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ON THE CONVEX HULL OF SYMMETRIC STABLE PROCESSES

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ABSTRACT. Let $\alpha \in (1,2]$ and X be an $\mathbb{R}^d$-valued symmetric α-stable Lévy process starting at 0. We consider the closure S_t of the path described by X on the interval $[0,t]$ and its convex hull Z_t. The first result of this paper provides a formula for certain mean mixed volumes of Z_t and in particular for the expected first intrinsic volume of Z_t. The second result deals with the asymptotics of the expected volume of the stable sausage $Z_t + B$ (where B is an arbitrary convex body with interior points) as $t \to 0$. For this we assume that X has independent components.

1. Introduction and main results

For fixed $\alpha \in (1,2]$ and fixed integer $d \geq 1$ we consider an $\mathbb{R}^d$-valued symmetric α-stable Lévy process $X \equiv (X(t))_{t\geq 0}$ starting at the origin. If $\alpha = 2$ and the components of X are independent with variance 1, then X is a standard Brownian motion.

Let Z be the closure of the convex hull of $\{X(t) : 0 \leq t \leq 1\}$. A classical result of [16] for planar standard Brownian motion says that

$$\mathbb{E}V_1(Z) = \sqrt{2\pi}, \tag{1.1}$$

where $V_1(Z)$ denotes half the perimeter of Z. Our first aim in this paper is to formulate and to prove such a result for arbitrary $\alpha \in (1,2]$ and arbitrary dimension d. In fact we also consider more general geometric functionals; see Theorem 1.1. To the best of our knowledge our paper is the first dealing with the convex hull Z in the case $\alpha \in (1,2)$.

Let $B \subset \mathbb{R}^d$ be a *convex body*, that is a non-empty compact and convex subset of $\mathbb{R}^d$. Then the (random) set $\bigcup_{s\leq t}(X(s)+B)$ is called *stable sausage* up to time t. The asymptotic behaviour of the volume of this sausage as $t \to \infty$ has been studied intensively; cf. [15] for the case of Brownian motion and [5, 11] for the case of more general stable processes. The second aim of our paper is to start an investigation of the asymptotic behaviour as $t \to 0$; see Theorem 1.7. For this, we assume that the process X has independent components. In the case $\alpha = 2$ the stable sausage is known as *Wiener sausage*. Even then our Theorem 1.7 constitutes a new result; see Corollary 1.8.

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By definition, the process X has stationary independent increments, and the distribution of $X(t)$ is symmetric α-stable for each $t \geq 0$. By [12, Theorem 2.4.3] there is a symmetric finite measure Γ (unique in case $\alpha < 2$) on the unit sphere S^{d-1} in $\mathbb{R}^d$ such that

$$\mathbb{E}\exp[i\langle u, X(t)\rangle] = \exp\Big[-t^\alpha \int |\langle u, v\rangle|^\alpha \Gamma(dv)\Big], \quad u \in \mathbb{R}^d, t \geq 0, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. The measure Γ is the *spectral measure* of $X(1)$. The Minkowski inequality implies that $(\int |\langle u, v\rangle|^\alpha \Gamma(dv))^{1/\alpha}$ is a sublinear function of u. Moreover, this function is also positively homogeneous. Theorem 1.7.1 in [13] implies that there is a convex body K (uniquely determined by Γ) such that

$$\mathbb{E}\exp[i\langle u, X(t)\rangle] = \exp[-th(K, u)^\alpha], \quad u \in \mathbb{R}^d, t \geq 0, \tag{1.3}$$

where

$$h(K, u) := \sup\{\langle u, x\rangle : \ x \in K\}, \quad u \in \mathbb{R}^d,$$

is the *support function* of K. We remark that K is also determined uniquely by the distribution of $X(1)$ in the case $\alpha = 2$. The body K is symmetric and contains the origin 0. However, 0 does not need to be an interior point of K. The set K is called the *associated zonoid* of $X(1)$; see [10]. Zonoids form a subfamily of convex bodies that are obtained as limits (with respect to the Hausdorff distance) of Minkowski (or elementwise) sums of segments, where the *Minkowski sum* is defined as $B + C := \{x + y : x \in B, y \in C\}$ for two sets $B, C \subset \mathbb{R}^d$. The set K from (1.3) belongs to the family of L_p-zonoids with $p = \alpha$, which appear as limits for L_p-sums (or Firey sums) of segments; see [4, 10].

As in [10] it is useful to introduce the *polar set*

$$F := \{u \in \mathbb{R}^d : h(K, u) \leq 1\}$$

of the associated zonoid K. This set is convex, closed and origin-symmetric and satisfies $\|u\|_F = h(K, u)$ for all $u \in \mathbb{R}^d$, where

$$\|u\|_F := \inf\{s \geq 0 : \ u \in sF\}$$

is the Minkowski functional of F; see [13, Sec. 1.6,1.7] for more details in case 0 is an interior point of K. Therefore,

$$\mathbb{E}\exp[i\langle u, X(t)\rangle] = \exp[-t\|u\|_F^\alpha], \quad u \in \mathbb{R}^d, t \geq 0.$$

If $\alpha = 2$, then X is a Gaussian process and both F and K are ellipsoids. In particular, if F (and also K) is a Euclidean ball, then the components of X are independent Brownian motions, which become standard if the radius of F is $\sqrt{2}$.

Note that the process X is *self-similar* in the sense that $(X(st))_{s\geq 0} \overset{d}{=} t^{1/\alpha}X$ for any $t > 0$; see [12, Ex. 7.1.3] and [6, Ch. 15]. For $t \geq 0$, let S_t be the closure of the path $S_t^0 := \{X(s) : 0 \leq s \leq t\}$ and let Z_t denote the convex hull of S_t. In Lemma 2.1 we will show that these are random closed sets; see [9] or [14] for the notion of a random closed set. We abbreviate $Z := Z_1$. By self-similarity

$$Z_t \overset{d}{=} t^{1/\alpha} Z, \quad t > 0. \tag{1.4}$$

In this paper we study geometric functionals of the random convex set Z, namely intrinsic and mixed volumes. The *intrinsic volumes* $V_j(K)$, $j = 0, \ldots, d$, of a convex body K are the unique real numbers satisfying the *Steiner formula*

$$V_d(K + tB^d) = \sum_{j=0}^{d} \kappa_{d-j} t^{d-j} V_j(K), \quad t \geq 0, \tag{1.5}$$

where B^d is the closed Euclidean unit ball in $\mathbb{R}^d$ and κ_j is the j-dimensional volume of B^j. In particular, $V_d(K)$ is the volume of K (justifying the use of the same symbol V_d for the Lebesgue measure), $V_{d-1}(K)$ is half the surface area, $V_{d-2}(K)$ is proportional to the integrated mean curvature, $V_1(K)$ is proportional to the mean width of K, and $V_0(K) = 1$. (If $d = 2$, then the first intrinsic volume V_1 appears in (1.1).) The *mixed volumes* $V(K[j], L[d-j])$, $j = 0, \ldots, d$, of two convex bodies $K, L \subset \mathbb{R}^d$ are the unique real numbers such that

$$V_d(K + tL) = \sum_{j=0}^{d} \binom{d}{j} t^{d-j} V(K[j], L[d-j]), \quad t \geq 0. \tag{1.6}$$

Taking $L := B^d$ in (1.6) and comparing with (1.5) shows that the intrinsic volumes are special mixed volumes:

$$V_j(K) = \frac{\binom{d}{j}}{\kappa_{d-j}} V(K[j], B^d[d-j]), \quad j = 0, \ldots, d. \tag{1.7}$$

A geometric interpretation of the mixed volume $V(K[d-1], L[1])$ can be derived from (1.6):

$$V(K[d-1], L[1]) = \lim_{t \downarrow 0} t^{-1}(V_d(K + tL) - V_d(K)).$$

For more information on intrinsic and mixed volumes the reader is referred to [13].

Theorem 1.1. *Let B be a convex body in $\mathbb{R}^d$. Then*

$$\mathbb{E}V(B[d-1], Z[1]) = \frac{\alpha}{\pi}\Gamma\left(1 - \frac{1}{\alpha}\right) V(B[d-1], K[1]), \tag{1.8}$$

where Γ is the Gamma-function and K is the associated zonoid of $X(1)$. In particular,

$$\mathbb{E}V_1(Z) = \frac{\alpha}{\pi}\Gamma\left(1 - \frac{1}{\alpha}\right) V_1(K)\,.$$

Remark 1.2. By the scaling relation (1.4) and the homogeneity property of mixed volumes [13, Eq. (5.1.24)], the identity (1.8) can be generalized to

$$\mathbb{E}V(B[d-1], Z_t[1]) = \frac{\alpha}{\pi}\Gamma\left(1 - \frac{1}{\alpha}\right) t^{1/\alpha} V(B[d-1], K[1]).$$

A similar remark applies to all results of this paper.

Remark 1.3. The mixed volumes $V(K_1, \ldots, K_d)$ of d convex bodies $K_1, \ldots, K_d$ are defined as coefficients in the linear expansion of the volume of $\lambda_1 K_1 + \cdots + \lambda_d K_d$; see [13, Sec. 5.1]. Since these volumes are linear in each argument, Theorem 1.1 and all further results of this paper hold also for $V(K_1, \ldots, K_{d-1}, Z)$ with arbitrary convex bodies $K_1, \ldots, K_{d-1}$, so that (1.8) is recovered if $K_1 = \cdots = K_{d-1} = B$.

The proof of Theorem 1.1 relies on the fact that

$$\mathbb{E}h(Z, u) = \frac{\alpha}{\pi}\Gamma\left(1 - \frac{1}{\alpha}\right) h(K, u),$$

for all u from the unit sphere S^{d-1} in $\mathbb{R}^d$. Therefore the rescaled K is the *mean body* of Z [14, p. 146] or the *selection expectation* of Z [9, Thm. 2.1.22].

It is known [10, Ex. 3.2] that $X(1)$ has i.i.d. components if and only if $F = \sigma B_\alpha^d$, where $\sigma > 0$ is a scaling parameter and

$$B_\alpha^d := \{x \in \mathbb{R}^d : \ (|x_1|^\alpha + \cdots + |x_d|^\alpha) \le 1\} \tag{1.9}$$

is the ℓ_α-unit ball in $\mathbb{R}^d$. In this case the polar set K to F is $\sigma^{-1}B_{\alpha'}^d$, where $B_{\alpha'}^d$ is the $\ell_{\alpha'}$-ball in $\mathbb{R}^d$ with $1/\alpha + 1/\alpha' = 1$. For $\alpha = 2$ and K being the Euclidean ball of radius $1/\sqrt{2}$, the following corollary provides a direct generalization of (1.1).

Corollary 1.4. *If X is a standard Brownian motion in $\mathbb{R}^d$, then*

$$\mathbb{E}V_1(Z) = \frac{2\sqrt{2}\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}.$$

A stable random vector is called *subgaussian* if it appears as a product of a Gaussian random vector and a power of an independent positive stable random variable; see [12]. It is noted in [10] that subgaussian stable distributions are characterised by the fact that their associated zonoid K is an ellipsoid. In particular, if $F = K = B^d$, then the process $X(t)$ has identically distributed (but not independent for $\alpha \ne 2$) components and

$$\mathbb{E}V_1(Z) = \frac{d\alpha}{\pi}\Gamma\left(1-\frac{1}{\alpha}\right)\frac{\kappa_d}{\kappa_{d-1}} = \frac{2\alpha}{\sqrt{\pi}}\Gamma\left(1-\frac{1}{\alpha}\right)\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}.$$

The general subgaussian case is handled in the following lemma by reducing it to a formula involving the surface area of an ellipsoid.

Lemma 1.5. *Let K be an ellipsoid in $\mathbb{R}^d$ with semi-axes $c_1, \ldots, c_d$. Then*

$$V_1(K) = \frac{2c_1\cdots c_d}{d\kappa_{d-1}}V_{d-1}(F)\,,$$

where $V_{d-1}(F)$ is the half surface area of the polar ellipsoid F to K.

Expressions for $V_{d-1}(F)$ by means of elliptical integral are given in [2] and [17].

In the case of Brownian motion it is possible to calculate the expectation of the second intrinsic volume $V_2(Z)$ of Z.

Proposition 1.6. *Assume that X is a standard Brownian motion in $\mathbb{R}^d$. Then*

$$\mathbb{E}V_2(Z) = (d-1)\frac{\pi}{2}.$$

Now consider a fixed convex body B with non-empty interior and turn to the asymptotic behaviour of the volume of the stable sausage $S_t + B$ as $t \to 0$. For this, we assume that the process X has independent components, i.e. the associated zonoid of $X(1)$,

$$K = \{(c_1x_1, \ldots, c_dx_d) : \ (x_1, \ldots, x_d) \in B_{\alpha'}^d\}, \tag{1.10}$$

is a scaled $\ell_{\alpha'}$-ball $B_{\alpha'}^d$; see (1.9).

Theorem 1.7. *Let B be a convex body with non-empty interior. If X is a symmetric α-stable Lévy process with independent components and the associated zonoid K is given by* (1.10), *then*

$$\lim_{t\to 0} t^{-1/\alpha}(\mathbb{E}V_d(S_t + B) - V_d(B)) = \frac{d\alpha}{\pi}\Gamma\left(1-\frac{1}{\alpha}\right)V(B[d-1], K[1]).$$

Theorem 1.7 also holds for processes obtained as a linear transformation of a symmetric α-stable Lévy process with independent components. Even in the case of a Wiener sausage, Theorem 1.7 yields a new result.

Corollary 1.8. *Assume that X is a Brownian motion and let B be a convex body with non-empty interior. Then*

$$\lim_{t\to 0} t^{-1/2}(\mathbb{E}V_d(S_t+B)-V_d(B)) = \frac{d\sqrt{2}}{\sqrt{\pi}} V(B[d-1], B^d[1]).$$

If $B = B^d$, the limit is $2\sqrt{2}\pi^{(d-1)/2}/\Gamma(d/2)$.

Remark 1.9. In the special case $d = 3$ and $\alpha = 2$ the classical result from [15] yields that

$$\mathbb{E}\, V_3(S_t + rB^3) = \frac{4}{3}\pi r^3 + 4\sqrt{2\pi} r^2\sqrt{t} + 2\pi r t \tag{1.11}$$

for any $r, t \geq 0$. The constant term in t can be interpreted geometrically as $V_3(rB^3)$. A comparison with Corollary 1.8 yields that the coefficient of $\sqrt{t}$ can be interpreted as

$$4\sqrt{2\pi} r^2 = 3\mathbb{E}V(rB^3[2], Z[1]) = r^2\kappa_2 \mathbb{E}V_1(Z)\,,$$

where we have used (1.7) and the homogeneity property of mixed volumes; see e.g. [13, Eq. (5.1.24)].

2. Proofs

We need the following measurability property of the closure S_t of $\{X(s) : 0 \leq s \leq t\}$ and its convex hull Z_t, referring to [9, 14] for the notion of a *random closed set*.

Lemma 2.1. *For any $t \geq 0$, S_t and Z_t are random closed sets.*

Proof. To prove the first assertion it suffices to show that $\{S_t \cap G = \emptyset\}$ is measurable for any open $G \subset \mathbb{R}^d$; see [14, Lemma 2.1.1]. Since X is right continuous with left limits, it is clear that $S_t \cap G = \emptyset$ if and only if $X(u) \notin G$ for all rational numbers $u \leq t$. The second assertion is implied by [14, Thms. 12.3.5, 12.3.2]. □

Lemma 2.1 implies that $V(B[d-1], Z_t[1])$ and $V_d(S_t + B)$ are random variables; see e.g. [13, p. 275] and [14, Thms. 12.3.5, 12.3.6].

Proof of Theorem 1.1. By [13, Eq. (5.1.18) and p. 277],

$$V(B[d-1], K[1]) = \frac{1}{d}\int_{S^{d-1}} h(K,u)\, S_{d-1}(B, du) \tag{2.1}$$

for all convex bodies $B, K \subset \mathbb{R}^d$, where $S_{d-1}(B,\cdot)$ is the *surface area measure* of B; see [13, Sec. 4.2]. Fubini's theorem and (2.1) yield that

$$\mathbb{E}V(B[d-1], Z[1]) = \frac{1}{d}\int_{S^{d-1}} \mathbb{E}h(Z,u)\, S_{d-1}(B, du).$$

For any $u \in S^{d-1}$,

$$\mathbb{E}h(Z,u) = \mathbb{E}\sup\{\langle x, u\rangle : x \in Z_1\} = \mathbb{E}\sup\{\langle X(s), u\rangle : s \in [0,1]\}.$$

It follows from the definition of α-stability that the one-dimensional Lévy process $Y = \langle X, u\rangle$ is symmetric α-stable. By [1, Thm. 4a], $\sup\{Y(s) : s \in [0,1]\}$ has a finite expectation. Differentiating equation (7b) in [1] (Spitzer's identity in continuous time), one can easily show that

$$\mathbb{E}\sup\{Y(s) : s \in [0,1]\} = \alpha \mathbb{E} Y(1)^+ = \alpha \mathbb{E}(\langle u, X(1)\rangle)^+,$$

where $a^+ := \max\{0, a\}$ denotes the positive part of a real number a. It is shown in [10, Sec. 6.4] that $\mathbb{E}(\langle u, X(1)\rangle)^+ = h(M, u)$ is the support function of a scaled variant of the associated zonoid K, namely

$$M = \frac{1}{\pi}\Gamma\left(1 - \frac{1}{\alpha}\right) K\,.$$

Together with (2.1), this yields assertion (1.8). □

Proof of Corollary 1.4. Since $K = B^d/\sqrt{2}$, Theorem 1.1 and (1.7) give

$$\begin{aligned}\mathbb{E}V_1(Z) &= \mathbb{E}\frac{d}{\kappa_{d-1}} V(B^d[d-1], Z[1]) \\ &= \frac{d}{\kappa_{d-1}}\,\frac{2}{\pi}\Gamma(1-\frac{1}{2}) V(B^d[d-1], \frac{B^d}{\sqrt{2}}[1]) \\ &= \sqrt{\frac{2}{\pi}}\frac{d\kappa_d}{\kappa_{d-1}}\,,\end{aligned}$$

where we have used the fact that $V(B^d[d-1], B^d[1]) = V_d(B^d)$. The well-known formula $\kappa_d = \pi^{d/2}/\Gamma(d/2+1)$ and the obvious identity $d/\Gamma(d/2+1) = 2/\Gamma(d/2)$ yield the result. □

Proof of Lemma 1.5. Without loss of generality assume that

$$K = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \left(\frac{x_1}{c_1}\right)^2 + \cdots + \left(\frac{x_d}{c_d}\right)^2 \le 1\}.$$

Consider the linear transform $x \mapsto f(x) = (c_1^{-1}x_1, \ldots, c_d^{-1}x_d)$. Then $f(K) = B^d$, while the affine equivariance of the mixed volumes and the fact that the determinant of f is $(c_1 \cdots c_d)^{-1}$ yield that

$$\begin{aligned}V_1(K) &= \frac{d}{\kappa_{d-1}} V(B^d[d-1], K[1]) \\ &= \frac{dc_1\cdots c_d}{\kappa_{d-1}} V(f(B^d)[d-1], f(K)[1]) \\ &= \frac{dc_1\cdots c_d}{\kappa_{d-1}} V(B^d[1], F[d-1]) \\ &= \frac{dc_1\cdots c_d}{\kappa_{d-1}}\frac{\kappa_1}{d} V_{d-1}(F)\,,\end{aligned}$$

where $F = f(B^d)$ is the ellipsoid with semi-axes $c_1^{-1}, \ldots, c_d^{-1}$, namely the polar body to K. It remains to note that $\kappa_1 = 2$. □

Proof of Proposition 1.6. Kubota's formula (see e.g. [13, Eq. (5.3.27)]) yields

$$V_2(Z) = \frac{d(d-1)\kappa_d}{2\kappa_2\kappa_{d-2}} \int_{G_2} V_2(Z|L)\,\nu_2(dL),$$

where G_2 denotes the set of all 2-dimensional linear subspaces of $\mathbb{R}^d$, ν_2 is the Haar measure on G_2 with $\nu_2(G_2) = 1$ and $Z|L$ denotes the image of Z under the orthogonal projection onto the linear subspace L. By Fubini's theorem,

$$\mathbb{E}V_2(Z) = \frac{d(d-1)\kappa_d}{2\kappa_2\kappa_{d-2}} \int_{G_2} \mathbb{E}V_2(Z|L)\,\nu_2(dL).$$

The spherical symmetry of Brownian motion implies that $\mathbb{E}V_2(Z|L)$ does not depend on L. Assume that $L = \{(x_1, x_2, 0, \dots, 0) : x_1, x_2 \in \mathbb{R}\}$. Now it is clear from the definition of the d-dimensional Brownian motion that the random closed set $Z|L$ is the convex hull of a Brownian path in L. By Remark (a) in [3, p. 149] (see also [8]) we have $\mathbb{E}V_2(Z|L) = \pi/2$. Therefore,

$$\mathbb{E}V_2(Z) = \frac{d(d-1)\kappa_d}{2\kappa_2\kappa_{d-2}} \frac{\pi}{2},$$

and the result follows by a straightforward calculation. □

Proof of Theorem 1.7. Note that a simple rescaling argument makes it possible to assume that $K = B^d_{\alpha'}$ is the unit $\ell_{\alpha'}$-ball. By self-similarity and the dominated convergence theorem we have

$$\begin{aligned}\lim_{t\to 0} t^{-1/\alpha}(\mathbb{E}V_d(S_t + B) - V_d(B)) &= \lim_{t\to 0} t^{-1/\alpha}(\mathbb{E}V_d(t^{1/\alpha}S_1 + B) - V_d(B)) \\ &= \lim_{t\to 0} t^{-1}(\mathbb{E}V_d(tS_1 + B) - V_d(B)) \\ &= \mathbb{E}\lim_{t\to 0} t^{-1}(V_d(tS_1 + B) - V_d(B)). \qquad (2.2)\end{aligned}$$

In order to justify the application of the dominated convergence theorem, define

$$\begin{aligned}Y_j &= \sup\{X_j(s) : s \in [0,1]\}, \\ \tilde{Y}_j &= \inf\{X_j(s) : s \in [0,1]\}, \quad j = 1, \dots, d.\end{aligned}$$

As noted in the proof of Theorem 1.1, Y_j has a finite expectation. Since $-\tilde{Y}_j$ has the same distribution as Y_j, $\tilde{Y}_j$ also has a finite expectation. From (1.6) we obtain for all $t \in (0,1]$ that

$$\begin{aligned}t^{-1}V_d(tS_1 + B) - V_d(B) &\le t^{-1}(V_d(tZ + B) - V_d(B)) \\ &= \sum_{j=0}^{d-1} \binom{d}{j} t^{d-j-1} V(B[j], Z[d-j]) \\ &\le \sum_{j=0}^{d} \binom{d}{j} V(B[j], Z[d-j]) \\ &= V_d(Z + B).\end{aligned}$$

Furthermore,

$$Z + B \subset \mathop{\times}_{j=1}^{d} [\tilde{Y}_j - h_B(-e_j), Y_j + h_B(e_j)],$$

where e_j denotes the jth unit vector. Thus

$$t^{-1}(V_d(tS_1 + B) - V_d(B)) \le \prod_{j=1}^{d} \Big(Y_j + h_B(e_j) - \tilde{Y}_j + h_B(-e_j) \Big)$$

for $t \in (0, 1]$. This is a product of integrable independent random variables and hence has finite expected value.

By [7, Cor. 3.2(2)],

$$\lim_{t \to 0} t^{-1}(V_d(tS_1 + B) - V_d(B)) = \int_{S^{d-1}} h(Z, u)\, S_{d-1}(B, du),$$

and using Theorem 1.1 and (2.1) we conclude from (2.2) that

$$\begin{aligned} \lim_{t \to 0} t^{-1/\alpha} \mathbb{E}(V_d(S_t + B) - V_d(B)) &= \mathbb{E} \int_{S^{d-1}} h(Z, u)\, S_{d-1}(B, du) \\ &= d\mathbb{E} V(B[d-1], Z[1]) \\ &= d \frac{\alpha}{\pi} \Gamma\left(1 - \frac{1}{\alpha}\right) V(B[d-1], K[1]). \end{aligned}$$

□

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