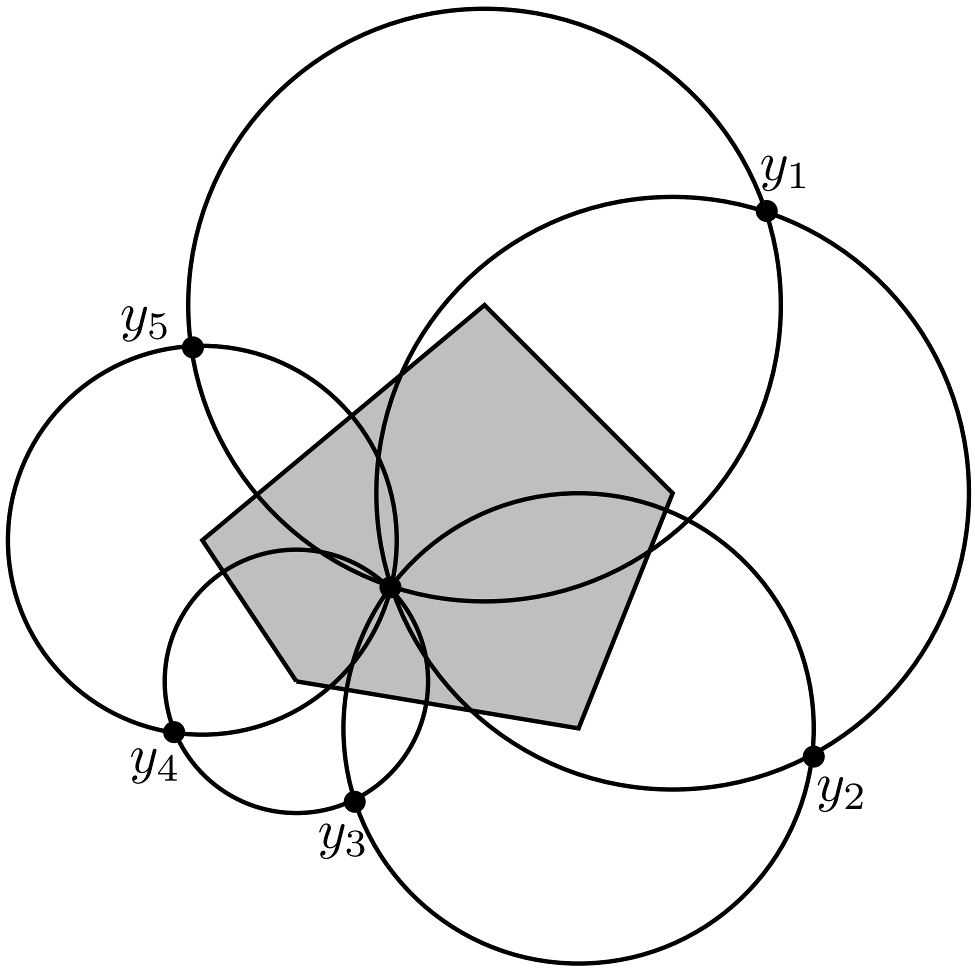
Example: (Miles and Maillardet 1982, Møller and Zuyev
C_d be one of the Voronoi cells picked at random. Let X
centre of X. Let

$$T_d^* := \bigcup_{y \text{ vertex of } C_d} B(y, \|y - X\|),$$

Consider T_d^* under the condition that $\Phi(T_d^*) = m + 1$
$m \geq d + 1$. Then the volume of T_d^* has a $\Gamma(m, \lambda)$-distrib

Remark: The random set T_d^* is called *Voronoi flower* (or
tal region) of the (typical) cell C_d.

Voronoi-cell with 5 edges generated by $y_1, \ldots, y_5 \in \Phi$:



Example: (Baumstark and L. 2009) Let $x \in \mathbb{R}_d$ and let
the Voronoi cell containing x. Let $X \in \Phi$ be the centr
and

$$T_d := \bigcup_{y \text{ vertex of } C_d(x)} B(y, \|y - X\|)$$

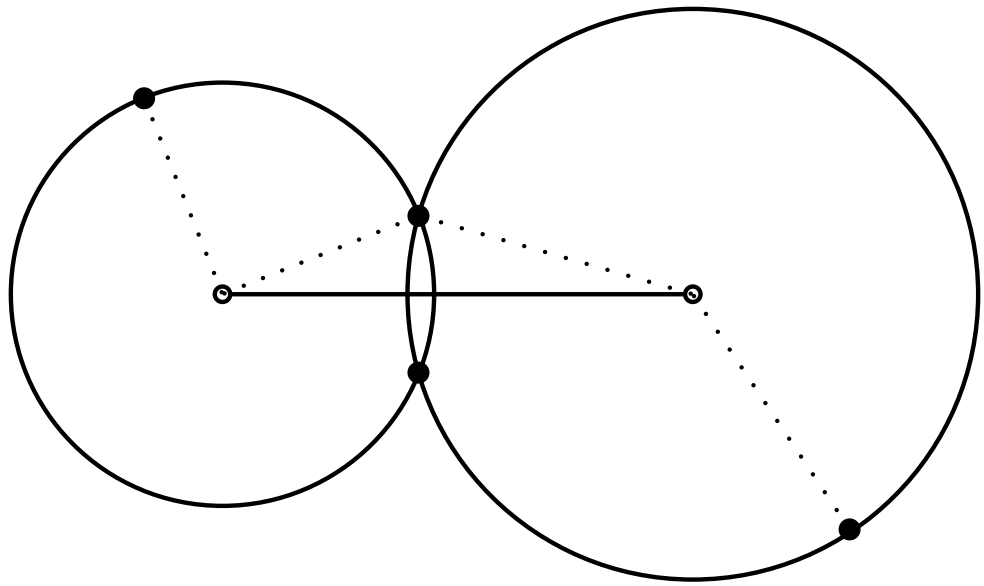
Consider the conditional probability measure given tha
$m+1$ for some $m \geq d+1$. Then the volume of T_d has a Γ
distribution.

Example: (Baumstark and L. 2009) Let $j \in \{0, \dots, d\}$. Pi
C_j of the Voronoi tessellation at random. Then there are a
$d - j + 1$ different Poisson points $X_1, \dots, X_{d-j+1}$, the *ne*
the j-face C_j, such that C_j is the intersection of the Vo
centred at those points. The *fundamental region* T_j of C_j
by

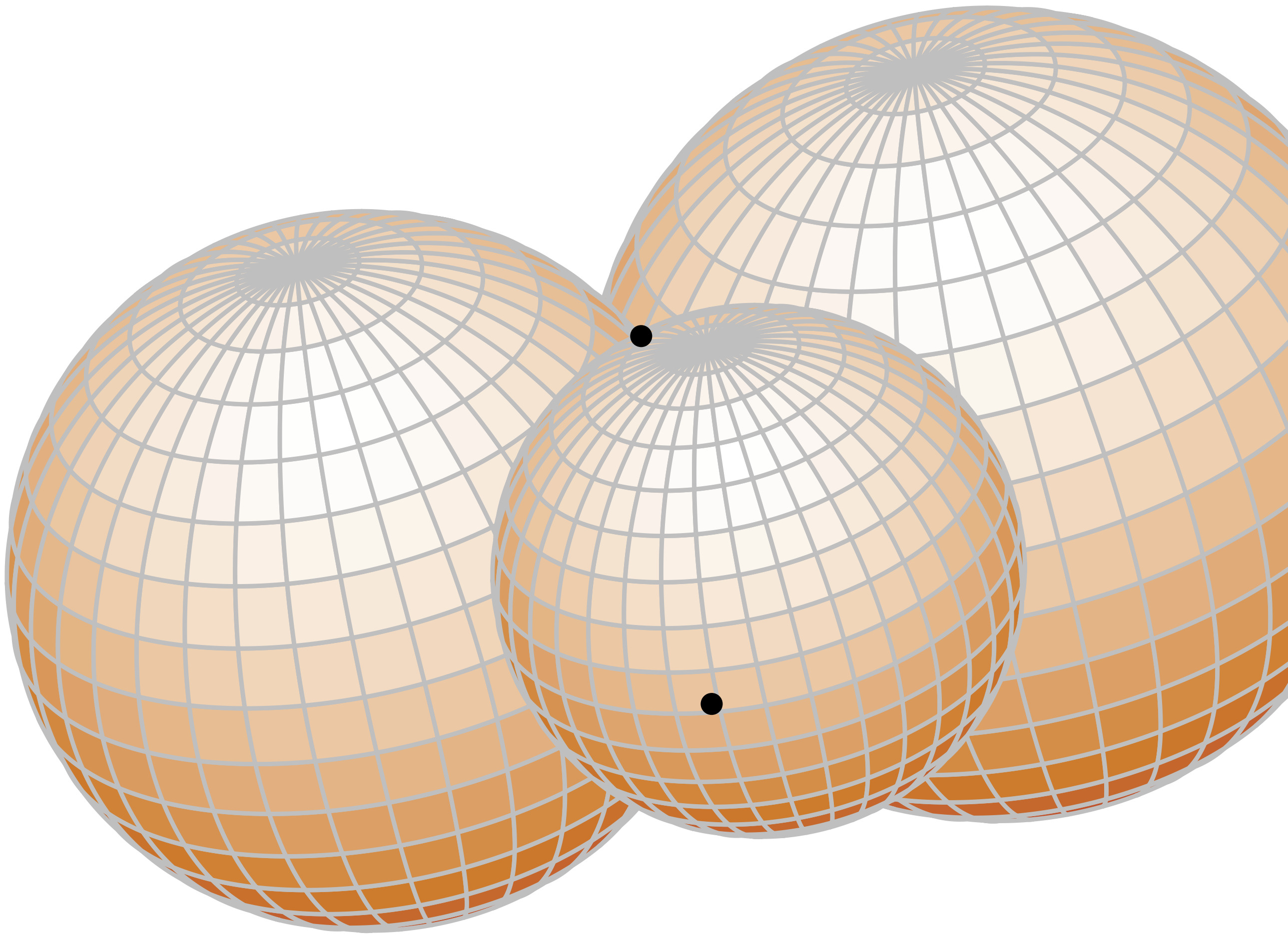
$$T_j := \bigcup_{y \text{ vertex of } C_j} B(x, \|x - X_1\|)$$

Consider T_j under the condition that $\Phi(T_j) = m + d - j$
$m = 0$ in case $j = 0$, $m = 2$ in case $j = 1$ and $m \geq j + 1$
Then the volume of the fundamental region T_j has $\Gamma(m$
distribution.

The fundamental region of C_1 in the case $d = 2$:



The fundamental region of C_2 in the case $d = 3$:



Example: (Baumstark and L. 2009) Let $j \in \{0, \ldots, d\}$. Pi
x on the j-faces at random and let $C_j(x)$ be the picked
X be one of the $d - j + 1$ neighbours of $C_j(x)$ and co
fundamental region T_j^* of $C_j(x)$: by

$$T_j^* := \bigcup_{y \text{ vertex of } C_j(x)} B(x, \|y - X\|)$$

Consider T_j^* under the condition that $\Phi(T_j^*) = m + c$
where $m = 0$ in case $j = 0$, $m = 2$ in case $j = 1$ and n
otherwise. Then the volume of the fundamental region
$\Gamma(m + d - j + j/d, \lambda)$-distribution.

5. Gamma distributions and stopping sets

Definition: Let $\mathbf{N}$ be the space of all locally finite sub
(identified with counting measures) and let $\mathcal{N}$ be the σ
erated by the mappings $\varphi \mapsto \varphi(B)$ for Borel sets $B \subset$
Borel set $K \subset \mathbb{R}^d$ let $\mathcal{N}_K$ be the σ-field generated by th
$\varphi \mapsto \varphi \cap K$.

Definition: A *stopping set* defined on $\mathbf{N}$ is a mapping T o
values in the space of closed subsets of $\mathbb{R}^d$ such that

$$\{\varphi \in \mathbf{N} : T(\varphi) \subset K\} \in \mathcal{N}_K, \qquad K \subset \mathbb{R}^d \text{ closed}$$

Proposition: (Kurtz 1980, Zuyev 1999) *Let* $\gamma > 0$ *and* *a probability measure such that* Φ *is (under* $\mathbb{P}_\gamma$*) a Poiss* *with intensity* γ*. Let* T *be a stopping set. Then we h* *measurable* $g : \mathbf{N} \to [0, \infty)$ *and* $m \geq 0$ *that*

$$\mathbb{E}_\gamma \mathbf{1}\{\Phi(T) = m\} g(\Phi \cap T)$$

$$= \frac{\rho^m}{\lambda^m} \mathbb{E}\Big[\exp((\lambda - \gamma)|T|)\mathbf{1}\{\Phi(T) = m\} g(\Phi \cap T$$

where $T \equiv T(\Phi)$*.*

Definition: Let m, n, j, R and μ be as before but assume
$S = S(x_1, \ldots, x_n, x, \varphi) \subset \mathbb{R}^d$ does also depend on $\varphi \in \mathbf{N}$
that S is equivariant. Define a stationary random meas
$\mathbb{R}^d \times [0, \infty)$ as follows. $M(B \times C)$ is given as the sum of

$$\int_B \mathbf{1}\{|S(x_1, \ldots, x_n, x, \Phi \setminus \{x_1, \ldots, x_n\})| \in C\}$$
$$R_m(x_1, \ldots, x_n, x, \Phi)\mu(x_1, \ldots, x$$

over all pairwise different Poisson points $x_1, \ldots, x_n \in \Phi$,
is defined as before.

Theorem: *If M has a positive and finite intensity, then distribution of M is $\Gamma(m + n - (d - j)/d, \lambda)$, where $j = d$ $n = 0$.*

6. Poisson flat processes

Definition: Let $d \in \mathbb{N}$ and $k \in \{0, \ldots, d-1\}$. A *k-flat* (i
k-dimensional affine subspace F of $\mathbb{R}^d$. The space of all su
denoted by $A(d, k)$. It is equipped with its standard Bo
$\mathcal{A}^k$. The space $G(d, k) \subset A(d, k)$ consists of all k-dimensio
subspaces of $\mathbb{R}^d$.

Definition: A *point process* of k-flats (*k-flat process*) is a
cess Φ on $A(d, k)$. It can be introduced as a random elem
space $\mathbf{N}^k$ of all *locally finite* counting measures on $A(d, k$
process Φ is called *stationary*, if $\Phi + x$ has the same distr
Φ for all $x \in \mathbb{R}^d$. Here

$$\varphi + x := \{F + x : F \in \varphi\}, \qquad \varphi \in \mathbf{N}^k.$$

Definition: The *intensity* of a stationary point process o
defined by

$$\lambda := \frac{1}{\kappa_{d-k}} \mathbb{E}\,\mathrm{card}\{F \in \Phi : F \cap B^d \neq \emptyset\}.$$

Proposition: *Let* Φ *be a stationary point process of* k*-flat*
itive and finite intensity λ*. Then*

$$\mathbb{E}\Phi(A) = \lambda \Lambda_{\mathbb{Q}}(A), \qquad A \in \mathcal{A}^k,$$

for some uniquely determined probability measure $\mathbb{Q}$ (the
distribution of Φ) *on* $G(d,k)$*, where*

$$\Lambda_{\mathbb{Q}}(A) := \int_{G(d,k)} \int_{F^\perp} \mathbf{1}\{F + x \in A\} \mathcal{H}^{d-k}(dx)\mathbb{Q}(d$$

Proposition: *A Poisson process* Φ *on* $A(d,k)$ *is station* *intensity measure is a multiple of* $\Lambda_{\mathbb{Q}}$ *for some* $\mathbb{Q}$*. It is* is $\mathbb{Q}$ *is the normalized invariant measure on* $G(d,k)$*.*

7. Gamma distributions for Poisson flat processes

Setting: Φ is a stationary Poisson process of k-flats, i.e.
process Φ on $A(d,k)$ with intensity measure $\lambda\Lambda_{\mathbb{Q}}$ for so
and a directional distribution $\mathbb{Q}$.

Definition: A *stopping set* defined on $\mathbf{N}^k$ is a mapping
taking values in the space of closed subsets of $A(d,k)$ suc

$$\{\varphi \in \mathbf{N} : T(\varphi) \subset K\} \in \mathcal{N}_K,$$

for any closed $K \subset A(d,k)$.

Definition: Let m, n, j, R and μ be as before but assum functions depend on $F_1, \ldots, F_n \in A(d,k)$ instead of x_1 $\mathbb{R}^d$). Asssume that S is equivariant and that R is in the la nent measurable with respect to the *stopping σ-field* gen S. Define a stationary random measure M on $\mathbb{R}^d \times [0,\infty)$ $M(B \times C)$ is the sum of

$$\int_B \mathbf{1}\{\Lambda_{\mathbb{Q}}(S(F_1,\ldots,F_n,x,\Phi \setminus \{F_1,\ldots,F_n\}))| \in C\}$$
$$R_m(F_1,\ldots,F_n,x,\Phi)\mu(F_1,\ldots$$

over all pairwise different Poisson flats $F_1, \ldots, F_n \in \Phi$.

Theorem: *If M has a positive and finite intensity, then distribution of M is $\Gamma(m+n-(d-j)/(d-k), \lambda)$, wher the case $n = 0$.*

Definition: For any closed set $K \subset \mathbb{R}^d$ define

$$\Lambda^*_{\mathbb{Q}}(K) := \Lambda_{\mathbb{Q}}(\{F \in A(d,k) : F \cap K \neq \emptyset\}).$$

Remark: If $k = 0$, then $\Lambda^*_{\mathbb{Q}}(K)$ is the volume of K. I
uniform distribution and K is compact and convex, th
is proportional to the surface area of K in the case k
proportional to the *mean breadth* of K in the case $k = d$

8. Poisson hyperplane tessellations

Setting: Φ is a stationary Poisson process of hyperpla
Poisson process Φ on $A(d, d-1)$ with intensity measur
some $\lambda > 0$ and a directional distribution $\mathbb{Q}$. Assume
nondegenerate.

Definition: The connected components of the complem
union

$$\bigcup_{F \in \Phi} F$$

is made up by open polyhedral sets. The closures of thes
forming the *hyperplane tessellation* X based on Φ.

Example: (Miles) Pick one of the cells at random and co
(maximal) inradius of this cell. This radius is exponen
tributed. (The theorem applies with $n = d + 1$, $m = 0$, a

Example: Pick an edge C_1 of the hyperplane tessellation a
Then $\Lambda^*_{\mathbb{Q}}(C_1)$ has an exponential distribution.

Example: Pick a point x on the edges at random (w.r.t. o
sional Hausdorff measure). Then x is in the relative inte
edge L_1. Then $\Lambda^*_{\mathbb{Q}}(L_1)$ has a $\Gamma(2, \lambda)$-distribution.

9. Tessellations and wireless networks

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