Günter Last
Institut für Mathematische Stochastik
Universität Karlsruhe (TH)

Filtering of marked point processes

Günter Last

National ICT Australia, Canberra

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Reference:

Last, G. and Brandt, A. (1995) *Marked Point Processes on the Real Line: The Dynamic Approach.* Springer-Verlag, New York.

1. Marked point processes

Let $\mathbf{X}$ be a Polish space and x_∞ a point external to $\mathbf{X}$. A marked point process (MPP) with mark space $\mathbf{X}$ is a sequence

$$\Phi = ((T_n, X_n))_{n\geq 1}$$

of random times $T_n \in (0,\infty]$ and random $(\mathbf{X} \cup \{x_\infty\})$-valued marks X_n such that

$$T_n < T_{n+1} \quad \text{if} \quad T_n < \infty,$$
$$T_n = T_{n+1} = \infty \quad \text{if} \quad T_n = \infty,$$

and

$$\{T_n = \infty\} = \{X_n = x_\infty\}.$$

A MPP Φ is identified with a random measure on $[0,\infty) \times \mathbf{X}$:

$$\Phi \equiv \sum_{n\geq 1} \mathbf{1}\{T_n < \infty\}\delta_{(T_n,X_n)}.$$

For any $t \geq 0$, Φ_t and Φ_{t-} denote the restrictions of Φ to $[0,t] \times \mathbf{X}$ resp. $[0,t] \times \mathbf{X}$. We denote

$$\mathcal{F}_t^\Phi := \sigma(\Phi_t), \qquad \mathcal{F}_{t-}^\Phi := \sigma(\Phi_{t-}).$$

$\{\mathcal{F}_t^\Phi : t \geq 0\}$ is the filtration generated by Φ. Given any filration $\{\mathcal{F}_t : t \geq 0\}$ satisfying

$$\mathcal{F}_t^\Phi \subset \mathcal{F}_t, \quad t \geq 0,$$

the $\{\mathcal{F}_t\}$-compensator is an $\{\mathcal{F}_t\}$-predictable random measure ν on $[0,\infty) \times \mathbf{X}$ satisfying the heuristic relationship

$$\nu(d(t,x)) = P(\Phi(d(t,x)) > 0|\mathcal{F}_{t-}).$$

2. Detecting change points

Let $\tau \in (0, \infty]$ be a random time with c.d.f. F having a density f (on $(0, \infty)$) and let $\lambda_1, \lambda_2 > 0$ with $\lambda_1 < \lambda_2$. Assume that $\Phi = (T_n)$ is a doubly stochastic Poisson process with respect to τ whose intensity switches from λ_1 to λ_2 at time τ. Hence, given τ, Φ is a Poisson process with intensity

$$\lambda(t) := \mathbf{1}\{t < \tau\}\lambda_1 + \mathbf{1}\{t \geq \tau\}\lambda_2, \quad t \geq 0.$$

Filtering problem:

Assume that Φ can be observed up to time t.

- Has the intensity switched by time t?
- When did the change happen?

Joint distribution of Φ_t and τ:

$$\mathbf{1}\{s \le t\}\mathbb{P}(\Phi_t \in d\varphi, \tau \in ds) = \mathbf{1}\{s \le t\}L(\varphi, s, t)f(s)e^t\mathbb{Q}^t(d\varphi)ds,$$

$$\mathbb{P}(\Phi_t \in d\varphi, \tau > t) = L(\varphi, t, t)(1 - F(t))f(s)e^t\mathbb{Q}^t(d\varphi),$$

where

$$L(\varphi, s, t) := \lambda_1^{\varphi(s)}e^{-\lambda_1 s}\lambda_2^{\varphi(t)-\varphi(s)}e^{-\lambda_2(t-s)},$$

$\varphi(s)$ is the number of points of φ in $[0, s]$ and $\mathbb{Q}^t$ is the distribution of a homogeneous Poisson process on $[0, t]$ of intensity 1.

Theorem: *We have*

$$\mathbf{1}\{s \le t\}\mathbb{P}(\tau \in ds|\Phi_t) = \frac{L(\Phi, s, t) f(s) ds}{\int_0^t L(\Phi, u, t) f(u) du + L(\Phi, t, t)(1 - F(t))}$$

and in particular

$$\hat{Z}(t) := \mathbb{P}(\tau \le t|\Phi_t) = \frac{\int_0^t L(\Phi, s, t) f(s) ds}{\int_0^t L(\Phi, u, t) f(u) du + L(\Phi, t, t)(1 - F(t))}.$$

Moreover, for $T_n < t < T_{n+1}$ *we have*

$$\hat{Z}(t) = \hat{Z}(T_n) + \int_{T_n}^t r(s)(1 - \hat{Z}(s)) ds + \int_{T_n}^t (\lambda_1 - \lambda_2)\hat{Z}(s)(1 - \hat{Z}(s)) ds,$$

where $r(s) := f(s)/(1 - F(s))$ *is the hazard rate of* F*, and*

$$\hat{Z}(T_n) = \frac{\lambda_2 \hat{Z}(T_n-)}{\lambda_1 + (\lambda_2 - \lambda_1)\hat{Z}(T_n-)}, \quad n \ge 1.$$

Stochastic $\{\mathcal{F}_t\}$-intensity of Φ:

$$\hat{\lambda}(t) := \mathbb{E}[\lambda(t)|\Phi_{t-}] = (1 - \hat{Z}(t-))\lambda_1 + \hat{Z}(t-)\lambda_2$$
$$= \lambda_1 + (\lambda_2 - \lambda_1)\hat{Z}(t-).$$

Semimartingale represention of $\hat{Z}(t)$:

$$\hat{Z}(t) = \int_0^t r(s)(1 - \hat{Z}(s))\, ds + \int_0^t Y_s \, d\hat{M}(s)$$

where

$$Y_s = \frac{(\lambda_2 - \lambda_1)\hat{Z}(s-)(1 - \hat{Z}(s-))}{\lambda_1 + (\lambda_2 - \lambda_1)\hat{Z}(s-)}$$

and

$$\hat{M}(t) := \Phi(t) - \int_0^t \hat{\lambda}(s)\, ds$$

is the innovations martingale.

2. Filtering of Semi-Markov processes

Consider a MPP $\Psi = ((T'_n, Y_n))$ with finite mark space $\{1, \ldots, m\}$. Let $T'_0 \equiv 0$ and Y_0 be a random element of $\{1, \ldots, m\}$. Let

$$(T^t, Z(t)) := (T'_n, Y_n) \quad \text{if} \quad T'_n \leq t < T'_{n+1}.$$

Define

$$\mathcal{F}_t := \sigma(Y_0, \Psi_t), \quad t \geq 0,$$

and assume that the $\{\mathcal{F}_t\}$-compensator ν of Ψ is given by

$$\nu(dt \times \{j\}) = r_{Z(t-)j}(t - T^{t-})dt,$$

where the r_{ij} are measurable functions such that

$$r_{ii}(t) \equiv 0, \quad i = 1, \ldots, m.$$

We can assume that

$$Y_n \neq Y_{n+1} \quad \text{if} \quad T'_n < \infty.$$

Let

$$q_{ij}(t) := r_{ij}(t) \exp\Big[-\int_0^t \sum_k r_{ik}(s)\, ds \Big], \quad t \geq 0.$$

We allow for

$$\sum_{j=1}^m \int_0^\infty r_{ij}(s) ds < \infty,$$

i.e.

$$\sum_{j=1}^m \int_0^\infty q_{ij}(s) ds < 1,$$

and define

$$\bar{Q}_i(t) := 1 - \sum_{j=1}^m \int_0^t q_{ij}(s) ds, \quad t \geq 0.$$

We assume that $\Phi = (T_n)$ is a doubly stochastic Poisson process with respect to $\{Z(t)\}$. More precisely we consider given intensities

$$\lambda_1, \ldots, \lambda_m > 0$$

and assume that the conditional distribution of Φ given the process $\{Z(t)\}$ is that of a Poisson process with intensity function

$$t \mapsto \lambda_{Z(t)}.$$

Let Z' be a $\sigma(Y_0)$-measurable random element of some measurable space and define the observations

$$\mathcal{G}_t := \sigma(Z', \Phi_t), \quad t \geq 0.$$

Theorem: *We have*

$$\mathbb{P}(T^t \in ds, Z(t) = j|\mathcal{G}_t) = \frac{Q(t;j,ds)}{\sum_k Q(t;k,[0,\infty]\times \mathbf{Y})} \quad \mathbb{P}-a.s.,$$

where

$$\begin{aligned} Q(t;j,ds) :=&\mathbb{P}(Y_0 = j|Z')\lambda_j^{\Phi(t)} e^{-\lambda_j t}\bar{Q}_j(t)\delta_0(ds) \\ &+ \mathbf{1}\{s \le t\}\lambda_j^{\Phi(t)-\Phi(s)} e^{-\lambda_j(t-s)}\bar{Q}_j(t-s)p(s,j)ds \end{aligned}$$

and $p : \Omega \times [0,\infty) \times \{1,\dots,m\} \to [0,\infty)$ *is a* $\{\mathcal{G}_t\}$*-optional process satisfying*

$$\begin{aligned} p(t,j) =& \sum_{k=1}^m \lambda_k^{\Phi(t)} e^{-\lambda_k t} q_{kj}(t)\mathbb{P}(Y_0 = k|Z') \\ &+ \sum_{k=1}^m \int_0^t \lambda_k^{\Phi(t)-\Phi(u)} e^{-\lambda_k(t-u)} q_{kj}(t-u)p(u,k)du. \end{aligned}$$

4. Generalizations

Consider a MPP $\Psi = ((T'_n, Y_n))$ with discrete mark space $\mathbf{Y}$ and let Y_0 be a random element of $\mathbf{Y}$. Denote

$$\mathcal{F}_t := \sigma(Y_0, \Psi_t), \quad t \geq 0,$$

and define the $\{\mathcal{F}_t\}$-adapted process $\{Y(t)\}$ by

$$Y(t) := Y_n \quad \text{if} \quad T'_n \leq t < T'_{n+1}$$

and $Y(t) := y_\infty$ if $t \geq T'_\infty := \lim_{n\to\infty} T'_n$. (Here $T'_0 := 0$.) Let

$$g_1 : [0,\infty) \times \mathbf{Y} \times [0,\infty) \times \mathbf{Y} \to \{0,1\}$$

be measurable and Ψ^0 the unique PP satisfying

$$\Psi^0(\cdot) = \int \mathbf{1}\{g_1(T^{t-}(\Psi^0), Y(t-), t, y) = 0\}\mathbf{1}\{t \in \cdot\}\Psi(d(t,y)),$$

where $T^{t-}(\Psi^0)$ is the last point of Ψ^0 strictly before t ($= 0$ if there is no such point).

The observed MPP $\Phi = ((T_n, X_n))$ with discrete mark space $\mathbf{X}$ is assumed to be given by

$$\Phi = \int g_1(\pi^{t-}(\Psi^0), Y(t-), t, y)$$
$$\mathbf{1}\{g_2(T^{t-}(\Psi^0), Y(t-), t, y) = x\}\mathbf{1}\{(t, x) \in \cdot\}\Psi(d(t, y)),$$

where

$$g_2 : [0, \infty) \times \mathbf{Y} \times [0, \infty) \times \mathbf{Y} \to \mathbf{X}$$

is a given measurable function. Let Z' be a $\sigma(Y_0)$-measurable random element of some measurable space and define the observations by

$$\mathcal{G}_t := \sigma(Z', \Phi_t), \quad t \geq 0.$$

Remark: The functions g_1 and g_2 could be allowed to depend on $\omega \in \Omega$ in a $\{\mathcal{G}_t\}$-predictable way.

Assume that the $\{\mathcal{F}_t\}$-compensator ν of Ψ is given by

$$\nu(dt \times \{y\}) = \mathbf{1}\{t < T'_\infty\} r(T^{t-}(\Psi^0), Y(t-), t, y)\, dt,$$

where

$$r : \Omega \times [0, \infty) \times \mathbf{Y} \times [0, \infty) \times \mathbf{Y} \to [0, \infty)$$

is a measurable function that is $\{\mathcal{G}_t\}$-predictable in the first and fourth argument. Let

$$\bar{r}(s, z, u) := \sum_{y \in \mathbf{Y}} r(s, z, u, y).$$

and assume that

$$\int_s^t \bar{r}(s, z, u)\, du < \infty, \quad s \leq t,$$
$$\mathbb{E} \int_0^t \mathbf{1}\{u < T'_\infty\} \bar{r}(T^u(\Psi^0), Y(u), u)\, du < \infty, \quad t \geq 0,$$

Define

$$U(s,y,t) := \mathbf{1}\{s \le t\} \exp\left[-\int_s^t \bar{r}(s,y,u)\,du\right].$$

Theorem: *We have*

$$\mathbb{P}(T^t(\Psi^0) \in ds, Y(t) = y | \mathcal{G}_t) = \frac{Q(t;y,ds)}{\sum_z Q(t;z,[0,\infty] \times \mathbf{Y})} \quad \mathbb{P} - a.s.,$$

where, for $n \ge 0$ *and* $T_n \le t < T_{n+1}$

$$Q(t;y,ds) = C_n(y)\frac{U(0,y,t)}{U(0,y,T_n)}\delta_0(ds) + \mathbf{1}\{s \le T_n\}p_n(s,y)\frac{U(s,y,t)}{U(s,y,T_n)}\,ds$$
$$+\mathbf{1}\{T_n < s \le t\}p(s,y)U(s,y,t)\,ds.$$

The $C_n(y)$ *are* $\mathcal{G}_{T_n}$*-measurable r.v.'s,* $p_n : \Omega \times [0,\infty) \times \mathbf{Y} \to [0,\infty)$ *is* $\mathcal{G}_{T_n}$*-measurable in the first argument, and* $p : \Omega \times [0,\infty) \times \mathbf{Y} \to [0,\infty)$ *is* $\{\mathcal{G}_t\}$*-optional in the first two arguments. These functions are given by:*

$$
\begin{aligned}
C_0(y) =& P(Y_0 = y|\mathcal{G}_0),\\
C_{n+1}(y) =& \sum_{z\in\mathbf{Y}} \mathbf{1}\{g_2(0,z,T_{n+1},y) = X_{n+1}\} g_1(0,z,T_{n+1},y)\\
& r(0,z,T_{n+1},y) C_n(z) \frac{U(0,z,T_{n+1})}{U(0,z,T_n)} \quad \text{on } \{T_{n+1} < \infty\},
\end{aligned}
$$

$$
\begin{aligned}
& p_0(s,y) = 0,\\
& p_{n+1}(s,y) = \mathbf{1}\{s \le T_n\} \sum_{z\in\mathbf{Y}} g_1(s,z,T_{n+1},y) \mathbf{1}\{g_2(s,z,T_{n+1},y) = X_{n+1}\}\\
& \quad r(s,z,T_{n+1},y) p_n(s,z) \frac{U(s,z,T_{n+1})}{U(s,z,T_n)}\\
& + \mathbf{1}\{T_n < s \le T_{n+1}\} \sum_{z\in\mathbf{Y}} g_1(s,z,T_{n+1},y) \mathbf{1}\{g_2(s,z,T_{n+1},y) = X_{n+1}\}\\
& \quad r(s,z,T_{n+1},y) p(s,z) U(s,z,T_{n+1}), \quad T_n \le t < T_{n+1}, n \ge 0.
\end{aligned}
$$

Furthermore, we have for all $n \geq 0$ and all t with $T_n \leq t < T_{n+1}$,

$$
\begin{aligned}
p(t,y) = &\sum_{z \in \mathbf{Y}} (1 - g_1(0,z,t,y)) r(0,z,t,y) C_n(z) \frac{U(0,z,t)}{U(0,z,T_n)} \\
&+ \sum_{z \in \mathbf{Y}} \int_0^{T_n} (1 - g_1(u,z,t,y)) r(u,z,t,y) p_n(u,z) \frac{U(u,z,t)}{U(u,z,T_n)}\, du \\
&+ \sum_{z \in \mathbf{Y}} \int_{T_n}^{t} (1 - g_1(u,z,t,y)) r(u,z,t,y) p(u,z) U(u,z,t)\, du.
\end{aligned}
$$