Generalized contact distributions of inhomogeneous Boolean models

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Abstract

The main purpose of this work is to study and apply generalized contact distributions of (inhomogeneous) Boolean models Z with values in the extended convex ring. Given a convex body $L \subset \mathbb{R}^d$ and a gauge body $B \subset \mathbb{R}^d$ such a generalized contact distribution is the conditional distribution of the random vector $(d_B(L,Z), u_B(L,Z), p_B(L,Z), l_B(L,Z))$ given that $Z \cap L = \emptyset$, where Z is a Boolean model, $d_B(L,Z)$ is the distance of L from Z with respect to B, $p_B(L,Z)$ is the boundary point in L realizing this distance (if it exists uniquely), $u_B(L,Z)$ is the corresponding boundary point of B (if it exists uniquely) and $l_B(L,\cdot)$ may be taken from a large class of locally defined functionals. In particular, we pursue the question to which extent the spatial density and the grain distribution underlying an inhomogeneous Boolean model Z are determined by the generalized contact distributions of Z.

1 Introduction

The *contact distribution functions* build a classical tool for the description and analysis of random closed sets Z in $\mathbb{R}^d$ $(d \geq 2)$. They can be expressed in geometric terms if the random set Z has a more specific structure. A common assumption, which we will require also throughout this work, is that Z can be represented as a (locally finite) union of (random) compact convex sets, hence the realizations of Z are assumed to be elements of the extended convex ring $\mathcal{S}^d$ (*polyconvex sets*). Such random sets provide a sufficiently general framework to cover most situations which arise in practical applications of stochastic geometry (see [20], [15]). To be more precise, we assume that

$$Z := \bigcup_{n\in\mathbb{N}} (\xi_n + Z_n), \tag{1.1}$$

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where $\Psi := \{(\xi_n, Z_n) : n \in \mathbb{N}\}$ is a marked point process on $\mathbb{R}^d$, with the marks Z_n, $n \in \mathbb{N}$, being random convex bodies in $\mathbb{R}^d$. The idea is that Z_n is a *grain* associated with the *germ* (or centre) ξ_n. Therefore Z is also referred to as a *grain model* (see [18], [20]). We remark that, for a random $\mathcal{S}^d$-set Z, a representation (1.1) is always possible, in fact one even can require that Ψ has the same invariance properties as Z (see [23] and [18]). The most important example of a grain model is the *Boolean model*; it arises if Ψ is a Poisson process. We will mainly focus on the Boolean model in the following, results for more general grain models Z will be presented briefly in the final section.

For a Boolean model, the intensity measure Θ of Ψ determines the distribution of Ψ and hence that of Z. Following recent developments in stochastic geometry (see [1], [2], [5], [21]) and its applications (see [13], [16]), we do not assume Z and Ψ to be stationary. Instead we require that Θ is absolutely continuous with respect to $\mathcal{H}^d \otimes \mathbb{Q}$ with Radon-Nikodym derivative f, where $\mathcal{H}^d$ is the d-dimensional Lebesgue measure on $\mathbb{R}^d$, $\mathbb{Q}$ is a probability measure on the set of convex bodies in $\mathbb{R}^d$, and f is a non-negative measurable function. The integral $\int f(x, K)\mathbb{Q}(dK)$ is finite for $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$, and the probability measure

$$\left(\int f(x, K)\mathbb{Q}(dK)\right)^{-1} f(x, K)\mathbb{Q}(dK)$$

can be interpreted as the conditional distribution of the grain associated with the centre x given the locations ξ_n, $n \in \mathbb{N}$, of the grains and given that x is one of these locations. If the function f does not depend on K, it is the intensity function of the Poisson process $\{\xi_n : n \in \mathbb{N}\}$ of *germs*, and then $\mathbb{Q}$ can be interpreted as the distribution of the *typical grain*. In this situation, Ψ is obtained from the Poisson process of germs by independent marking.

The contact distribution functions of a random closed set Z are defined, with respect to a convex and compact *gauge body* (or structuring element) B containing the origin $0 \in \mathbb{R}^d$, as the conditional distribution functions of the random variables

$$d_B(\{x\}, Z) := \inf\{t \geq 0 : (x + tB) \cap Z \neq \emptyset\}, \quad x \in \mathbb{R}^d,$$

given that $x \notin Z$. The set in brackets may be empty if 0 is not an interior point of B or if $Z = \emptyset$, then we define $\inf \emptyset := \infty$. If Z is stationary, then these functions are independent of x. The monograph [20] contains important properties and applications of contact distribution functions in the stationary case and for random polyconvex sets, while in [3] this concept is applied to general stationary random closed sets without a convexity assumption. The inhomogeneous (i.e. non-stationary) case is investigated in the recent paper [5]. A survey on contact distributions is provided in [6].

The principal aim, which we pursue here, is to introduce and study a more general notion of contact distributions and to apply these to Boolean models Z, without any stationarity assumption. To describe our concept, we start from a convex body $L \subset \mathbb{R}^d$ and let

$$d_B(L, Z) := \inf\{t \geq 0 : (L + tB) \cap Z \neq \emptyset\}$$

denote the (relative) distance of Z from L. If there is a unique point x in the boundary of L and a unique point y in the boundary of Z such that $d_B(\{x\}, \{y\}) = d_B(L, Z) > 0$, then we define $(p_B(L, Z), p'_B(L, Z)) := (x, y)$. The point $u_B(L, Z) := (y - x)/d_B(L, Z)$ then lies in the boundary point of B. Assume, moreover, that $l_B(L, Z)$ is a random variable that does only depend on $Z \cap U$ for some arbitrary small neighbourhood U of $p'_B(L, Z)$. Important examples of such locally defined quantities are the principal curvatures of Z at the boundary point $p'_B(L, Z)$, if the latter is a smooth boundary point of Z. Our main result provides a formula for the conditional distribution of the general random vector

$$(d_B(L, Z), u_B(L, Z), p_B(L, Z), l_B(L, Z))$$

given that $Z \cap L = \emptyset$. These conditional distributions, which are obtained for different choices of B, L and the functional $l_B(L, \cdot)$, will be called *generalized contact distributions* of Z. A main ingredient of the formula for the generalized contact distributions of a Boolean model are the mixed curvature measures introduced in [8].

Our results generalize some of the findings in [5], which in turn extended previous work in [10]. The latter was devoted to the study of the random vectors $(d_{B^d}(\{0\}, Z), u_{B^d}(\{0\}, Z))$ for a stationary random set Z and the Euclidean unit ball B^d. This investigation was then continued in [5] for random vectors $(d_B(\{x\}, Z), u_B(\{x\}, Z))$, where B is a general gauge body and Z is allowed to be non-stationary. Here we extend these results in two directions. First, we treat general convex sets L rather than singletons $\{x\}$, second we also include a locally defined random variable $l_B(L, Z)$. The first generalization provides a unified framework for random distances and is interesting from a mathematical point of view. Our results show close relationships between the distribution of $d_B(L, Z)$ and the distributions of $d_{B+s(L-x)}(\{x\}, Z)$ where $x \in L$ and $s > 0$. The second generalization seems to be more important, if one has statistical applications in mind, and we will present some related uniqueness results in Section 4, similar in spirit to the recent contributions in [21], [22]. To be more specific, we deal with a Boolean model parametrized by the pair $(f, \mathbb{Q})$, where $f(x, K)$ is independent of K and Z therefore is an independently marked grain model. Taking B as the Euclidean unit ball and varying L in the set of all singletons $\{x\}$, we will give some partial answers to the question which properties of $(f, \mathbb{Q})$ are determined by the generalized contact distributions. The results concerning the grain distribution $\mathbb{Q}$ are new even in the stationary case and can be considered as a further small step towards the estimation of $\mathbb{Q}$. According to [14] this is the ultimate goal in the statistics of the Boolean model.

The paper is organized as follows. In Section 2 we recall basic concepts from convex geometry. In particular we introduce mixed (relative) support measures and generalize an integral-geometric formula from [8]. In Section 3 we prove our main results for the Boolean model and in particular a representation of the generalized contact distributions in terms of mixed curvature measures and the pair $(f, \mathbb{Q})$. Section 4 contains some applications under specific assumptions on $(f, \mathbb{Q})$. The final section uses Palm probabilities as in [5] to generalize the main result in Section 3 to a quite arbitrary marked point process Ψ.

2 Geometrical foundations

Throughout the following, we work in Euclidean space $\mathbb{R}^d, d \geq 2$, with norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$. We write $\mathcal{H}^d$ for the d-dimensional Lebesgue measure, B^d for the unit ball in $\mathbb{R}^d$, and S^{d-1} for the unit sphere. The ball of radius $r \geq 0$, centred at $x \in \mathbb{R}^d$, is denoted by $B^d(x, r)$. For a set $F \subset \mathbb{R}^d$, $\dim F$ is the dimension of the affine hull of F, ∂F denotes the boundary and $\check{F}$ is the reflection of F with respect to the origin.

Let $\mathcal{K}^d$ be the set of convex bodies in $\mathbb{R}^d$, i.e. the set of all non-empty compact convex subsets of $\mathbb{R}^d$. For properties of convex bodies and further standard notions in convex geometry, which we use in the following without explanation, we refer to [17]. If $K \in \mathcal{K}^d$ and $u \in S^{d-1}$, $F(K,u)$ is the support set of K in direction u. For $r \in \{2,3,\ldots\}$, we denote by $\mathcal{K}^{d,r}_{gp}$ the set of all $(K_1,\ldots,K_r) \in (\mathcal{K}^d)^r$ which are in general relative position, that is, for which

$$\dim F(K_1+\ldots+K_r,u) = \dim F(K_1,u)+\ldots+\dim F(K_r,u) \tag{2.1}$$

is satisfied for all $u \in S^{d-1}$ (see [8] for more details). We will need this concept for $r=2,3$. Another way to express condition (2.1) is to say that the sum of the linear subspaces parallel to the affine hulls of $F(K_1,u),\ldots,F(K_r,u)$ is direct, for all $u \in S^{d-1}$. For instance, $(K_1,K_2) \in \mathcal{K}^{2,2}_{gp}$ if and only if the convex bodies K_1 and K_2 do not have parallel segments in their boundaries with the same exterior unit normal vectors; hence, $(K,K) \in \mathcal{K}^{2,2}_{gp}$ if and only if $K \in \mathcal{K}^2$ is strictly convex, whereas smoothness of K is not relevant here. More generally, $(K_1,\ldots,K_r) \in \mathcal{K}^{d,r}_{gp}$ if all but possibly one of the convex bodies $K_1,\ldots,K_r$ are strictly convex. On the other hand, strict convexity is a sufficient but not a necessary requirement. For instance, we may choose a square as $K_1 \in \mathcal{K}^2$ and a small rotation of this square as $K_2 \in \mathcal{K}^2$ to obtain $(K_1,K_2) \in \mathcal{K}^{2,2}_{gp}$. In fact, for arbitrary $K_1,K_2 \in \mathcal{K}^d$, we have $(K_1,\rho K_2) \in \mathcal{K}^{d,2}_{gp}$ for almost all rotations ρ of $\mathbb{R}^d$; compare Lemma 5.3 for a more general assertion concerning three convex bodies.

A set $S \subset \mathbb{R}^d$ is an element of the extended convex ring $\mathcal{S}^d$, if it can be represented as a union

$$S = \bigcup_{i\in\mathbb{N}} K_i \tag{2.2}$$

of convex sets $K_i \in \mathcal{K}^d$, which form a locally finite system of sets, i.e. which are such that each bounded set is intersected by only a finite number of the sets K_i. In this case the set S is closed. For the purpose of this paper it is convenient to allow also empty unions, i.e. to include the empty set $\emptyset$ into $\mathcal{S}$.

Subsequently, we fix a convex body $B \in \mathcal{K}^d$ which contains the origin 0, but is otherwise arbitrary. B serves as a gauge body (structuring element) relative to which distances are measured. For $S \in \mathcal{S}^d$ and $L \in \mathcal{K}^d$ we define

$$d_B(L,S) := \inf\{t \geq 0 : (L+tB)\cap S \neq \emptyset\}$$

(with $\inf\emptyset := \infty$) as the relative distance of S from L. Note that $d_B(L,S) > 0$ if and only if $L \cap S = \emptyset$. Generalizing some of the notions in [5] we define, for each $L \in \mathcal{K}^d$ and $S \in \mathcal{S}^d$ satisfying $0 < d_B(L,S) < \infty$,

$$\Pi_B(L,S) := \{(x,y) \in \partial L \times \partial S : d_B(\{x\},\{y\}) = d_B(L,S)\},$$

and then the *skeleton class* $\mathcal{K}^d_B(S)$ of S with respect to B by

$$\mathcal{K}^d_B(S) := \{L \in \mathcal{K}^d : 0 < d_B(L,S) < \infty, \operatorname{card}\Pi_B(L,S) \geq 2\}.$$

$\mathcal{K}^d_B(S)$ contains the singleton sets from the exoskeleton $\operatorname{exo}_B(S)$ introduced in [5], but also sets $L \in \mathcal{K}^d$ which are in special position with respect to S. If $0 < d_B(L,S) < \infty$ and $L \notin \mathcal{K}^d_B(S)$, then there are uniquely determined boundary points $x \in \partial L$ and $y \in$

∂S such that $d_B(\{x\},\{y\}) = d_B(L,S)$. We then define $p_B(L,S) := x$, $p'_B(L,S) := y$ and $u_B(L,S) := (y-x)/d_B(L,S) \in \partial B$. Note that under these assumptions $p_B(L,S)$, $p'_B(L,S)$, $d_B(L,S)$ and $u_B(L,S)$ are related by

$$p'_B(L,S) = p_B(L,S) + d_B(L,S)u_B(L,S). \tag{2.3}$$

In case $d_B(L,S) \in \{0,\infty\}$ or $L \in \mathcal{K}^d_B(S)$, we give $(p_B(L,S), u_B(L,S))$ some fixed value in $\partial L \times \partial B$ and set $p'_B(L,S) := 0$. We mention one special situation where we slightly deviate from these definitions. Namely, if B is the unit ball B^d, we can define $u_{B^d}(L,S)$ even if the boundary points $x \in \partial L$ and $y \in \partial S$, which fulfill $d_{B^d}(\{x\},\{y\}) = d_{B^d}(L,S) > 0$, are not unique. The only assumption we need in the Euclidean case is that, for given L, there is a unique K_i in the representation (2.2) such that $d_{B^d}(L,K_i) = d_{B^d}(L,S)$. Then, for all $x \in \partial L$, $y \in \partial S$ as described above, the direction $u_{B^d}(L,S) := (y-x)/d_{B^d}(L,S)$ is the same. In fact, assume that there exist $(x_1,y_1),(x_2,y_2) \in \partial L \times \partial K_i$ such that $d_{B^d}(\{x_j\},\{y_j\}) = d_{B^d}(L,K_i) > 0$ for $j = 1,2$. (This situation arises, for instance, if $L = [0,1]^d$ and $K_i = [0,1]^d + 2e_1$.) Set $u := (y_1 - x_1)/d_{B^d}(\{x_1\},\{y_1\})$. Then we deduce that

$$x_2 \in L \subset \{x \in \mathbb{R}^d : \langle x - x_1, u\rangle \le 0\}, \quad y_2 \in K_i \subset \{y \in \mathbb{R}^d : \langle y - y_1, u\rangle \ge 0\}.$$

But then $d_{B^d}(\{x_1\},\{y_1\}) = d_{B^d}(\{x_2\},\{y_2\})$ implies $\langle x_2 - x_1, u\rangle = 0$, $\langle y_2 - y_1, u\rangle = 0$ and $y_2 = x_2 + d_{B^d}(\{x_1\},\{y_1\})u$, which yields the desired conclusion. Hence, in such cases, we have $d_{B^d}(L,S)$ and $u_{B^d}(L,S)$, whereas $p_{B^d}(L,S)$ and $p'_{B^d}(L,S)$ remain undefined, in general.

Let $L, K \in \mathcal{K}^d$ satisfy $(L, \check{K}, B) \in \mathcal{K}^{d,3}_{gp}$ and $0 < d_B(L,K) < \infty$. It is not difficult to see that then $L \notin \mathcal{K}^d_B(K)$. Assume now, more generally, that S is an element of the extended convex ring, represented as in (2.2). If there is an $n \in \mathbb{N}$ such that $(L, \check{K}_n, B) \in \mathcal{K}^{d,3}_{gp}$ and $0 < d_B(L,K_n) < d_B(L, \cup_{i\neq n} K_i)$, then $L \notin \mathcal{K}^d_B(S)$ and

$$(d_B(L,S), p_B(L,S), p'_B(L,S), u_B(L,S)) = (d_B(L,K_n), p_B(L,K_n), p'_B(L,K_n), u_B(L,K_n)).$$

This simple fact will be needed in the proof of our main result in Section 3. Note that here the assumption $(L, \check{K}_n, B) \in \mathcal{K}^{d,3}_{gp}$ is used in an essential way.

Subsequently, we use the *relative support measures* $\Theta_{i;d-i}(K;B;\cdot)$, for $i \in \{0,\dots,d-1\}$ and $(K,B) \in \mathcal{K}^{d,2}_{gp}$, and the *mixed relative support measures* $\Theta_{i,j;k+1}(K,L;B;\cdot)$, for $i,j,k \in \{0,\dots,d-1\}$ with $i+j+k = d-1$ and $(K,L,B) \in \mathcal{K}^{d,3}_{gp}$. These measures have been introduced in [8] and [5] and can be obtained in the following way.

For $(K,B) \in \mathcal{K}^{d,2}_{gp}$ with $0 \in B$, $\rho \ge 0$, and a measurable set $C \subset (\mathbb{R}^d)^2$ we define the local parallel set

$$M_\rho(K;B;C) := \{x \in (K+\rho B)\setminus K : (p_B(K,\{x\}), u_B(K,\{x\})) \in C\}.$$

It was shown in [8] (see also [5]) that there exist finite measures $\Theta_{i;d-i}(K;B;\cdot)$, $i \in \{0,\dots,d-1\}$, on $(\mathbb{R}^d)^2$ such that

$$\mathcal{H}^d(M_\rho(K;B;\cdot)) = \frac{1}{d}\sum_{j=0}^{d-1} \rho^{d-j}\binom{d}{j}\Theta_{j;d-j}(K;B;\cdot)$$

for $\rho \geq 0$. Now let $(K, L, B) \in \mathcal{K}^{d,3}_{gp}$, $0 \in B$, $\rho > 0$, and let $A_1 \subset K$, $A_2 \subset L$, $C \subset B$ be measurable. Then we obtain from [8, (5.12)] and from a special case of Theorem 5.6 in [8] that

$$\begin{aligned}
&\mathcal{H}^d(M_1(\rho K + \rho L; B; (\rho A_1 + \rho A_2) \times C)) \\
&= \frac{1}{d} \sum_{j=0}^{d-1} \rho^j \binom{d}{j} \Theta_{j;d-j}(K + L; B; (A_1 + A_2) \times C) \\
&= \frac{1}{d} \sum_{i,l,r=0,r>0}^{d} \binom{d}{i, l, r} \rho^{i+l} \Theta_{i,l;r}(K, L; B; A_1 \times A_2 \times C),
\end{aligned}$$

where the multinomial coefficient is defined by

$$\binom{d}{i, l, r} := \frac{d!}{i!l!r!},$$

if i, l, r are non-negative integers with $i + l + r = d$, and as zero otherwise. A comparison of coefficients yields that

$$\begin{aligned}
&\frac{1}{d} \binom{d}{j} \Theta_{j;d-j}(K + L; B; (A_1 + A_2) \times C) \qquad (2.4) \\
&= \frac{1}{d-j} \sum_{i,l=0}^{d-1} \binom{d-1}{i, l, d-1-j} \Theta_{i,l;d-j}(K, L; B; A_1 \times A_2 \times C),
\end{aligned}$$

where $j \in \{0, \ldots, d-1\}$. For $i, l, k \in \{0, \ldots, d-1\}$ with $i+l+k = d-1$ and $(K, L, B) \in \mathcal{K}^{d,3}_{gp}$, the mixed relative support measure $\Theta_{i,l;k+1}(K, L; B; \cdot)$ is a finite measure on $(\mathbb{R}^d)^3$ which is concentrated on $\partial K \times \partial L \times \partial B$. The total measure $\Theta_{i,l;k+1}(K, L; B; (\mathbb{R}^d)^3)$ is a special *mixed volume*,

$$\Theta_{i,l;k+1}(K, L; B; (\mathbb{R}^d)^3) = dV(K\,[i], L\,[l], B\,[k+1]).$$

If the gauge body B is the unit ball B^d and $(K, L) \in \mathcal{K}^{d,2}_{gp}$, then the classical mixed surface area measures appear as marginal measures,

$$\Theta_{i,l;k+1}(K, L; B^d; \mathbb{R}^d \times \mathbb{R}^d \times \cdot) = S(K\,[i], L\,[l], B^d\,[k]; \cdot);$$

see [8, p. 328 and (4.8)]. Here, and in the following, we identify measures on $\mathbb{R}^d$ which have their support in S^{d-1} with measures on S^{d-1}.

As another special case, we consider

$$\Theta_j(K; \cdot) := \Theta_{j;d-j}(K; B^d; \cdot),$$

the Euclidean support measures of $K \in \mathcal{K}^d$ for $j \in \{0, \ldots, d-1\}$, as well as the surface area measures $S_j(K; \cdot)$ which are obtained as the image measures of $\Theta_j(K; \cdot)$ under the projection $\mathbb{R}^d \times S^{d-1} \to S^{d-1}$, $(x, u) \mapsto u$.

The next theorem will be used to prove our main result in Section 3.

Theorem 2.1. *Let L, K be convex bodies such that $(L, \check{K}, B) \in \mathcal{K}^{d,3}_{gp}$. If $g : \mathbb{R}^d \to [0, \infty)$ is a measurable function, then*

$$\int \mathbf{1}\{0 < d_B(L, z + K) < \infty\} g(z) \mathcal{H}^d(dz)$$

$$= \sum_{i,j,k=0}^{d-1} \binom{d-1}{i,j,k} \int_0^\infty \int t^k g(x + y + tb) \Theta_{i,j;k+1}(L, \check{K}; B; d(x, y, b)) dt.$$

Proof. From the definition of the relative support measures and by an argument similar to the one leading to formula (2.4) in [5], we obtain

$$\int \mathbf{1}\{0 < d_B(L, z + K) < \infty\} g(z) \mathcal{H}^d(dz)$$

$$= \int \mathbf{1}\{0 < d_B(L + \check{K}, \{z\}) < \infty\}$$

$$\times\, g(p_B(L + \check{K}, \{z\}) + d_B(L + \check{K}, \{z\}) u_B(L + \check{K}, \{z\})) \mathcal{H}^d(dz)$$

$$= \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^\infty \int t^{d-1-j} g(z + tb) \Theta_{j;d-j}(L + \check{K}; B; d(z, b)) dt.$$

By (2.4) and by Lemma 3.2 in [8], the latter sum is equal to

$$\sum_{i,l,j=0}^{d-1} \binom{d-1}{i, l, d-1-j} \int_0^\infty \int t^{d-1-j} g(x + y + tb) \Theta_{i,l;d-j}(L, \check{K}; B; d(x, y, b)) dt.$$

The substitution $k = d - 1 - j$ now yields the assertion. □

As a consequence we obtain the following result which generalizes Theorem 4.3 in [8]. We also use this opportunity to correct a misprint in Theorem 4.3 of [8], which was carried over to Corollary 4.4 of that paper, namely a missing minus sign in one of the arguments.

Theorem 2.2. *Let L, K be convex bodies such that $(L, \check{K}, B) \in \mathcal{K}^{d,3}_{gp}$. If $g : [0, \infty] \times \partial B \times \partial L \times \partial K \to [0, \infty)$ is a measurable function, then*

$$\int \mathbf{1}\{0 < d_B(L, z + K) < \infty\} g(d_B(L, z + K), u_B(L, z + K), p_B(L, z + K),$$

$$p'_B(L, z + K) - z) \mathcal{H}^d(dz)$$

$$= \sum_{i,j,k=0}^{d-1} \binom{d-1}{i,j,k} \int_0^\infty \int t^k g(t, b, x, -y) \Theta_{i,j;k+1}(L, \check{K}; B; d(x, y, b)) dt.$$

Proof. Let $i, j, k \in \{0, \dots, d-1\}$ satisfy $i + j + k = d - 1$, and let $x \in \partial L$, $y \in \partial \check{K}$, $b \in \partial B$ be such that (x, y, b) is in the support of $\Theta_{i,j;k+1}(L, \check{K}; B; \cdot)$. We may assume that the pair $(x + y, b)$ is a B-support element of $L + \check{K}$ (see [8], [5]), and then it follows that

$$d_B(L, x + tb + y + K) = t.$$

This and Lemma 3.2 in [8] imply that

$$(p_B(L, x+tb+y+K), u_B(L, x+tb+y+K), p'_B(L, x+tb+y+K)) = (x, b, x+tb).$$

Inserting these relations into the result of Theorem 2.1, we obtain the asserted formula. □

3 Contact distributions of Boolean models

Let $\mathcal{S}^d$ and $\mathcal{K}^d$ be endowed with the σ-field generated by the standard topology (see [11] or [18]).

In the following, we consider processes of convex particles, i.e. point processes on $\mathcal{K}^d$. For convenience, we represent a point process on $\mathcal{K}^d$ as a *marked point process* on $\mathbb{R}^d$ with marks in $\mathcal{K}^d$. By the latter we mean a random measure Ψ with values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ defined on an abstract probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and such that $\Psi(M \times \mathcal{K}^d) < \infty$, for all compact sets $M \subset \mathbb{R}^d$. By definition the given point process on $\mathcal{K}^d$ is obtained from Ψ as the image measure under the map $\mathbb{R}^d \times \mathcal{K}^d \to \mathcal{K}^d$, $(x, K) \mapsto x + K$. Hence, Ψ is not uniquely determined by the underlying particle process as long as the marks of Ψ are not normalized in a suitable way. At this point we do not introduce any normalization in order to avoid an unnecessary restriction of the generality of our results. We refer to [7] for more details on random measures and point processes and to [18] for processes of geometric objects.

The *intensity measure* Θ of Ψ is defined (as usual) by $\Theta := \mathbb{E}\,\Psi$, it is a Borel measure on $\mathbb{R}^d \times \mathcal{K}^d$. We assume that Θ is *locally finite* in the sense that

$$\Theta(\{(x, K) : (x+K) \cap M \neq \emptyset\}) < \infty \tag{3.1}$$

for all compact sets $M \subset \mathbb{R}^d$. Condition (3.1) implies that

$$\Psi(\{(x, K) : (x+K) \cap M \neq \emptyset\}) < \infty \qquad \mathbb{P}\text{ - a.s.} \tag{3.2}$$

If Ψ is a *Poisson process*, then the random variable in (3.2) has a Poisson distribution, and therefore (3.2) is equivalent to (3.1), in this case.

As announced in Section 1, we will also assume that Θ can be represented in the form

$$\Theta = \iint \mathbf{1}\{(x, K) \in \cdot\} f(x, K) \mathcal{H}^d(dx) \mathbb{Q}(dK), \tag{3.3}$$

where $\mathbb{Q}$ is a probability measure on $\mathcal{K}^d$ and f is a non-negative, real-valued, measurable function on $\mathbb{R}^d \times \mathcal{K}^d$. Note that in general f and $\mathbb{Q}$ are not uniquely determined by Θ. In the special case where the point process Ψ is stationary (here and in the following stationarity refers to the first component), (3.3) is satisfied with a constant function f, which is called the *intensity* γ of Ψ.

We make use of the fact that a point process Ψ on $\mathbb{R}^d \times \mathcal{K}^d$ can be represented in the form

$$\Psi = \sum_{n=1}^{\tau} \delta_{(\xi_n, Z_n)}, \tag{3.4}$$

where (ξ_n, Z_n), $n \in \mathbb{N}$, is a random variable in $\mathbb{R}^d \times \mathcal{K}^d$ and τ is a random variable taking values in $\mathbb{N}_0 \cup \{\infty\}$. The *second factorial moment measure* $\Theta^{(2)}$ of Ψ is then defined as

$$\Theta^{(2)} := \mathbb{E}\Big[\sum_{m \neq n} \mathbf{1}\{(\xi_m, Z_m, \xi_n, Z_n) \in \cdot\}\Big].$$

Clearly, this definition is independent of the particular representation (3.4) of Ψ. In addition to (3.1) and (3.3), we require that there exists a σ-finite measure β on $\mathcal{K}^d \times \mathbb{R}^d \times \mathcal{K}^d$ such that

$$\Theta^{(2)} \ll \mathcal{H}^d \otimes \beta. \tag{3.5}$$

It follows that $\xi_n \neq \xi_m$ $\mathbb{P}$ - almost surely for all $n \neq m$, i.e. $\Psi(\cdot \times \mathcal{K}^d)$ is a *simple* point process. If Ψ is a Poisson process, then we have $\Theta^{(2)} = \Theta \otimes \Theta$ and (3.5) is a consequence of (3.3).

Given a marked point process Ψ fulfilling (3.1), we define the associated closed union set

$$Z := \bigcup_{(x,K) \in \Psi} (x + K), \tag{3.6}$$

where we write $(x, K) \in \Psi$ if $\Psi(\{(x, K)\}) > 0$. Note that we have not excluded the case $Z = \emptyset$ which might occur with positive probability even in the stationary case. For $L \in \mathcal{K}^d$ and the given gauge body B, we define the *contact distribution function* $H_B(L, \cdot)$ of Z by

$$H_B(L, t) := \mathbb{P}(d_B(L, Z) \leq t \mid Z \cap L = \emptyset), \quad t \geq 0, \tag{3.7}$$

provided that $\mathbb{P}(Z \cap L = \emptyset) > 0$. For stationary Z and $L = \{0\}$ this coincides with the classical notion (see e.g. [20]); in this case we use the abbreviation $H_B(t) := H_B(\{0\}, t)$. Obviously, we have

$$\mathbb{P}(d_B(L, Z) > t) = \mathbb{P}(Z \cap L = \emptyset)(1 - H_B(L, t)). \tag{3.8}$$

If Ψ is a Poisson process, then we call Z an (inhomogeneous) *Boolean model.* Note that the point process $\Psi(\cdot \times \mathcal{K}^d)$ need not be independent of the sequence (Z_n). The latter can be achieved if and only if Θ is a product measure. We will collect some further comments on Boolean models with independent grains in Section 4. For a Boolean model Z,

$$\mathbb{P}(Z \cap M = \emptyset) = \exp\left[-\Theta(\{(x, K) : (x + K) \cap M \neq \emptyset\})\right], \tag{3.9}$$

which is positive if M is a compact set and (3.1) is satisfied; in particular, $H_B(L, t) < 1$ for $t \geq 0$. It is well-known (see e.g. [18]) that the probabilities $\mathbb{P}(Z \cap M = \emptyset)$ in (3.9), with M running through all compact subsets of $\mathbb{R}^d$, uniquely determine the distribution of the Poisson point process on $\mathcal{K}^d$ from which Z is obtained by forming the union set. The transition from this particle process to a marked point process Ψ on $\mathbb{R}^d \times \mathcal{K}^d$ then depends on a suitable (and in general non-unique) decomposition of convex bodies into 'location' and 'shape'. We will discuss this and related topics in the beginning of the next section.

Although we mainly focus on the Boolean model we will present some more general results in the final section.

The function g appearing in our main theorem below has to fulfill a certain property which we now define. Given $L \in \mathcal{K}^d$, we say that a measurable function g on $\mathcal{S}^d$ is *L-admissible* if $g(S) = g(S')$ whenever $S, S' \in \mathcal{S}^d$ have the following three properties:

$$
\begin{aligned}
&0 < d_B(L, S), d_B(L, S') < \infty,\\
&L \notin \mathcal{K}_B^d(S) \cup \mathcal{K}_B^d(S'),\\
&B^d(p'_B(L, S), \varepsilon) \cap S = B^d(p'_B(L, S'), \varepsilon) \cap S' \quad \text{for some } \varepsilon > 0.
\end{aligned}
$$

In a certain sense, this condition ensures that g is locally defined. A measurable function g on a product space of the form $\mathcal{X} \times \mathcal{S}^d$ is called L-admissible, if $g(x, \cdot)$ is L-admissible for all $x \in \mathcal{X}$. Clearly, the admissibility of a function also depends on the choice of the gauge body. We do not indicate this by our terminology, because B will usually be fixed in the sequel.

Theorem 3.1. *Let Z be the Boolean model defined by a Poisson process Ψ satisfying* (3.1) *and* (3.3)*. Let $L \in \mathcal{K}^d$ be such that $(L, \check{K}, B) \in \mathcal{K}_{gp}^{d,3}$ for $\mathbb{Q}$ - almost all $K \in \mathcal{K}^d$. Then*

$$\mathbb{P}(0 < d_B(L, Z) < \infty, L \in \mathcal{K}_B^d(Z)) = 0. \tag{3.10}$$

If $g : \mathcal{S}^d \to [0, \infty)$ is an L-admissible function, then

$$
\begin{aligned}
&\mathbb{E}\left[\mathbf{1}\{d_B(L, Z) < \infty\} g(Z) \mid Z \cap L = \emptyset\right]\\
&\quad = \sum_{i,j,k=0}^{d-1} \binom{d-1}{i, j, k} \int_0^\infty t^k (1 - H_B(L, t)) \iint g(x + tb + y + K) \qquad (3.11)\\
&\qquad \times f(x + tb + y, K) \Theta_{i,j;k+1}(L, \check{K}; B; d(x, y, b)) \mathbb{Q}(dK) dt.
\end{aligned}
$$

The proof of Theorem 3.1 requires two auxiliary results which we formulate and prove in greater generality (without the Poisson assumption), since we will use them again in the final section.

Lemma 3.2. *Let Ψ be a marked point process on $\mathbb{R}^d \times \mathcal{K}^d$ fulfilling* (3.1), (3.3) *and* (3.5)*, represented as in* (3.4)*. Then*

$$\mathbb{P}(0 < d_B(L, \xi_m + Z_m) = d_B(L, \xi_n + Z_n) < \infty) = 0, \quad m \neq n.$$

Proof. From (3.5), we obtain

$$\mathbb{P}\left(\bigcup_{m\neq n}\{0<d_B(L,\xi_m+Z_m)=d_B(L,\xi_n+Z_n)<\infty\}\right)$$

$$\leq \mathbb{E}\left[\sum_{m\neq n}\mathbf{1}\{0<d_B(L,\xi_m+Z_m)=d_B(L,\xi_n+Z_n)<\infty\}\right]$$

$$=\int \mathbf{1}\left\{x_1\in\partial(L+\check{K}_1+d_B(L,x_2+K_2)B)\right\}$$

$$\times\,\mathbf{1}\{0<d_B(L,x_2+K_2)<\infty\}\Theta^{(2)}(d(x_1,K_1,x_2,K_2))$$

$$=\iint \mathbf{1}\left\{x_1\in\partial(L+\check{K}_1+d_B(L,x_2+K_2)B)\right\}h(x_1,K_1,x_2,K_2)$$

$$\times\,\mathbf{1}\{0<d_B(L,x_2+K_2)<\infty\}\mathcal{H}^d(dx_1)\beta(d(K_1,x_2,K_2)),$$

where h denotes the density of $\Theta^{(2)}$ with respect to $\mathcal{H}^d\otimes\beta$. The last expression vanishes, since the boundary of a convex body has $\mathcal{H}^d$ - measure zero. □

Lemma 3.3. *Let Ψ be a marked point process on $\mathbb{R}^d\times\mathcal{K}^d$ fulfilling* (3.1) *and* (3.3), *represented as in* (3.4). *If L is such that $(L,\check{K},B)\in\mathcal{K}^{d,3}_{gp}$ for $\mathbb{Q}$ - almost all $K\in\mathcal{K}^d$, then $(L,\check{Z}_n,B)\in\mathcal{K}^{d,3}_{gp}$ is satisfied $\mathbb{P}$ - almost surely for all $n\in\mathbb{N}$.*

Proof. Let $n\in\mathbb{N}$ be fixed. Then

$$\mathbb{P}((L,\check{Z}_n,B)\notin\mathcal{K}^{d,3}_{gp})\leq\mathbb{E}\left[\int\mathbf{1}\{(L,\check{K},B)\notin\mathcal{K}^{d,3}_{gp}\}\Psi(d(y,K))\right]$$

$$=\int\mathbf{1}\{(L,\check{K},B)\notin\mathcal{K}^{d,3}_{gp}\}\Theta(d(y,K))$$

$$=\iint\mathbf{1}\{(L,\check{K},B)\notin\mathcal{K}^{d,3}_{gp}\}f(y,K)\mathbb{Q}(dK)\mathcal{H}^d(dy).$$

This equals 0, since by assumption $\mathbb{Q}\left(\left\{K\in\mathcal{K}^d:(L,\check{K},B)\notin\mathcal{K}^{d,3}_{gp}\right\}\right)=0.$ □

Proof of Theorem 3.1. First, we note that the set

$$\{S\in\mathcal{S}^d:0<d_B(L,S)<\infty,L\in\mathcal{K}^d_B(S)\}$$

is Borel measurable, since it can be written as a countable union of closed sets (compare the proofs of Lemmas 3.11 and 3.12 in [5]). We let T denote the measurable mapping (on a suitable space of locally finite counting measures on $\mathbb{R}^d\times\mathcal{K}^d$) which is implicitly defined by (3.6), i.e. which satisfies $T(\Psi)=Z$. Then Lemmas 3.2 and 3.3 entail the following partition

$$\{0<d_B(L,Z)<\infty\}=\bigcup_{n=1}^{\infty}(A_n\cap B_n\cap C_n)\quad\mathbb{P}\text{ - a.s.},$$

where

$$
\begin{aligned}
A_n &:= \{0 < d_B(L, \xi_n + Z_n) < \infty\}, \\
B_n &:= \{d_B\left(L, T\left(\Psi - \delta_{(\xi_n, Z_n)}\right)\right) > d_B(L, \xi_n + Z_n)\}, \\
C_n &:= \{(L, \check{Z}_n, B) \in \mathcal{K}^{d,3}_{gp}\}.
\end{aligned}
$$

This implies (3.10). Moreover, for all $n \in \mathbb{N}$ we have

$$
(d_B(L, Z), p_B(L, Z), u_B(L, Z)) = (d_B(L, \xi_n + Z_n), p_B(L, \xi_n + Z_n), u_B(L, \xi_n + Z_n))
$$

on $A_n \cap B_n \cap C_n$. Since g is L-admissible, we obtain

$$
g(Z) = g(\xi_n + Z_n) \quad \text{on } A_n \cap B_n \cap C_n.
$$

Using this together with well-known properties of the Poisson process Ψ (see e.g. Satz 3.1 in [12]), as well as (3.3) and (3.8), we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\mathbf{1}\{0 < d_B(L, Z) < \infty\} g(Z)\right] = \mathbb{E}\left[\sum_{n=1}^{\infty} g(\xi_n + Z_n) \mathbf{1}_{A_n \cap B_n \cap C_n}\right] \\
&= \mathbb{E}\Bigg[\int g(z + K) \mathbf{1}\{0 < d_B(L, z + K) < \infty\} \\
&\qquad \times \mathbf{1}\{d_B(L, T(\Psi - \delta_{(z,K)})) > d_B(L, z + K)\} \mathbf{1}\{(L, \check{K}, B) \in \mathcal{K}^{d,3}_{gp}\} \Psi(d(z, K))\Bigg] \\
&= \mathbb{E}\Bigg[\int g(z + K) \mathbf{1}\{0 < d_B(L, z + K) < \infty\} \\
&\qquad \times \mathbf{1}\{d_B(L, T(\Psi)) > d_B(L, z + K)\} \mathbf{1}\{(L, \check{K}, B) \in \mathcal{K}^{d,3}_{gp}\} \Theta(d(z, K))\Bigg] \\
&= \mathbb{P}(L \cap Z = \emptyset) \iint g(z + K)(1 - H_B(L, d_B(L, z + K))) \mathbf{1}\{0 < d_B(L, z + K) < \infty\} \\
&\qquad \times \mathbf{1}\{(L, \check{K}, B) \in \mathcal{K}^{d,3}_{gp}\} f(z, K) \mathcal{H}^d(dz) \mathbb{Q}(dK).
\end{aligned}
$$

For each $K \in \mathcal{K}^d$ with $(L, \check{K}, B) \in \mathcal{K}^{d,3}_{gp}$, we can now apply Theorem 2.1 to the function

$$
g_K(z) := (1 - H_B(L, d_B(L, z + K))) g(z + K) f(z, K)
$$

and obtain as in the proof of Theorem 2.2 that

$$
\begin{aligned}
&\mathbb{E}\left[\mathbf{1}\{d_B(L, Z) < \infty\} g(Z) \mid Z \cap L = \emptyset\right] \\
&\quad = \sum_{i,j,k=0}^{d-1} \binom{d-1}{i, j, k} \int_0^{\infty} t^k (1 - H_B(L, t)) \iint g(x + tb + y + K) \\
&\qquad \times f(x + tb + y, K) \Theta_{i,j;k+1}(L, \check{K}; B; d(x, y, b)) \mathbb{Q}(dK) dt.
\end{aligned}
$$

This finally proves the theorem. □

It is often more convenient to apply Theorem 3.1 in the following form:

Theorem 3.4. *Let the assumptions of Theorem 3.1 be satisfied, and let* $g : [0,\infty] \times \partial B \times \partial L \times \mathcal{S}^d \to [0,\infty)$ *be an* L*-admissible function. Then*

$$\mathbb{E}\left[\mathbf{1}\{d_B(L,Z) < \infty\} g(d_B(L,Z), u_B(L,Z), p_B(L,Z), Z) \mid Z \cap L = \emptyset\right]$$

$$= \sum_{i,j,k=0}^{d-1} \binom{d-1}{i,j,k} \int_0^\infty t^k (1 - H_B(L,t)) \iint g(t, b, x, x + tb + y + K) \tag{3.12}$$

$$\times f(x + tb + y, K) \Theta_{i,j;k+1}(L, \check{K}; B; d(x,y,b)) \mathbb{Q}(dK) dt.$$

Proof. The function $S \mapsto g(d_B(L,S), u_B(L,S), p_B(L,S), S)$ from $\mathcal{S}^d$ to $[0,\infty)$ is L-admissible and Theorem 3.1 can be applied. It remains to transform the right-hand side of (3.11) using the arguments in the proof of Theorem 2.2. □

For an L-admissible function g, Theorems 3.1 and 3.4 in particular describe the distributions of $g(Z)$ and $g(d_B(L,Z), u_B(L,Z), p_B(L,Z), Z)$, respectively, conditionally to $Z \cap L = \emptyset$. If $l_B(L,\cdot)$ is an L-admissible function describing a local geometric quantity, we get, as a special case, the (conditional) distribution of the random vector $W := (d_B(L,Z), u_B(L,Z), p_B(L,Z), l_B(L,Z))$. These conditional distributions, which are obtained for different choices of B, L and $l_B(L,\cdot)$, are the *generalized contact distributions* mentioned in the title of this paper and in the introduction. As a further specialization, we may consider the random vector $W' := (u_B(L,Z), p_B(L,Z), l_B(L,Z))$ and fix a Borel set C' in the space where W' takes its values. Then Theorem 3.4 shows that the *generalized contact distribution function*

$$\tilde{F}(t) := \mathbb{P}(d_B(L,Z) \le t, W' \in C' \mid Z \cap L = \emptyset), \quad t \ge 0,$$

is absolutely continuous, and this theorem also gives an expression for the density of $\tilde{F}$ in terms of f, $\mathbb{Q}$, and the 'ordinary' contact distribution function $H_B(L,\cdot)$. For the latter, we can obtain a more explicit representation as follows.

Corollary 3.5. *Let the assumptions of Theorem 3.1 be satisfied. Then*

$$H_B(L,t) = 1 - \exp\left\{-\int_0^t \lambda_B(L,s)\, ds\right\}, \quad t \ge 0,$$

where

$$\lambda_B(L,s) = \sum_{i,j,k=0}^{d-1} \binom{d-1}{i,j,k} s^k \iint f(x + sb + y, K) \Theta_{i,j;k+1}(L, \check{K}; B; d(x,y,b)) \mathbb{Q}(dK). \tag{3.13}$$

If the Poisson process Ψ *is stationary with intensity* γ*, then*

$$H_B(L,t) = 1 - \exp\left\{-\sum_{i,j,k=0}^{d-1} \binom{d}{i,j,k+1} t^{k+1} \gamma \int V(L\,[i], \check{K}\,[j], B\,[k+1]) \mathbb{Q}(dK)\right\}$$

(without the assumption on the general relative position).

Proof. For $t \geq 0$, Theorem 3.4 implies that

$$H_B(L,t) = \int_0^t \lambda_B(L,s)(1 - H_B(L,s))ds.$$

We already know that $H_B(L,s) < 1$ for $s \geq 0$ (see the corresponding remark following (3.9)). Moreover, $H_B(L,\cdot)$ is a continuous function that satisfies $H_B(L,0) = 0$. Using the monotonicity of $H_B(L,\cdot)$, we obtain that

$$\int_0^t \lambda_B(L,s)\,ds \leq \frac{H_B(L,t)}{1 - H_B(L,t)} < \infty$$

for all $t \geq 0$. Hence the first assertion immediately follows from the exponential formula of Lebesgue-Stieltjes calculus (see e.g. [9, Theorem A4.12]).

The result for the stationary case is a direct consequence of (3.13) and the formula for the total mixed relative support measures given in Section 2. □

Remark 3.6. We emphasize the special case $L = \{x\}$, $x \in \mathbb{R}^d$. The assumptions of Theorem 3.4 then amount to requiring that $(\check{K}, B) \in \mathcal{K}^{d,2}_{gp}$ for $\mathbb{Q}$ - almost all $K \in \mathcal{K}^d$ and that $g : [0,\infty] \times \partial B \times \mathcal{S}^d \to [0,\infty)$ is an $\{x\}$-admissible function. We recall that Theorem 5.6 in [8] implies that

$$\Theta_{i,j;k+1}(\{x\}, \check{K}; B; \cdot) = 0, \quad i > 0,$$

and

$$\Theta_{0,j;k+1}(\{x\}, \check{K}; B; C_1 \times C_2 \times C_3) = \mathbf{1}\{x \in C_1\}\Theta_{j;k+1}(\check{K}; B; C_2 \times C_3)$$

for measurable sets $C_1, C_2, C_3 \subset \mathbb{R}^d$. Inserting these formulas into Theorem 3.4, we deduce that

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}\{d_B(\{x\}, Z) < \infty\} g(d_B(\{x\}, Z), u_B(\{x\}, Z), Z) \mid x \notin Z\right] \\
&= \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^\infty t^{d-1-j}(1 - H_B(\{x\}, t)) \iint g(t, b, x + tb + y + K) \\
&\qquad \times f(x + tb + y, K)\Theta_{j;d-j}(\check{K}; B; d(y,b))\mathbb{Q}(dK)dt;
\end{aligned}$$

moreover,

$$H_B(\{x\}, t) = 1 - \exp\left\{-\int_0^t \lambda_B(\{x\}, s)\,ds\right\}$$

and

$$\lambda_B(\{x\}, s) = \sum_{j=0}^{d-1} \binom{d-1}{j} s^{d-1-j} \iint f(x + sb + y, K)\Theta_{j;d-j}(\check{K}; B; d(y,b))\mathbb{Q}(dK).$$

In the stationary case the formula for the contact distribution function transforms into the classical representation (without the assumption on the general relative position)

$$H_B(t) = 1 - \exp\left\{-\sum_{j=0}^{d-1} \binom{d}{j} t^{d-j}\, \gamma \int V(\check{K}\,[j], B\,[d-j])\mathbb{Q}(dK)\right\}.$$

Our results simplify further if the structuring element B is the Euclidean ball B^d. In that case, we write $d(L,S)$, $u(L,S)$, $H(L,t)$ instead of $d_{B^d}(L,S)$, $u_{B^d}(L,S)$, $H_{B^d}(L,t)$, etc. In the Euclidean case, Theorem 3.4 leads to the following result, which holds without an assumption on the general position of L and the particles (compare the remarks on $u(L,S) = u_{B^d}(L,S)$ in Section 2). In order to obtain this extension, it seems to be necessary to repeat the proof of Theorem 3.1 and to use equations (4.2.9) and (5.1.17) in [17] (alternatively, see [19, Theorem 4.4]) instead of Theorem 2.1 (compare also [8, p. 316]).

Theorem 3.7. *Let Z be the stationary Boolean model defined by a stationary Poisson process Ψ with intensity γ and satisfying* (3.1)*. Let $L \in \mathcal{K}^d$, let $C \subset \mathbb{R}^d$ be a Borel set and $r \geq 0$. Then*

$$\begin{aligned}&\mathbb{P}\left(d(L,Z) \leq r, u(L,Z) \in C \mid L \cap Z = \emptyset\right)\\ &= \sum_{i,j,k=0}^{d-1} \binom{d-1}{i,j,k} \gamma \int_0^r t^k (1 - H(L,t))dt \int S(L\,[i], \check{K}\,[j], B^d\,[k]; C)\mathbb{Q}(dK)\,.\end{aligned}$$

In particular,

$$\begin{aligned}&\mathbb{P}\left(d(\{0\},Z) \leq r, u(\{0\},Z) \in C \mid 0 \notin Z\right)\\ &= \sum_{j=0}^{d-1} \binom{d-1}{j} \gamma \int_0^r t^{d-1-j} (1 - H(t))dt \int S_j(\check{K}; C)\mathbb{Q}(dK)\,.\end{aligned} \tag{3.14}$$

Remark 3.8. It is easy to see that the contact distributions in (3.14) determine the mean value $\int S_j(K;C)\mathbb{Q}(dK)$ for all measurable C. We refer to [4, p. 156] for a more detailed discussion of these mean values.

Finally, we consider generalized contact distributions under geometric constraints on L and on the particles in Ψ. More specifically, we investigate the situation where L is a polytope and the probability measure $\mathbb{Q}$ is supported by the set of polytopes in $\mathcal{K}^d$ with the help of Theorem 3.1. Let $\mathcal{T}_i^\circ(L)$, $i \in \{0,\dots,d-1\}$, be the union of the relative interiors of the i-dimensional faces of L. We denote by $\mathcal{P}^d \subset \mathcal{K}^d$ the set of convex polytopes in $\mathbb{R}^d$, we let C^d be the d-dimensional unit cube centred at 0, and we set $C^d(x,\epsilon) := x + \epsilon C^d$, for $x \in \mathbb{R}^d$ and $\epsilon > 0$. Then, for $S \in \mathcal{S}^d$ and $i \in \{0,\dots,d-1\}$, we write $z \in \mathcal{T}_i^\circ(S)$ if there is some $\epsilon > 0$ such that

$$C^d(z,\epsilon) \cap S \in \mathcal{P}^d \qquad \text{and} \qquad z \in \mathcal{T}_i^\circ(C^d(z,\epsilon) \cap S).$$

Note that this definition is independent of the special choice of $\epsilon > 0$ and it is also consistent with the previous definition. Finally, for $S \in \mathcal{S}^d$, we put

$$\xi_B(L,S) := \sum_{i=0}^{d-1} i\mathbf{1}\{p_B(L,S) \in \mathcal{T}_i^\circ(L)\},$$

$$\eta_B(L,S) := \sum_{j=0}^{d-1} j\mathbf{1}\{p'_B(L,S) \in \mathcal{T}_j^\circ(S)\}.$$

Using the arguments employed in the proof of Corollary 3.14 in [5] and on p. 236 in [18], we deduce that the functions $\xi_B(L,\cdot), \eta_B(L,\cdot) : \mathcal{S}^d \to [0,\infty)$ are measurable.

We then obtain the following result.

Theorem 3.9. *Let the assumptions of Theorem 3.1 be satisfied. Assume that L is a polytope and $\mathbb{Q}$ is concentrated on the set of polytopes. Furthermore, let $g : [0,\infty] \times \partial B \times \{0,\dots,d-1\}^2 \times \mathcal{S}^d \to [0,\infty)$ be an L-admissible function. Then*

$$\mathbb{E}\left[\mathbf{1}\{d_B(L,Z)<\infty\} g(d_B(L,Z), u_B(L,Z), \xi_B(L,Z), \eta_B(L,Z), Z) \mid Z \cap L = \emptyset\right]$$

$$= \sum_{i,j,k=0}^{d-1} \binom{d-1}{i,j,k} \int_0^\infty t^k (1-H_B(L,t)) \iint g(t,b,i,j,x+tb+y+K)$$

$$\times f(x+tb+y,K)\Theta_{i,j;k+1}(L,\check{K};B;d(x,y,b))\mathbb{Q}(dK)dt.$$

Proof. In order to apply Theorem 3.1, we define a measurable function $\bar{g} : \mathcal{S}^d \to [0,\infty)$ in the following way. If $0 < d_B(L,S) < \infty$ and $L \notin \mathcal{K}^d_B(S)$, then we set

$$\bar{g}(S) := g(d_B(L,S), u_B(L,S), \xi_B(L,S), \eta_B(L,S), S),$$

and otherwise we define $\bar{g}$ to be zero. Clearly, $\bar{g}$ is an L-admissible function. Substituting $\bar{g}$ into the left-hand side of equation (3.11) and using (3.10), we find

$$\mathbb{E}\left[\mathbf{1}\{d_B(L,Z)<\infty\}\bar{g}(Z) \mid Z \cap L = \emptyset\right]$$

$$= \mathbb{E}\left[\mathbf{1}\{d_B(L,Z)<\infty\} g(d_B(L,Z), u_B(L,Z), \xi_B(L,Z), \eta_B(L,Z), Z) \mid Z \cap L = \emptyset\right].$$

For the right-hand side of (3.11), we obtain

$$\sum_{i,j,k=0}^{d-1} \binom{d-1}{i,j,k} \int_0^\infty t^k (1-H_B(L,t))$$

$$\times \iint g(t,b,\xi_B(L,x+tb+y+K), \eta_B(L,x+tb+y+K), x+tb+y+K)$$

$$\times f(x+tb+y,K)\Theta_{i,j;k+1}(L,\check{K};B;d(x,y,b))\mathbb{Q}(dK)dt.$$

For $\Theta_{i,j;k+1}(L,\check{K};B;\cdot)$ - almost all (x,y,b) we have $(x,y) \in \mathcal{T}_i^\circ(L) \times \mathcal{T}_j^\circ(\check{K})$, and thus

$$p_B(L,x+tb+y+K) \in \mathcal{T}_i^\circ(L),\ p'_B(L,x+tb+y+K) \in \mathcal{T}_j^\circ(x+tb+y+K),$$

from which we get

$$g(t,b,\xi_B(L,x+tb+y+K), \eta_B(L,x+tb+y+K), x+tb+y+K)$$

$$= g(t,b,i,j,x+tb+y+K).$$

If B is a polytope as well, the above-mentioned support property follows from formula (5.14) in [8]. Using the special form of these measures for polytopes and the fact that the mixed relative support measures are weakly continuous, one can deduce the general case by an approximation argument. □

Theorem 3.9 admits an interesting interpretation in the stationary case:

Remark 3.10. Let the assumptions of Theorem 3.9 be satisfied and assume moreover that the Poisson process Ψ is stationary with intensity γ. Define

$$\lambda_B(L,t,i,j) := \binom{d-1}{i,j,d-1-i-j} t^{d-1-i-j}\gamma d \int V(L\,[i], \check{K}\,[j], B\,[d-i-j])\mathbb{Q}(dK)$$

and note that

$$\lambda_B(L,t) = \sum_{i,j=0}^{d-1} \lambda_B(L,t,i,j)$$

is just the function in (3.13) (under the assumption of stationarity) which is also called *hazard rate* of $H_B(L,\cdot)$. By Theorem 3.9

$$t \mapsto (1 - H_B(L,t))\lambda_B(L,t,i,j)$$

is a density of $\mathbb{P}(d_B(L,Z) \in \cdot, \xi_B(L,Z) = i, \eta_B(L,Z) = j \mid Z \cap L = \emptyset)$. Under $\mathbb{P}(\cdot \mid Z \cap L = \emptyset)$ we may interpret $(d_B(L,Z), (\xi_B(L,Z), \eta_B(L,Z)))$ as a random marked point with *hazard measure*

$$\sum_{i,j=0}^{d-1} \int \mathbf{1}\{(t,i,j) \in \cdot\}\lambda_B(L,t,i,j)dt,$$

see e.g. Appendix A5.3 in [9]. It is well known and easy to prove (on the basis of the preceding results) that

$$\mathbb{P}((\xi_B(L,Z), \eta_B(L,Z)) = (i,j) \mid d_B(L,Z) = t) = \frac{\lambda_B(L,t,i,j)}{\lambda_B(L,t)}$$

for $\mathbb{P}(d_B(L,Z) \in \cdot \mid Z \cap L = \emptyset)$ - a.e. $t \geq 0$. (Here we define $0/0 := 0$.) Suppose that the set Z starts growing at time 0 in such a way that it covers $Z + t\check{B}$ at time t. Then $d_B(L,Z)$ is just the time of the first contact of the growing set with L. For small $h > 0$ the number $\lambda_B(L,t,i,j)h$ can then be interpreted as the conditional probability that the first contact occurs in the time interval $(t, t+h]$ at an i-dimensional face of L and a j-dimensional face of Z (more accurately: at a point of $\partial(Z + t\check{B})$ corresponding to a point in a j-dimensional face of Z) given that the contact has not occured yet by time t. For large values of t the hazard rate $\lambda_B(L,\cdot)$ is determined essentially by the position of the vertices of L and the vertices of the typical grain of Ψ, i.e.

$$\lim_{t\to\infty} \frac{\lambda_B(L,t,i,j)}{\lambda_B(L,t)} = \begin{cases} 1 & \text{if } i = j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For small values of t, however, the position and orientation of all pairs of faces of L and faces of the typical grain of Ψ whose dimensions add up to $d-1$ are the determining factor, i.e.

$$\lim_{t\to 0} \frac{\lambda_B(L,t,i,j)}{\lambda_B(L,t)} = \frac{\lambda_B(L,0,i,j)}{\lambda_B(L,0)} = 0,$$

for $i + j < d - 1$, provided that $\lambda_B(L,0) > 0$.

4 Statistical analysis of Boolean models

Let Z be a Boolean model as defined by (3.6) in terms of a Poisson process Ψ with an intensity measure satisfying (3.1) and (3.3). In the last section, we have seen that certain conditional expectations of Z can be expressed in terms of the characteristic quantities f and $\mathbb{Q}$ of Ψ. In this section, we investigate to what extent f and $\mathbb{Q}$ are determined by the generalized contact distribution functions of Z. We have already mentioned that the distribution of Z determines the intensity measure and hence the distribution of the associated Poisson particle process $X = \{\xi_n + Z_n : n \in \mathbb{N}\}$. However it does not determine the intensity measure of Ψ. Therefore it does not seem to be reasonable to pursue the above question in full generality.

A standard way of transforming a particle process into a marked point process is to use a *centre function*, i.e. a measurable function $c : \mathcal{K}^d \to \mathbb{R}^d$ satisfying $c(K + x) = c(K) + x$, for all $x \in \mathbb{R}^d$ and $K \in \mathcal{K}^d$. Common choices of such centre functions are the centre of the circumscribed ball, the centre of mass, the Steiner point of the convex hull, the lower left tangent point or the lower left corner (compare [18]). Any centre function can be used to obtain a normalized representation of a given Boolean model Z in terms of the marked point process $\Psi_c := \{(c(K) + x, K - c(K)) : (x, K) \in \Psi\}$ taking its marks in the set $\mathcal{K}^d_c := \{K \in \mathcal{K}^d : c(K) = 0\}$. (If $c(K) \in K$ for all $K \in \mathcal{K}^d$, then the technical condition $\Psi_c(M \times \mathcal{K}^d_c) < \infty$ for all compact sets $M \subset \mathbb{R}^d$ is a consequence of (3.2); otherwise it has to be assumed.) The distribution of Ψ_c is uniquely determined by that of Z. In general Ψ_c will not be independently marked. However, if Ψ is stationary, then Ψ_c is stationary as well. In this case Ψ_c is an independent marking of the stationary Poisson process $\Psi_c(\cdot \times \mathcal{K}^d)$.

In this section we will not assume any special centering. However, we assume that the function f in (3.3) does not depend on K. Then Ψ is an independently marked Poisson process on $\mathbb{R}^d \times \mathcal{K}^d$ for which the Poisson process of germs $\Psi(\cdot \times \mathcal{K}^d)$ has the intensity function f. This assumption is crucial for all results in this section and it should be noticed that it is defined in terms of Ψ and not of Z. Not only does Z not determine the distribution of Ψ, but it is quite possible that Z can be represented in terms of two Poisson processes, where the first is an independent marking and the second not (see Example 4.5 below). Notwithstanding these facts it is an interesting and challenging task to find general conditions under which the generalized contact distributions of Z already determine the *spatial density* f and the *mark distribution* $\mathbb{Q}$ of Ψ (and therefore also Θ). As far as the determination of f is concerned, the main problem is that the points of $\Psi(\cdot \times \mathcal{K}^d)$ are not directly accessible, at least not in the general case, via generalized contact distributions involving admissible functions of Z. In the sequel, however, we shall discuss several situations in which f, and sometimes even $\mathbb{Q}$, are determined. Our results require certain additional properties of f and $\mathbb{Q}$, which vary according to the special situation considered.

Let us fix now an independently marked Poisson process Ψ with spatial density f and mark distribution $\mathbb{Q}$ and the associated Boolean model

$$Z = \bigcup_{(x,K)\in\Psi} (x + K).$$

Here we exclude the trivial case $\Theta \equiv 0$. It is convenient to consider a *typical grain* of Ψ, i.e. a random convex body Ξ with distribution $\mathbb{Q}$. In order to unify the presentation, we postulate two regularity assumptions which we assume to hold for the whole section. For the spatial density f, we assume that

$$f \text{ is continuous and bounded.} \tag{4.1}$$

For the typical grain we require that

$$\Xi \text{ is almost surely of class } C^2_+. \tag{4.2}$$

Here we say that a convex body $K \in \mathcal{K}^d$ is of class C^2_+ if ∂K is a hypersurface of differentiability class C^2 with everywhere positive Gauss curvature (see §2.5 in [17]). For this and other basic notions of convex geometry such as support functions, second order differentiability of convex functions or principal radii of curvature we refer to [17, §§1.5, 1.7, 2.5]. If (4.1) and (4.2) are satisfied, we say (in this section) that the Boolean model Z is *smooth*.

We start our investigation of generalized contact distributions by introducing a suitable class of admissible functions. The definition of these functions is based on the local (second order) information which can be expressed in terms of radii of curvature. Let $j \in \{1, \dots, d-1\}$, $K \in \mathcal{K}^d$ and $u \in S^{d-1}$. Then $s_j(K, u)$ is defined as the j-th normalized elementary symmetric function of the principal radii of curvature of K at u, if the support function $h(K, \cdot)$ of K is second order differentiable at u; otherwise we set $s_j(K, u) := 0$. In addition, we define $s(K, u) := (s_1(K, u), \dots, s_{d-1}(K, u))$. Note that if $h(K, \cdot)$ is of class C^2, then, for each $j \in \{1, \dots, d-1\}$, the surface area measure $S_j(K; \cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure ν^{d-1} on S^{d-1} and the density function is just $s_j(K, \cdot)$. Further, if $h(K, \cdot)$ is of class C^1, then K is strictly convex and we let $\tau(K, u)$ denote the unique boundary point of K with exterior unit normal vector u.

From the map $s : \mathcal{K}^d \times S^{d-1} \to \mathbb{R}^{d-1}$ we now derive a measurable map $\bar{s} : \mathbb{R}^d \times \mathcal{S}^d \to \mathbb{R}^{d-1}$. Let $x \in \mathbb{R}^d$ and $S \in \mathcal{S}^d$ be given. If $0 < d(\{x\}, S) < \infty$, $\{x\} \notin \mathcal{K}^d_{B^d}(S)$ and $B^d(p'(\{x\}, S), \epsilon) \cap S$ is convex for some $\epsilon > 0$, then we set

$$\bar{s}(x, S) := s(B^d(p'(\{x\}, S), \epsilon) \cap S, -u(\{x\}, S));$$

if these conditions are not satisfied, then we set $\bar{s}(x, S) := 0 \in \mathbb{R}^{d-1}$. Using this map, we consider generalized contact distributions of the form

$$\mathbb{E}\left[\mathbf{1}\{d(\{x\}, Z) \le r\} G(u(\{x\}, Z), \bar{s}(x, Z)) \mid x \notin Z\right], \tag{4.3}$$

where $G : S^{d-1} \times \mathbb{R}^{d-1} \to [0, \infty)$ is measurable, $x \in \mathbb{R}^d$ and $r \ge 0$. Subsequently, we say that a quantity is *determined* by (4.3) if it is determined provided that (4.3) is known for all $x \in \mathbb{R}^d$, all $r \ge 0$, and all measurable functions $G : S^{d-1} \times \mathbb{R}^{d-1} \to [0, \infty)$.

Theorem 4.1. *Let Z be a Boolean model which is smooth in the sense of* (4.1) *and* (4.2)*. Then the generalized contact distributions of the form* (4.3) *determine the expectations*

$$\mathbb{E}\left[f(x - \tau(\Xi, u)) g(s(\Xi, u))\right] \tag{4.4}$$

for all $x \in \mathbb{R}^d$, all $u \in S^{d-1}$, and all measurable functions $g : \mathbb{R}^{d-1} \to [0, \infty)$.

Proof. We fix $x \in \mathbb{R}^d$ and take a bounded measurable function $G : S^{d-1} \times \mathbb{R}^{d-1} \to [0, \infty)$. The function

$$S^{d-1} \times \mathcal{S}^d \to [0, \infty), \qquad (u, S) \mapsto G(u, \bar{s}(x, S)), \tag{4.5}$$

is $\{x\}$-admissible. An application of a special case of Theorem 3.4 (compare Remark 3.6) then shows that the generalized contact distributions of the form (4.3) determine the expressions

$$\begin{aligned}
&\sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^r t^{d-1-j}(1 - H(\{x\}, t)) \iint G(u, \bar{s}(x, x + tu + y + K)) \\
&\qquad \times f(x + tu + y)\Theta_j(\check{K}; d(y, u))\mathbb{Q}(dK)dt \\
&= \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^r t^{d-1-j}(1 - H(\{x\}, t)) \iint G(-u, s(K, u)) \\
&\qquad \times f(x - tu - y)\Theta_j(K; d(y, u))\mathbb{Q}(dK)dt.
\end{aligned} \tag{4.6}$$

By assumption (4.2), for $j \in \{0, \dots, d-1\}$ and $\mathbb{Q}$ - almost every $K \in \mathcal{K}^d$ we get

$$\Theta_j(K; \cdot) = \int \mathbf{1}\{(\tau(K, u), u) \in \cdot\} s_j(K, u)\nu^{d-1}(du),$$

where $s_0(K, u) := 1$ for all $u \in S^{d-1}$. Substituting this into (4.6), we obtain

$$\begin{aligned}
&\sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^r t^{d-1-j}(1 - H(\{x\}, t)) \iint G(-u, s(K, u)) \\
&\qquad \times f(x - tu - \tau(K, u))s_j(K, u)\nu^{d-1}(du)\mathbb{Q}(dK)dt.
\end{aligned}$$

Replacing in (4.5) the map G by

$$(u, (s_1, \dots, s_{d-1})) \mapsto \frac{G(u, s_1, \dots, s_{d-1})}{(1 + s_1)^d},$$

we obtain that the expression

$$\begin{aligned}
&\sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^r t^{d-1-j}(1 - H(\{x\}, t)) \iint G(-u, s(K, u)) \\
&\qquad \times f(x - tu - \tau(K, u))\frac{s_j(K, u)}{(1 + s_1(K, u))^d}\nu^{d-1}(du)\mathbb{Q}(dK)dt
\end{aligned} \tag{4.7}$$

is also determined for all bounded measurable functions $G : S^{d-1} \times \mathbb{R}^{d-1} \to [0, \infty)$. Using that G, f and the ratio $s_j(K, u)/(1 + s_1(K, u))^d$ are bounded, that all functions involved are continuous as functions of t, and applying $\frac{\partial}{\partial r}\big|_{r=+0}$ to (4.7), we find that

$$\iint f(x - \tau(K, u))G(-u, s(K, u))\frac{s_{d-1}(K, u)}{(1 + s_1(K, u))^d}\mathbb{Q}(dK)\nu^{d-1}(du)$$

is determined for all $x \in \mathbb{R}^d$ and all bounded measurable functions $G : S^{d-1} \times \mathbb{R}^{d-1} \to [0, \infty)$. Since f is continuous and bounded, and $\tau(K, \cdot), s(K, \cdot)$ are continuous for $\mathbb{Q}$ - almost every $K \in \mathcal{K}^d$, we finally see that

$$\int f(x - \tau(K, u))g(s(K, u))\mathbb{Q}(dK)$$

is determined for all $u \in S^{d-1}$, and all continuous and bounded functions $g : \mathbb{R}^{d-1} \to [0, \infty)$. The Riesz representation theorem (applied for fixed u) implies that the latter integrals are even determined for arbitrary measurable $g : \mathbb{R}^{d-1} \to [0, \infty)$. $\square$

Remark 4.2. In the proof of Theorem 4.1, the boundedness of f is used to justify the application of the dominated convergence theorem. A weaker sufficient condition would be to assume the existence of a function $F : \mathbb{R}^d \times S^{d-1} \times \mathcal{K}^d \to [0, \infty)$ such that

$$f(x - tu - \tau(K, u)) \leq F(x, u, K)$$

for all $x \in \mathbb{R}^d$, ν^{d-1} - a.e. $u \in S^{d-1}$, $\mathbb{Q}$ - a.e. $K \in \mathcal{K}^d$, and for all $t \in [0, 1]$, and such that F is integrable with respect to $\nu^{d-1} \otimes \mathbb{Q}$ for all $x \in \mathbb{R}^d$. Especially, if we know that Ξ is $\mathbb{P}$ - a.s. contained in a fixed bounded set, then it is sufficient to assume that f is continuous.

In the particular case where Z can be represented as an independently marked Poisson process Ψ which is concentrated on convex bodies having their upper (say) tangent point at the origin, Theorem 4.1 has the following interesting consequences.

Theorem 4.3. *Let Z be a smooth Boolean model which can be derived from an independently marked Poisson process Ψ concentrated on $\mathbb{R}^d \times \mathcal{K}_0^d$, where $\mathcal{K}_0^d := \{K \in \mathcal{K}^d : \tau(K, u_0) = 0\}$ for some fixed $u_0 \in S^{d-1}$. Then, f and the distribution of $s(\Xi, u_0)$ are determined by* (4.3).

Corollary 4.4. *Let Z be a stationary and smooth Boolean model. Then the intensity γ of $\Psi(\cdot \times \mathcal{K}^d)$ is determined by* (4.3). *Moreover, the distribution of $s(\Xi, u_0)$ is determined for all $u_0 \in S^{d-1}$.*

Proof. We fix $u_0 \in S^{d-1}$ and define $c(K) := \tau(K, u_0)$ for $K \in \mathcal{K}^d$. As noticed earlier the Poisson process Ψ_c defined in the first paragraph of this section is an independent marking of $\Psi_c(\cdot \times \mathcal{K}^d)$. Theorem 4.3 with Ψ replaced with Ψ_c shows that γ and the distribution of $s(\Xi - c(\Xi), u_0) = s(\Xi, u_0)$ are determined by (4.3). $\square$

The following example shows that the property of independent marking is usually destroyed by switching to another centre function. Therefore the preceding proof does not apply in the non-stationary case.

Example 4.5. Let Z be a Boolean model which is derived from an independently marked Poisson process Ψ with spatial density f and for which the typical grain is almost surely a ball centred at the origin. Now choose $c(\cdot) := \tau(\cdot, u_0)$, for some fixed $u_0 \in S^{d-1}$, as a centre function. Then Ψ_c in general will only be independently marked, if f is translation invariant in direction u_0. Hence, we can only expect Ψ_c to be independently marked for every choice of u_0, when f is constant and therefore Z is stationary.

Now we return to the general situation from the beginning of this section. Then we can deduce several further results from Theorem 4.1.

Theorem 4.6. *Let Z be a smooth Boolean model. Assume that the spatial density f depends only on the first k arguments, for some $k \in \{0, \dots, d\}$, and is integrable on $\mathbb{R}^k$. Then $\int_{\mathbb{R}^k} f(y,0)\mathcal{H}^k(dy)$ is determined by* (4.3)*. Moreover,* (4.3) *determines the distribution of $s(\Xi, u)$ for all $u \in S^{d-1}$.*

Proof. Write $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ in (4.4). Integrating (4.4) with respect to $y \in \mathbb{R}^k$, for fixed $z \in \mathbb{R}^{d-k}$, and using Fubini's theorem as well as the translation invariance of Lebesgue measure, we deduce that

$$\int_{\mathbb{R}^k} f(y,0)\mathcal{H}^k(dy) \int g(s(K,u))\mathbb{Q}(dK)$$

is determined for all $u \in S^{d-1}$ and all measurable functions $g : \mathbb{R}^{d-1} \to [0, \infty)$. This determines the first factor, and then, since $y \mapsto f(y, 0)$ is in $L_1(\mathbb{R}^k)$ and $\Theta \not\equiv 0$, also the distribution of $s(\Xi, u)$. □

Remark 4.7. The case $k = 0$ of Theorem 4.6 is just Corollary 4.4. If $k = d$ in Theorem 4.6, then we have $f \in L^1(\mathbb{R}^d)$. Therefore the intensity measure of Ψ is finite in this case. Other classes of densities f can be treated if e.g. one has some prior information (such as boundedness) about the typical grain. We will not pursue this further.

Remark 4.8. As a very special consequence of Theorem 4.6 we obtain that under the assumptions of this theorem the mean values

$$\int S_j(K,C)\mathbb{Q}(dK) = \int_C \mathbb{E}\left[s_j(\Xi,u)\right]\mathcal{H}^{d-1}(du)$$

are determined for all measurable sets $C \subset S^{d-1}$. As we have already noticed in Remark 3.8, the stationary case of this result holds without any smoothness assumptions.

We continue with a rather general result on the spatial density f. For its proof, we need an additional condition on the distribution $\mathbb{Q}$ of the typical grain which is of a geometric and probabilistic type. The condition guarantees that at least in one direction $u_0 \in S^{d-1}$, the distribution of $s_1(\Xi, u_0)$ has the left end point of its support at the origin (hence the radii of curvature of the particles are not bounded away from 0), and, on the other hand, the origin approaches the boundaries of the particles in this direction if the corresponding radii of curvature are getting small. More precisely, we assume that there are $u_0 \in S^{d-1}$ and $c > 0$ such that

(i) $\|\tau(\Xi, u_0)\| \le c\, s_1(\Xi, u_0)$ $\mathbb{P}$ - a.s.,

(ii) $\mathbb{P}(s_1(\Xi, u_0) \le \epsilon) > 0$, for all $\epsilon > 0$.

Now we can state our next result.

Theorem 4.9. *Let Z be a smooth Boolean model satisfying* (i) *and* (ii) *for some $u_0 \in S^{d-1}$, $c > 0$. Assume that f is as in Theorem 4.6. Then f is uniquely determined by* (4.3)*.*

Proof. We fix $x \in \mathbb{R}^d$ and show that $f(x)$ is determined by (4.3). Let $\epsilon > 0$ be given. Since f is continuous, there is some $\delta > 0$ such that $\|y - x\| < \delta$ implies $|f(y) - f(x)| < \epsilon$. Moreover, if $s_1(\Xi, u_0) \le \delta/c$, then $\|\tau(\Xi, u_0)\| \le \delta$, $\mathbb{P}$ - a.s. Therefore, we find

$$\begin{aligned}
&\left| f(x) - \frac{1}{\mathbb{P}(s_1(\Xi, u_0) \le \delta/c)} \mathbb{E}\left[f(x - \tau(\Xi, u_0))\mathbf{1}\{s_1(\Xi, u_0) \le \delta/c\}\right] \right| \\
&\le \frac{1}{\mathbb{P}(s_1(\Xi, u_0) \le \delta/c)} \mathbb{E}\left[|f(x) - f(x - \tau(\Xi, u_0))|\mathbf{1}\{s_1(\Xi, u_0) \le \delta/c\}\right] \\
&\le \epsilon.
\end{aligned}$$

Hence,

$$f(x) = \lim_{\delta \downarrow 0} \frac{1}{\mathbb{P}(s_1(\Xi, u_0) \le \delta/c)} \mathbb{E}\left[f(x - \tau(\Xi, u_0))\mathbf{1}\{s_1(\Xi, u_0) \le \delta/c\}\right],$$

so that Theorems 4.1 and 4.6 imply that $f(x)$ is indeed determined by (4.3). $\square$

Remark 4.10. Clearly, conditions (i) and (ii) could be replaced by various other conditions involving, for instance, $s_j(\Xi, u_0)$ or certain powers of these random variables. In fact, we could even require estimates for a function of $s(\Xi, u_0)$. These more general cases can be treated in the same way as Theorem 4.9, hence we do not go into the details.

The class of grain distributions satisfying conditions (i) and (ii) is still very large. In the remainder of this section we will impose more specific assumptions on $\mathbb{Q}$ in order to obtain more accurate information on f and $\mathbb{Q}$. From now on, it is convenient to rewrite the typical grain in the form

$$\Xi = \eta \Xi', \tag{4.8}$$

where $\eta > 0$ is a random variable and Ξ' is a random convex body that is almost surely of class C^2_+. At first glance such a representation does not seem to provide any progress. The idea, however, is that Ξ' and η model the shape and size of Ξ, respectively. Although the additional hypothesis $\mathcal{H}^d(\Xi') = 1$ would force the factorization (4.8) to be unique, we will *not* make this or a similar assumption. The simplest special case of (4.8) is a deterministic Ξ', i.e. a Boolean model with a randomly scaled grain. Our final result will show that in this case the contact distributions determine $\mathbb{Q}$ as well as the convex body Ξ' up to a constant multiple and up to a translation. In more general cases some prior information on the distribution of Ξ' is needed to identify $\mathbb{Q}$. Before going into the details we first adapt Theorem 4.9 to the current setting.

Corollary 4.11. *Let Z be a smooth Boolean model whose typical grain is given by* (4.8). *Assume that there are $u_0 \in S^{d-1}$, $c > 0$ such that $\|\tau(\Xi', u_0)\| \le c\, s_1(\Xi', u_0)$ $\mathbb{P}$ - a.s. and that $\mathbb{P}(\mathbb{P}(\eta \le \varepsilon \mid \Xi') > 0$ for all $\varepsilon > 0) > 0$. Assume, finally, that f is as in Theorem 4.6. Then f is uniquely determined by* (4.3).

Remark 4.12. If the realizations of Ξ' are all contained in a compact subset of the set of convex bodies of class C^2_+, then the first assumption of Corollary 4.11 is automatically satisfied. On the other hand, if η and Ξ' are stochastically independent, then the second condition boils down to the requirement that $\mathbb{P}(\eta \le \epsilon) > 0$ for all $\epsilon > 0$.

Example 4.13. Let $\mathbb{P}(\Xi' \in \cdot)$ be concentrated on m convex bodies $K_1, \dots, K_m \in \mathcal{K}^d$ of class C^2_+ and assume that there is some $i \in \{1, \dots, m\}$ such that $\mathbb{P}(\Xi' = K_i) > 0$ and $\mathbb{P}(\eta \le \varepsilon \mid \Xi' = K_i) > 0$ for all $\varepsilon > 0$. Then the assumptions of Corollary 4.11 are satisfied for any $u_0 \in S^{d-1}$.

To prepare our next main result we need some more terminology. Let μ be a measure on $\mathcal{K}^d$ and let $\mathbf{F}$ be a family of measurable functions $h : \mathcal{K}^d \to \mathbb{R}$. We call $\mathbf{F}$ a *μ-separating class* of functions if for any measurable function $g : \mathcal{K}^d \to \mathbb{R}$ with $\mu(\{K : g(K) \neq 0\}) > 0$ there is some $h_0 \in \mathbf{F}$ such that $\int |gh_0| d\mu < \infty$ and $\int gh_0 d\mu \neq 0$.

Example 4.14. Let μ be concentrated on m convex bodies $K_1, \dots, K_m \in \mathcal{K}^d$ and assume that $\mu(\{K_i\}) > 0$ for all $i \in \{1, \dots, m\}$. It is easy to check that a family $\mathbf{F}$ of measurable functions $h : \mathcal{K}^d \to [0, \infty)$ is a μ-separating class if and only if there are m functions $h_1, \dots, h_m \in \mathbf{F}$ such that the vectors $(h_1(K_i), \dots, h_m(K_i))$, $i = 1, \dots, m$, are linearly independent.

Theorem 4.15. *Let Z be a smooth Boolean model whose typical grain is given by* (4.8) *and assume that f is as in Theorem 4.6. Let μ be a fixed σ-finite measure on $\mathcal{K}^d$ concentrated on the set of convex bodies of class C^2_+ and assume that there is some $j \in \{1, \dots, d-1\}$ such that $\{s_j(\cdot, u)^{\frac{n}{j}} : u \in S^{d-1}\}$ is a μ-separating class of functions for all $n \in \mathbb{N}$. Assume that the distribution of Ξ' is absolutely continuous with respect to μ with density α and that $\mathbb{E}\left[s_j(\Xi, u)^n\right] < \infty$ for all $n \in \mathbb{N}$ and all $u \in S^{d-1}$. Assume finally that*

$$\limsup_{n \to \infty} n^{-1} (\mathbb{E}\left[\eta^n \mid \Xi'\right])^{1/n} < \infty \quad \mathbb{P} \text{ - } a.s. \tag{4.9}$$

Then, $\alpha(K)$ is determined by (4.3) for μ - almost all K. Moreover, the conditional distributions $\mathbb{P}(\eta \in \cdot \mid \Xi' = K)$ are determined for μ - almost all K with $\alpha(K) > 0$. If μ is known, then $\mathbb{Q}$ is determined as well.

Proof. By Theorem 4.6 we get for all $n \in \mathbb{N}$ and all $u \in S^{d-1}$ that

$$\begin{aligned} \mathbb{E}\left[s_j(\Xi, u)^{\frac{n}{j}}\right] &= \mathbb{E}\left[\eta^n s_j(\Xi', u)^{\frac{n}{j}}\right] \\ &= \int \mathbb{E}\left[\eta^n \mid \Xi' = K\right] s_j(K, u)^{\frac{n}{j}} \alpha(K) \mu(dK) \end{aligned}$$

is uniquely determined by (4.3). Since this expression is finite by assumption and since $\{s_j(\cdot, u)^{\frac{n}{j}} : u \in S^{d-1}\}$ is a μ-separating class we conclude that $\alpha(K)\mathbb{E}\left[\eta^n \mid \Xi' = K\right]$ is determined for μ - almost all K. As it is well known, assumption (4.9) implies that the Laplace transforms

$$\varphi(K, t) := \mathbb{E}\left[\exp[-t\eta] \mid \Xi' = K\right], \quad t \ge 0,$$

are for $\mathbb{P}(\Xi' \in \cdot)$ - almost all K given by

$$\varphi(K, t) - 1 = \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} \mathbb{E}\left[\eta^n \mid \Xi' = K\right]$$

for all sufficiently small $t > 0$. But $\varphi(K, \cdot)$ is analytic on $(0, \infty)$, so that $\alpha(K)(\varphi(K, \cdot) - 1)$ is determined for μ - almost all K. Since $\eta > 0$ we have $\varphi(K, x) \to 0$ as $x \to \infty$, and therefore $\alpha(K)$ is determined for μ - almost all K. But then $\varphi(K, \cdot)$, and hence also $\mathbb{P}(\eta \in \cdot \mid \Xi' = K)$, is determined for μ - almost all K with $\alpha(K) > 0$. The final assertion follows from $\mathbb{P}((\eta, \Xi') \in d(y, K)) = \mathbb{P}(\eta \in dy \mid \Xi' = K)\alpha(K)\mu(dK)$. $\square$

Example 4.16. Let μ be concentrated on m convex bodies $K_1, \ldots, K_m \in \mathcal{K}^d$ of class C^2_+ with $\mu(\{K_i\}) = 1$ for all $i \in \{1, \ldots, m\}$. If the functions $s_j(K_1, \cdot)^{\frac{n}{j}}, \ldots, s_j(K_m, \cdot)^{\frac{n}{j}}$ are linearly independent for some $j \in \{1, \ldots, m\}$ and all $n \in \mathbb{N}$, then, according to Example 4.14, $\{s_j(\cdot, u)^{\frac{n}{j}} : u \in S^{d-1}\}$ is a μ-separating class of functions. Hence μ satisfies the assumptions of Theorem 4.15. The distribution of Ξ' is absolutely continuous with respect to μ if and only if it is concentrated on $\{K_1, \ldots, K_m\}$, that is, if and only if $\mathbb{P}(\Xi' \in \cdot) = \sum_{i=1}^m \alpha_i \delta_{K_i}$ for some $\alpha_1, \ldots, \alpha_m \geq 0$. If

$$\limsup_{n\to\infty} n^{-1}(\mathbb{E}\,[\eta^n \mid \Xi' = K_i])^{1/n} < \infty$$

for all $i \in \{1, \ldots, m\}$ with $\mathbb{P}(\Xi' = K_i) > 0$, then also η satisfies the assumptions of Theorem 4.15. Under all these assumptions we conclude that (4.3) determines the probabilities α_i and $\mathbb{P}(\eta \in \cdot \mid \Xi' = K_i)$ whenever $\alpha_i > 0$. We also note that f is determined as well, provided that there is some $i \in \{1, \ldots, m\}$ with $\alpha_i > 0$ such that $\mathbb{P}(\eta \leq \varepsilon \mid \Xi' = K_i) > 0$ for all $\varepsilon > 0$ (see Example 4.13).

In the situation of Example 4.16 and for the special case of two convex bodies $K_1, K_2 \in \mathcal{K}^d$, the assumption of the existence of a suitable separating class of functions is satisfied if and only if K_1 and K_2 are not homothetic. We state the corresponding result separately as a corollary.

Corollary 4.17. *Let Z be a smooth Boolean model whose typical grain is given by* (4.8) *and assume that f is as in Theorem 4.6. Let $K_1, K_2 \in \mathcal{K}^d$ be two convex bodies of class C^2_+ which are not homothetic, and assume that $\mathbb{P}(\Xi' \in \{K_1, K_2\}) = 1$. Further, assume that $\limsup_{n\to\infty} n^{-1}\mathbb{E}\,[\eta^n \mid \Xi' = K_i]^{1/n} < \infty$ for all $i \in \{1, 2\}$ with $\alpha_i := \mathbb{P}(\Xi' = K_i) > 0$. Then α_1, α_2 are determined by* (4.3)*. Moreover, the conditional distributions $\mathbb{P}(\eta \in \cdot \mid \Xi' = K_i)$ are determined whenever $\alpha_i > 0$. If K_1, K_2 are known, then $\mathbb{Q}$ is determined as well. Finally, f is determined if there is some $i \in \{1, 2\}$ with $\alpha_i > 0$ such that $\mathbb{P}(\eta \leq \varepsilon \mid \Xi' = K_i) > 0$ for all $\varepsilon > 0$.*

Remark 4.18. In general, it does not seem to be an easy task to find, for $m \geq 3$, a simple condition on convex bodies $K_1, \ldots, K_m \in \mathcal{K}^d$ of class C^2_+ which ensures that, say, $s_1(K_1, \cdot)^n, \ldots, s_1(K_m, \cdot)^n$ are linearly independent for all $n \in \mathbb{N}$. In order to obtain an example involving three convex bodies in $\mathbb{R}^2$, we consider ellipses $\mathcal{E}(a_i, b_i) := \{x \in \mathbb{R}^2 : (x_1/a_i)^2 + (x_2/b_i)^2 \leq 1\}$ with $a_i, b_i > 0$ for $i = 1, 2, 3$. It can be shown that the functions $s_1(\mathcal{E}(a_i, b_i), \cdot)^n$, $i = 1, 2, 3$, are linearly independent for all $n \in \mathbb{N}$, for $\mathcal{H}^6$ - almost all $(a_1, b_1, a_2, b_2, a_3, b_3) \in (\mathbb{R}^+)^6$. Therefore a random choice of the lengths of the semi-axes (a_i, b_i), $i = 1, 2, 3$, according to an absolutely continuous distribution on $(\mathbb{R}^+)^6$ will almost surely lead to the desired example.

We wish to emphasize, that the measure μ in Theorem 4.15 has been fixed in advance. The same remark applies to the convex bodies $K_1, \dots, K_m$ in Example 4.16 and Corollary 4.17. If μ is known, then $\mathbb{Q} = \mathbb{P}(\Xi \in \cdot)$ is determined under the assumptions of Theorem 4.15.

We finally consider the special case of a deterministic Ξ', i.e. a Boolean model with a randomly scaled (deterministic) grain. In contrast to Theorem 4.15 we do not assume that the shape of this grain is known. In fact, part of the problem consists in determining precisely this shape. Moreover, in comparison with Example 4.16 we can slightly improve the result regarding f.

Theorem 4.19. *Let Z be a smooth Boolean model with typical grain $\Xi = \eta K_0$ for some $K_0 \in \mathcal{K}^d$ of class C^2_+ and a positive random variable η with $\mathbb{E}\,\eta = 1$. Assume that f is as in Theorem 4.6. Then the distribution of η is determined by* (4.3)*; moreover, the convex body K_0 and the intensity of germs f are determined up to a translation by* (4.3)*.*

Proof. By Theorem 4.6, the expectations

$$\mathbb{E}\left[g(s_1(\eta K_0, u))\right] = \mathbb{E}\left[g(\eta s_1(K_0, u))\right]$$

are determined for all $u \in S^{d-1}$ and all measurable functions $g : \mathbb{R} \to [0, \infty)$. Hence, in particular, $s_1(K_0, u)$ is determined by (4.3) for all $u \in S^{d-1}$. Since K_0 is of class C^2_+, this yields that K_0 is determined up to translation. We define a probability measure on $(0, \infty)$ by setting $\mu(\cdot) := \mathbb{P}(\eta \in \cdot)$. It follows that

$$\mathbb{E}\left[g(\eta s_1(K_0, u))\right] = \int g(s_1(K_0, u)t)\mu(dt)$$

is determined. Since $s_1(K_0, \cdot)$ is already known, we conclude that μ is determined, and it remains to consider f. Theorem 4.1 and the results obtained so far show that

$$\int f(x - t\tau(K_0, u))g(t)\mu(dt)$$

is determined for all $x \in \mathbb{R}^d$, all $u \in S^{d-1}$, and all measurable functions $g : \mathbb{R} \to [0, \infty)$. We fix $u \in S^{d-1}$ arbitrarily. Since $\mu((0, \infty)) = 1$, there is some $t_0 > 0$ such that

$$\mu([t_0 - \epsilon, t_0 + \epsilon]) > 0$$

for all $\epsilon > 0$. We set $g_\epsilon := \mathbf{1}_{[t_0 - \epsilon, t_0 + \epsilon]}$ and deduce that

$$\lim_{\epsilon \downarrow 0} \frac{1}{\mu([t_0 - \epsilon, t_0 + \epsilon])} \int f(x - t\tau(K_0, u))g_\epsilon(t)\mu(dt) = f(x - t_0\tau(K_0, u))$$

is determined for all $x \in \mathbb{R}^d$. Hence f is determined up to a translation by (4.3). $\square$

5 The general case

In this final section we provide a general version of Theorem 3.4 which does not require any specific distributional assumptions.

The appropriate tool for formulating and proving general results about contact distributions of a random closed set Z, which is derived from a point process Ψ on $\mathbb{R}^d \times \mathcal{K}^d$, are the *Palm probabilities* $\{\mathbb{P}_{(x,K)} : (x, K) \in \mathbb{R}^d \times \mathcal{K}^d\}$ of Ψ. Their definition requires that the intensity measure Θ of Ψ is σ-finite, which is satisfied as a consequence of our assumption (3.1) if Θ is concentrated on the set of all $(x, K) \in \mathbb{R}^d \times \mathcal{K}^d$ for which $0 \in K$. The intensity measure Θ is also σ-finite if relation (3.3) is satisfied with a real-valued function f. The Palm probabilities of Ψ are defined by the relation

$$\iint H(\omega, x, K)\Psi(\omega, d(x, K))\mathbb{P}(d\omega) = \iint H(\omega, x, K)\mathbb{P}_{(x,K)}(d\omega)\Theta(d(x, K))\,, \tag{5.1}$$

which holds for an arbitrary measurable function $H : \Omega \times \mathbb{R}^d \times \mathcal{K}^d \to [0, \infty]$. Hence, $(x, K) \mapsto \mathbb{P}_{(x,K)}(A)$ is, for all $A \in \mathcal{A}$, a Radon-Nikodym derivative of the measure $\mathbb{E}\,[\mathbf{1}_A\Psi(\cdot)]$ with respect to Θ. As in Kallenberg ([7], p. 84) we can assume without restricting generality that $(x, K) \mapsto \mathbb{P}_{(x,K)}(\cdot)$ is a stochastic kernel, since all of our random elements take their values in Polish spaces. By Lemma 10.2 in [7] we can also assume that $\mathbb{P}_{(x,K)}(\Psi(\{(x, K)\}) \geq 1) = 1$ for all (x, K). Moreover, $\mathbb{P}_{(x,K)}(A)$ can be interpreted as the conditional probability of A given that $\Psi(\{(x, K)\}) = 1$.

A straightforward extension of the proofs for Theorems 3.1 and 3.4 implies the following general result, which is again based on Lemmas 3.2 and 3.3. We recall that the mapping T, which is used in the statement of Theorem 5.1 below, has been introduced at the beginning of the proof of Theorem 3.1.

Theorem 5.1. *Let Z be a random closed set in $\mathcal{S}^d$ such that $Z = T(\Psi)$ for a point process Ψ on $\mathbb{R}^d \times \mathcal{K}^d$ satisfying* (3.2), (3.3) *and* (3.5)*. Let $L \in \mathcal{K}^d$ be such that $(L, \check{K}, B) \in \mathcal{K}^{d,3}_{gp}$ for $\mathbb{Q}$ - almost all $K \in \mathcal{K}^d$. Further, let $g : [0, \infty] \times \partial B \times \partial L \times \mathcal{S}^d \to [0, \infty)$ be L-admissible. Then*

$$\begin{aligned}
&\mathbb{E}\,[\mathbf{1}\{0 < d_B(L, Z) < \infty\}g(d_B(L, Z), u_B(L, Z), p_B(L, Z), Z)] \\
&= \sum_{i,j,k=0}^{d-1} \binom{d-1}{i, j, k} \int_0^\infty t^k \iint g(t, b, x, x + tb + y + K) \\
&\qquad \times \mathbb{P}_{(x+tb+y,K)}(d_B(L, T(\Psi - \delta_{(x+tb+y,K)})) > t) \\
&\qquad \times f(x + tb + y, K)\Theta_{i,j;k+1}(L, \check{K}; B; d(x, y, b))\mathbb{Q}(dK)dt.
\end{aligned}$$

This theorem can be specified in various ways; compare [5]. In fact, the discussion in [5, pp. 835–842] of Gibbs processes, Cox processes and Poisson cluster processes can be transferred to the present more general framework. For instance, Theorem 3.4 is implied by Theorem 5.1, since for a Poisson process Ψ we have $\mathbb{P}_{(x,K)}(\Psi - \delta_{(x,K)} \in \cdot) = \mathbb{P}(\Psi \in \cdot)$ for Θ - almost all $(x, K) \in \mathbb{R}^d \times \mathcal{K}^d$.

As another immediate consequence of the previous general theorem, we obtain an extension of Theorem 4.16 in [5], where the case $L = \{x\}$ was considered.

Corollary 5.2. *Let Z be a random closed set in $\mathcal{S}^d$ such that $Z = T(\Psi)$ for a point process Ψ on $\mathbb{R}^d \times \mathcal{K}^d$ satisfying* (3.2), (3.3) *and* (3.5)*. Assume that $(\check{K}, B) \in \mathcal{K}^{d,2}_{gp}$ for $\mathbb{Q}$*

- *almost all* $K \in \mathcal{K}^d$. *Further, let* $g : [0,\infty] \times \partial B \to [0,\infty)$ *be measurable. Then*

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}\{0 < d_B(\{x\}, Z) < \infty\} g(d_B(\{x\}, Z), u_B(\{x\}, Z))\right] \\
&= \sum_{i=0}^{d-1} \binom{d-1}{i} \int_0^\infty t^{d-1-i} \iint g(t,b) \mathbb{P}_{(x+tb+y,K)}(d_B(\{x\}, T(\Psi \setminus \delta_{(x+tb+y,K)})) > t) \\
&\qquad \times f(x+tb+y, K) \Theta_{i;d-i}(\check{K}; B; d(y,b)) \mathbb{Q}(dK) dt.
\end{aligned}$$

The assumption that L, $\check{K}$ and B should be in general relative position is fulfilled, for instance, if B and L are strictly convex. In fact, this condition is satisfied generically in the following strong sense. Let $\mathbf{SO}(d)$ denote the group of proper rigid rotations of $\mathbb{R}^d$, and let ν be the Haar probability measure on $\mathbf{SO}(d)$.

Lemma 5.3. *Let* $L, B \in \mathcal{K}^d$ *be fixed. Then, for* $\nu \otimes \nu$ *- almost all* $(\rho_1, \rho_2) \in \mathbf{SO}(d) \times \mathbf{SO}(d)$, *the condition* $(\rho_1 B, \check{K}, \rho_2 L) \in \mathcal{K}^{d,3}_{gp}$ *is satisfied for* $\mathbb{Q}$ *- almost all* $K \in \mathcal{K}^d$.

Proof. The set

$$\{(K, \rho_1, \rho_2) \in \mathcal{K}^d \times \mathbf{SO}(d) \times \mathbf{SO}(d) : (\rho_1 B, \check{K}, \rho_2 L) \notin \mathcal{K}^{d,3}_{gp}\}$$

is Borel measurable. To see this, observe that

$$(K_1, K_2, K_3) \in \mathcal{K}^{d,3}_{gp}$$

if and only if

$$(K_1, K_2) \in \mathcal{K}^{d,2}_{gp} \qquad \text{and} \qquad (K_1 + K_2, K_3) \in \mathcal{K}^{d,2}_{gp}.$$

Moreover, $(K_1, K_2) \notin \mathcal{K}^{d,2}_{gp}$ if and only if K_1 and K_2 contain non-degenerate parallel segments lying in parallel and equally oriented supporting hyperplanes. Finally, one has to use that the set of all pairs $(K_1, K_2) \in (\mathcal{K}^d)^2$ such that K_1 and K_2 contain parallel segments of length greater or equal $1/m$ lying in parallel and equally oriented supporting hyperplanes is closed, for all $m \in \mathbb{N}$.

Hence the assertion follows by a repeated application of Theorem 2.3.10 in [17] and by means of Fubini's theorem. □

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