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On a Class of Lévy Stochastic Networks *

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Abstract. We consider a Lévy stochastic network as a regulated multidimensional Lévy process. The reflection direction is constant on each boundary of the positive orthant and the corresponding reflection matrix corresponds to a single-class network. We use the representation of the Lévy process and Itô's formula to arrive at some equations for the steady-state process; the latter is shown to exist, under natural stability conditions. We specialize first to the class of Lévy processes with non-negative jumps and then add the assumption of self-similarity. We show that the stationary distribution of the network corresponding the the latter process does not has product form (except in trivial cases). Finally, we derive asymptotic bounds for two-dimensional Lévy stochastic network.

Keywords: Lévy process, Skorokhod reflection, stochastic network, Itô's formula, Loynes' scheme

1. Introduction

During the past decade, several advances have been made in understanding traffic models in broadband high-speed communication networks. Detailed measurements, statistical analyses, simulation, and theoretical results point out that traffic exhibits certain unconventional features: self-similarity, heavy-tails, and long-range dependence [5,6,8,26]. Evidently, traditional stochastic models do not suffice, for they do not capture all the above characteristics, as far as description of traffic at the packet level is concerned. (At the session level, conventional models, such as Poisson processes may suffice.) The search and justification of reasonable stochastic models have thus far led to two basic models that can be used, depending on the situation. The first is a fractional Brownian motion, and the second a stable-Lévy process. A heuristic explanation of how these models arise is along these lines: consider communication sessions arriving at a com-

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munication node according to a Poisson process. Each session transmits data at, say, a constant rate, and its duration is a heavy-tailed random variable. The cumulative traffic A_t on an interval of length t is a random process that exhibits long-range dependence. Now, look at the model at a different time scale. Assume that sessions appear faster and faster (multiply the Poisson rate by a large integer n) and rescale the centered cumulative process appropriately. Then, in the limit as $n \to \infty$, one obtains a stable Lévy motion. If, however, sessions are scaled even faster, then a (different) rescaling converges to a fractional Brownian motion. The first observation was made in [25]. A rigorous comparison of the two limiting regimes can be found in the recent paper [29]. Other papers supporting theoretical results on the obtaining of limits are [13,30,34,35,39,41]. Different limits, other than Lévy and fractional Brownian motion, are also possible; see, e.g., [14,27].

Several researchers have used some of the aforementioned models in building and analyzing stochastic models of queues and networks. For instance, in [31] a queue driven by a fractional Brownian motion is studied, and in [24] a network driven by fractional Brownian motions is described as a limit of single-class queueing networks with heavy-tailed renewal arrival processes. Of course, Brownian networks, i.e., networks driven by Brownian motions have been studied extensively, both as heavy-traffic limits [33] but also as models per se [10–12,40]. In this paper, we focus on one of the above models, namely the Lévy process, in order to describe and explain some properties of stochastic networks driven by such processes. Lévy processes form a large class of stochastic models that include compound Poisson processes (possibly heavy-tailed), and Brownian motion. The use of Lévy processes in applied probability is not new. Traditional use of such models has found applications in storage and dam applications (see [32, and references therein]). More recently, Kella and co-workers (see, e.g., [17–21]) have studied several properties of classes of networks driven by Lévy processes. Here, we consider a Lévy process X in d dimensions and define a stochastic network by means of reflecting X on the boundary of the positive orthant $\mathbb{R}^d_+$ along a reflection matrix R that can be written as $I - P^{\mathrm{T}}$, where I is the identity matrix, and P is a strictly substochastic matrix. It is possible to prove that such a stochastic model is the limit, in a certain sense, of a simpler network driven by processes of the type $\{A_t\}$ described in the beginning of this section. We shall not prove such limit theorems here. Rather, we will deal with the reflected Lévy process as a model per se. A rigorous formulation of the model can be done pathwise, using the Skorokhod reflection mapping. This is described in section 2. It should be noted that, throughout, we work with a multi-dimensional Lévy process X with values in $\mathbb{R}^d$, whose coordinates $X^1, \ldots, X^d$, may be dependent processes. This allows modeling networks with correlated inputs. In section 3 we assume that the Lévy process X has finite first moment and give a proof for the existence of a stationary reflected process, under the natural stability condition $R^{-1}\mu < 0$ (vector inequalities are always interpreted componentwise). The stationarity proof for the case of Brownian networks is given by Harrison and Williams [12]; we generalize the proof to the case of Lévy networks, taking into account the discontinuities of the processes. In section 4, we start from the Itô representation of X as a stochastic integral against a Poisson random measure in $1 + d$ dimensions, and derive a "balance equation" for functionals of the reflected process (section 4.1). We move from the most general case to the specific one we are interested in. This is a spectrally positive Lévy process in the sense that its coordinate processes X^i have only nonnegative jumps, i.e., $X^i_t - X^i_{t-} \geqslant 0$ for all i, t, a.s. The reason for this assumption is that it is supported by the limit theorems that we have in mind [25]. Equivalently, a spectrally positive Lévy process in d dimensions is one whose Lévy measure (always denoted by ν in this paper) is supported on the positive orthant $\mathbb{R}^d_+$ (section 4.2). This model, for special topology networks, has been studied by Kella [16]. A key quantity that plays a role in the remaining of the paper is the "boundary measure" $\beta_j(A)$, which is defined as the expected amount of time spent by the reflected process Z in the Borel set A, when time runs according to the jth reflector L^j. Since, by the Skorokhod reflection problem, L^j is supported only on the face F_j defined by $\{x \in \mathbb{R}^d_+ \colon x^j = 0\}$, it follows that β_j is also supported on the same face. An interesting special case, which is rather well-known, is that of a single queue with Lévy input. We briefly summarize the solution of the balance equation for this case in section 4.3. In section 4.4 we add the assumption that the Lévy process is *(weakly) α-stable* for some $\alpha \in (1, 2)$. By this we mean the following: a Lévy process with finite mean can be written as $X_t = \mu t + \sigma B_t + M_t$ (see (5)), where $\mu t = \mathbb{E}X_t$, σB_t is a Brownian motion in $\mathbb{R}^d$, and M_t is a zero-mean martingale (we refer to it as the *centered jumppart* of X) which we require to be *α-self-similar* in the sense that $\{M_{\lambda t}\} \stackrel{d}{=} \{\lambda^{1/\alpha} M_t\}$, for all $\lambda > 0$. Again, this assumption is supported by limit theorems [25,29]. It turns out that the Lévy measure ν is not only supported on $\mathbb{R}^d_+$ but is also defined by its values on the the unit sphere (a simple consequence of the scaling property). Thus, the model considered here, and in the following section, is a multi-dimensional generalization of an α-stable spectrally positive Lévy motion [1]. In section 5 we ask the question of whether the distribution of the stationary reflected process Z_t, at some (and hence any) t has independent components $Z^1_t, \ldots, Z^d_t$. This question has been asked several times before, starting from traditional queueing networks [22,38], to Brownian networks [12,40] and special classes of Lévy networks [17,19]. Several classes of classical queueing networks possessing this property have been found. For Brownian networks, it has been shown [12] that product-form exists if and only if the covariance matrix of the Brownian motion relates to the transition probability matrix P in a special way. For Lévy networks, the answer is always negative. This is still the case for our model too, and this is shown in section 5. In section 6 we derive asymptotic bounds for the tail of the stationary distribution for a 2-dimensional Lévy network. Finally, some open problems are mentioned in section 7.

2. A Lévy stochastic network

Let ν be a Lévy measure on $\mathbb{R}^d$, i.e., a Borel measure satisfying

$$\nu(\{0\}) = 0, \qquad \int_{\mathbb{R}^d} |x|^2 \wedge 1 \; \nu(\mathrm{d}x) < \infty.$$

Here, and throughout the paper, $|x|$ denotes the Euclidean norm of of $x = (x^1, \dots, x^d) \in \mathbb{R}^d$. We consider a Lévy process $\{X_t = (X_t^1, \dots, X_t^d),\ t \geqslant 0\}$, with characteristic function

$$\mathbb{E}e^{i\theta^{\mathrm{T}}X_t} = \exp\left\{t\left[i\theta^{\mathrm{T}}a - \frac{1}{2}\theta^{\mathrm{T}}C\theta + \int_{\mathbb{R}^d}\left(e^{i\theta^{\mathrm{T}}x} - 1 - i\theta^{\mathrm{T}}x\mathbb{1}\{|x| \leqslant 1\}\right)\nu(\mathrm{d}x)\right]\right\}, \quad \theta \in \mathbb{R}^d, \tag{1}$$

where $a \in \mathbb{R}^d$, and C a $d \times d$ symmetric positive semidefinite matrix. This is of course the famous Lévy–Khinchine formula. (See [1,36] for background on Lévy processes.) We denote by $x^{\mathrm{T}}y$ the usual inner product in $\mathbb{R}^d$. All vectors are assumed to be column vectors, and x^{T} denotes the transpose of a column, whereas A^{T} denotes the transpose of a matrix. The process X admits a càdlàg version, and this will be assumed throughout. It is known that X can be constructed by means of Poisson random measure as follows: let η be a Poisson random measure on $\mathbb{R} \times \mathbb{R}^d$ with intensity measure

$$\mathbb{E}\eta(\mathrm{d}t, \mathrm{d}x) = \mathrm{d}t\,\nu(\mathrm{d}x),$$

let B be a standard d-dimensional Brownian motion. Then

$$X_t = at + \sigma B_t + \int_0^t\int_{|x|\leqslant 1} x\left(\eta(\mathrm{d}u, \mathrm{d}x) - \mathrm{d}u\,\nu(\mathrm{d}x)\right) + \int_0^t\int_{|x|>1} x\eta(\mathrm{d}u, \mathrm{d}x), \tag{2}$$

where σ is a matrix such that $C = \sigma\sigma^{\mathrm{T}}$. Here and in what follows, we use the convention $\int_s^t = \int_{(s,t]}$. Note that the first integral in the above display converges because $\int_{|x|\leqslant 1} x^2\nu(\mathrm{d}x) < \infty$, while the last integral is actually a sum of a finite (but random) number of terms, due to $\nu(\{x \in \mathbb{R}^d\colon\ |x| > 1\}) < \infty$.

This representation of the Lévy process in terms of a Brownian motion B and a Poisson random measure η can be found, e.g., in [15]; it is one of its possible realizations of a general Lévy process X with values in $\mathbb{R}^d$. The process X is a semimartingale with respect to the filtration generated by the restriction of η to $[0, t] \times \mathbb{R}^d$ and $\{B_s\colon\ 0 \leqslant s \leqslant t\}$. In fact, the process

$$M_t = \int_0^t\int_{|x|\leqslant 1} x\left(\eta(\mathrm{d}u, \mathrm{d}x) - \mathrm{d}u\,\nu(\mathrm{d}x)\right), \quad t \geqslant 0, \tag{3}$$

is a martingale while

$$K_t = \int_0^t\int_{|x|>1} x\eta(\mathrm{d}u, \mathrm{d}x) \tag{4}$$

is a compound Poisson process with piecewise constant sample paths.

The semimartingale X is *special* [15] iff $\int \mathbb{1}\{|x| > 1\}|x|\nu(\mathrm{d}x) < \infty$. In this case X_t is integrable. Setting

$$\mu := \mathbb{E}X_1 = a + \int_{|x|>1} x\nu(\mathrm{d}x),$$

we can then write

$$X_t = \mu t + \sigma B_t + \int_0^t \int_{\mathbb{R}} x\big(\eta(\mathrm{d}u, \mathrm{d}x) - \mathrm{d}u\,\nu(\mathrm{d}x)\big). \tag{5}$$

If X is a special Lévy process as in (5), then we refer to $M_t = X_t - \mu t - \sigma B_t$ as the *centered jump part* of X.

We now define a Lévy network, by analogy to a Brownian network [12]. Such stochastic networks have been considered in the work of Kella and co-workers [17–21]. They can be seen to be heavy-traffic type of approximations of networks whose input is long-range dependent, in the spirit of [25]. In this paper, we shall not deal with formulating or proving such limit theorems. Rather, we work directly with the concept of a Lévy network.

Consider a communication or processing network consisting of d nodes. Each node has a server that releases bits in a fluid-like fashion. We are given a $d \times d$ transitive substochastic matrix P, i.e., a nonnegative matrix with entries p_{ij} such that $\sum_j p_{ij} \leqslant 1$, for all i, whose spectral radius is strictly less than one. The element p_{ij} represents a routing probability from node i to node j. The ith component X_t^i of the Lévy process "represents" or, rather, approximates the total traffic entering the node from the outside world, plus traffic from other nodes, minus traffic leaving the node (provided the servers work at full capacity). It is then well-known that the amount of bits residing in node i at time t is given by Z_t^i, where the vector $Z_t = (Z_t^1, \ldots, Z_t^d)$ satisfies the following Skorokhod reflection problem [11,12,23] on the positive orthant $\mathbb{R}_+^d = \{x \in \mathbb{R}^d\colon x^i \geqslant 0,\ i = 1, \ldots, d\}$. We let

$$F_i := \big\{x \in \mathbb{R}_+^d\colon x_i = 0\big\}, \quad i = 1, \ldots, d,$$

be the closed $(d-1)$-dimensional faces of $\mathbb{R}_+^d$. Let $r^1, \ldots, r^d$ be the columns of the matrix

$$R = I - P^{\mathrm{T}}, \tag{6}$$

where I is the $d \times d$ identity matrix. These are reflection vectors associated with the faces $F_1, \ldots, F_d$, respectively. We then require that

$$Z_t = Z_0 + X_t + \sum_{i=1}^d r_i L_t^i, \quad t \geqslant 0, \tag{7}$$

where Z_0 is a given initial random variable with values in $\mathbb{R}_+^d$, and L^i, $i = 1, \ldots, d$, are $\mathbb{R}_+$-valued increasing processes, starting from 0, such that

$$\sum_{i=1}^d \int_0^\infty \mathbb{1}\big(Z_s^i > 0\big)\,\mathrm{d}L_s^i = 0. \tag{8}$$

The process Z may also be called "regulated Lévy process". The following theorem ensures that, under our assumptions on the matrix R, the regulated Lévy process is uniquely defined path-by-path.

Theorem 1. Let P be a substochastic matrix with spectral radius strictly less than one. Let $R := I - P^{\mathrm{T}}$. Let $x : \mathbb{R}_+ \to \mathbb{R}^d$ be continuous on the right with left limits. Then there is a uniquely defined $\ell_t = (\ell_t^1, \ldots, \ell_t^d)$, where, for all $i = 1, \ldots, d$, $\ell^i : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing with $\ell_0^i = 0$, such that

$$z_t = x_t + R\ell_t$$

lies in $\mathbb{R}_+^d$ for all t, and

$$\sum_{i=1}^{d} \int_0^\infty \mathbb{1}\big(z_s^i > 0\big)\, \mathrm{d}\ell_s^i = 0.$$

This theorem is proved in [11,33] for continuous x (see also [4]) and generalized in [23] for càdlàg x.

There is a crucial characterization of ℓ that will be used in the proof of theorem 3 below. Let $\mathfrak{I}$ be the set of increasing càdlàg functions $\ell : \mathbb{R}_+^d \to \mathbb{R}_+$ such that $\ell(0) = 0$. Given x and R as above, let

$$\mathfrak{L}(x, R) := \{\ell \in \mathfrak{I}\colon\ x_t + R\ell_t \geqslant 0 \text{ for all } t \geqslant 0\}.$$

The function ℓ of theorem 1 is clearly an element of $\mathfrak{L}(x, R)$. But more is true:

Proposition 1. Equip $\mathfrak{L}(x, R)$ with the natural pointwise partial ordering. Under this ordering, the function ℓ is its unique minimal element.

The proof of a special case of this proposition (that is, when x is continuous) can be found in [33], and is based on a theorem of George Birkhoff. Without the continuity assumption, we can find a somewhat more direct proof that does not use Birkhoff's theorem in [23].

There is a much simpler version of the above, formulated for vectors instead of functions, and this we call the static reflection problem or the problem of oblique reflection. It says that:

Theorem 2. Let R be as above. Then for any $x \in \mathbb{R}^d$ there is a unique $\ell \in \mathbb{R}_+^d$, such that

$$z := x + R\ell \in \mathbb{R}_+^d, \qquad \sum_{i=1}^{d} z^i \ell^i = 0.$$

We shall denote the mapping $x \mapsto z$ defined by this theorem by the symbol

$$z = \pi_R(x).$$

This is used just once in the formulation of the balance equations in section 4.

We just sketch the proof of this for completeness. For $d = 1$, the theorem is trivial: If $x \geqslant 0$, choose $\ell = 0$ and so $z = x$; if $x < 0$, choose $\ell = -x$, and so $z = 0$. Hence, for $d = 1$, we have $\ell = -(x \wedge 0)$. If $d > 1$, but $P = 0$, then the problem reduces to d one-dimensional problems; still, we can write $\ell = -(x \wedge 0)$, where the operation $\wedge$ is performed component-wise. For general P, we note that ℓ must satisfy

$$\ell = -\big[\big(x + P^{\mathrm{T}}\ell\big) \wedge 0\big].$$

But the mapping $\varphi(\ell) := -[(x + P^{\mathrm{T}}\ell) \wedge 0]$, maps $\mathbb{R}_+^d$ into itself and is a contraction. Hence ℓ is obtained as the unique fixed point of this contraction, and so $z = x + R\ell$.

3. Stationarity of the Lévy network

Constructing a stationary process Z satisfying (7) and (8), for a given Lévy process X, is our next concern. This is possible under the condition $R^{-1}\mu < 0$, which is the natural stability condition for the stochastic network. This is the content of the next theorem.

Theorem 3. Suppose that X is a Lévy process in $\mathbb{R}^d$ with finite mean, and let $\mu \in \mathbb{R}^d$ satisfy

$$\mathbb{E}X_t = \mu t.$$

Suppose also that R is a reflection matrix, as in (6), such that

$$R^{-1}\mu < 0. \tag{9}$$

Then there exists a stationary process Z satisfying (7) and (8).

This theorem is proved, under more general conditions indeed, in [21]. An alternative proof, based on the ideas of Harrison and Williams [12], is presented below.

Proof. Consider (7) with $Z_0 = 0$. That is,

$$Z_t = X_t + RL_t, \quad t \geqslant 0.$$

By definition, $X_0 = 0$, $L_0 = 0$. Let C_t be the event that X is continuous at t. We have $\mathbb{P}(C_t) = 1$, for all t. We first show that the probability

$$p(t, z) := \mathbb{P}(Z_t \leqslant z)$$

has a limit as $t \to \infty$, where inequalities are interpreted componentwise. To do so, we argue as in [12]. Since R^{-1} is a non-negative matrix,

$$p(t, z) = \mathbb{P}\big(L_t \leqslant R^{-1}z - R^{-1}X_t\big).$$

Now use the minimality characterization of L of proposition 1 to write this as

$$\begin{aligned} p(t, z) &= \mathbb{P}\big(\exists \ell \in \mathfrak{L}(X, R)\colon\ \ell_t \leqslant R^{-1}z - R^{-1}X_t\big) \\ &= \mathbb{P}\big(\exists \ell \in \mathfrak{I}\colon\ X_s + R\ell_s \geqslant 0 \text{ for all } s \in [0, t];\ \ \ell_t \leqslant R^{-1}z - R^{-1}X_t\big). \end{aligned}$$

It is easy to see that the last inequality can be replaced by equality, so that

$$p(t,z) = \mathbb{P}\big(\exists \ell \in \mathfrak{I}\colon\ X_s + R\ell_s \geqslant 0 \text{ for all } s \in [0,t];\ \ X_t + R\ell_t = z\big).$$

Let $\mathfrak{I}_t$ be the set of all $\ell \in \mathfrak{I}$ such that ℓ is continuous at t. By intersecting with the a.s. event C_t, and by noting that continuity of X at t implies continuity of L at t, we may write

$$p(t,z) = \mathbb{P}\big(\exists \ell \in \mathfrak{I}_t\colon\ X_s + R\ell_s \geqslant 0 \text{ for all } s \in [0,t];\ \ X_t + R\ell_t = z\big).$$

The next step is a time reversal on $[0,t]$. Consider that transformation Φ_t, taking càdlàg functions into càdlàg functions, defined by

$$\Phi_t f(s) := \begin{cases} f(t) - f\big((t-s)-\big) & 0 \leqslant s < t, \\ f(s) & s \geqslant t. \end{cases}$$

Note that Φ_t is a bijection on $\mathfrak{I}_t$. Indeed, $\Phi_t \ell$ is increasing for an $\ell \in \mathfrak{I}_t$; the fact that an $\ell \in \mathfrak{I}_t$ is 0 at 0 implies that $\Phi_t \ell$ is continuous at t; similarly, continuity of ℓ at t implies $\Phi_t \ell(0) = 0$. Also note that $\Phi_t X$ is a Lévy process, identical in distribution to X. Hence

$$\begin{aligned} p(t,z) &= \mathbb{P}\big(\exists \ell \in \mathfrak{I}_t\colon\ \Phi_t X_s + R\Phi_t \ell_s \geqslant 0 \text{ for all } s \in [0,t];\ \ \Phi_t X_t + R\Phi_t \ell_t = z\big) \\ &= \mathbb{P}\big(\exists \ell \in \mathfrak{I}_t\colon\ X_t - X_{(t-s)-} + R\ell_t - R\ell_{(t-s)-} \geqslant 0 \text{ for all } s \in [0,t]; \\ &\qquad X_t + R\ell_t = z\big) \\ &= \mathbb{P}\big(\exists \ell \in \mathfrak{I}_t\colon\ X_{u-} + R\ell_{u-} \leqslant z \text{ for all } u \in [0,t];\ \ X_t + R\ell_t \leqslant z\big). \end{aligned}$$

Note that we interchanged again the inequality and equality in the last condition $X_t + R\ell_t \leqslant z$. Since $\mathbb{P}(C_t) = 1$, we can write the above as

$$p(t,z) = \mathbb{P}\big(\exists \ell \in \mathfrak{I}\colon\ X_s + R\ell_s \leqslant z \text{ for all } s \in [0,t]\big).$$

Thus $p(t,z)$ decreases in t:

$$p(t,z) = \mathbb{P}(Z_t \leqslant z) \downarrow p(\infty,z) := \mathbb{P}\big(\exists \ell \in \mathfrak{I}\colon\ X_s + R\ell_s \leqslant z \text{ for all } s \geqslant 0\big).$$

We need to show that $p(\infty,z)$ is an honest distribution. Since, by assumption, $R^{-1}\mu < 0$, the function

$$\hat{\ell}_t := -\frac{t}{2} R^{-1}\mu, \quad t \geqslant 0,$$

is a member of the set $\mathfrak{I}$; and so we have

$$\begin{aligned} p(\infty,z) &\geqslant \mathbb{P}\big(X_s + R\hat{\ell}_s \leqslant z \text{ for all } s \geqslant 0\big) \\ &= \mathbb{P}\left(R^{-1}X_s - \frac{s}{2}R^{-1}\mu \leqslant R^{-1}z \text{ for all } s \geqslant 0\right), \end{aligned} \tag{10}$$

where we obtained the latter by multiplying by the non-negative matrix R^{-1} throughout. Note that

$$R^{-1}X_s - \frac{s}{2}R^{-1}\mu, \quad s \geqslant 0,$$

is a Lévy process with mean $\mathbb{E}R^{-1}X_s - (s/2)R^{-1}\mu = sR^{-1}\mu - (s/2)R^{-1}\mu = (s/2)R^{-1}\mu < 0$. Hence the random variable

$$W^* := \sup_{s \geqslant 0} \left\{ R^{-1}X_s - \frac{s}{2}R^{-1}\mu \right\}, \tag{11}$$

where the supremum is taken componentwise, is a.s. finite. From (10) and (11) we have

$$p(\infty, z) \geqslant \mathbb{P}\big(W^* \leqslant z\big),$$

and so $\lim_{z\to\infty} p(\infty, z) = 1$. Next we recall the Skorokhod reflection problem – see (7), (8) – and observe that, for Z_0 any given element of $\mathbb{R}^d_+$, the process $Z = \{Z_t,\, t \geqslant 0\}$ has the Markov property. So the limiting distribution $p(\infty, z)$ is also a stationary distribution, provided that Z has the Feller property. This property is a direct consequence of the uniform continuity of the reflection mapping on compacta [23]. We thus construct a stationary Markov process $\{Z_t\colon\, t \geqslant 0\}$ by letting $\mathbb{P}(Z_0 \leqslant z) = p(\infty, z)$. □

Note. The extension of Z to the whole time axis $(-\infty, \infty)$ is a standard argument on stationary processes. We will thus tacitly assume that both X and Z are defined on the whole time axis.

4. Balance equations in a stationary Lévy network

In this section, we will be working with a process Z obtained by reflecting a Lévy process X along a reflection matrix R. We shall assume that Z is stationary. A sufficient condition for the latter was given in section 3. Also, we shall assume that Z_t has been defined for all $t \in \mathbb{R}$. Even though our principal interest is in Lévy processes with finite mean and spectrally positive Lévy measure, we will, for the sake of generality, deal first with balance equations for a general Lévy process.

Basically, we are interested in obtaining information about the distribution of Z_0 (which is the same as that of Z_t, for any t, by stationarity). We consider smooth functions $f : \mathbb{R}^d \to \mathbb{R}$ and denote by $\partial_i f$ and $\partial_{ij} f$ the first and second order partial derivatives, respectively. We further let Df be the gradient of f and $D^2 f$ be its Hessian matrix. The trace of a matrix A is denoted by $\mathrm{tr}(A)$.

4.1. General Lévy network

Theorem 4. Let X be a general Lévy process, and Z a stationary process defined as the reflection of X on the positive orthant $\mathbb{R}^d_+$ along the matrix R. Let $f : \mathbb{R}^d_+ \to \mathbb{R}$ be bounded and twice continuously differentiable with bounded first and second order partial derivatives. Assume that

$$\mathbb{E} \int_{|x| \leqslant 1} \big| f\big(\pi_R(Z_0 + x)\big) - f(Z_0) - x^{\mathrm{T}} Df(Z_0) \big| \nu(\mathrm{d}x) < \infty. \tag{12}$$

Then

$$a^{\mathrm{T}}\mathbb{E}Df(Z_0) + \frac{1}{2}\operatorname{tr}\big(\sigma^{\mathrm{T}}\mathbb{E}D^2 f(Z_0)\sigma\big) + \mathbb{E}\int_0^1 Df(Z_{s-})^{\mathrm{T}}\,\mathrm{d}\big(RL^{\mathrm{c}}\big)_s$$
$$+\,\mathbb{E}\int \big(f\big(\pi_R(Z_0+x)\big) - f(Z_0) - \mathbb{1}\big\{|x| \leqslant 1\big\}x^{\mathrm{T}}Df(Z_0)\big)\nu(\mathrm{d}x) = 0. \quad (13)$$

Proof. Let f be as in the theorem statement. Write Itô's formula [15] for the d-dimensional semimartingale Z:

$$f(Z_1) - f(Z_0) = \sum_i \int_0^1 \partial_i f(Z_{s-})\,\mathrm{d}Z_s^i + \frac{1}{2}\sum_{i,j}\int_0^1 \partial_{ij} f(Z_{s-})\,\mathrm{d}\big[Z^i, Z^j\big]_s^{\mathrm{c}}$$
$$+ \sum_0^1 \Big[\Delta f(Z_s) - \sum_i \partial_i f(Z_{s-})\Delta Z_s^i\Big], \quad (14)$$

where $[Z^i, Z^j]^{\mathrm{c}}$, is the continuous part of the quadratic covariation between Z^i and Z^j. We have:

$$\big[Z^i, Z^j\big]_t^{\mathrm{c}} = \big[X^i, X^j\big]_t^{\mathrm{c}} = \big(\sigma\sigma^{\mathrm{T}}\big)_{ij}.$$

The three terms of the r.h.s. of (14) are denoted by J_1, J_2, and J_3. The second term can be written as

$$J_2 = \frac{1}{2}\int_0^1 \operatorname{tr}\big(\sigma^{\mathrm{T}}D^2 f(Z_{s-})\sigma\big)\,\mathrm{d}s.$$

For the first integral, recall that Z relates to X via (7), while X is given by (2). So

$$\mathrm{d}Z_s = a\mathrm{d}s + \sigma\mathrm{d}B_s + \mathrm{d}M_s + \mathrm{d}K_s + \mathrm{d}(RL)_s,$$

where M is the square integrable martingale (3), and K is the compound Poisson process (4). Thus

$$\Delta K_s = \Delta X_s \mathbb{1}\big\{|X_s| > 1\big\}.$$

Hence the first integral of (14) is

$$J_1 = \int_0^1 a^{\mathrm{T}}Df(Z_{s-})\,\mathrm{d}s + \int_0^1 Df(Z_{s-})^{\mathrm{T}}(\sigma\mathrm{d}B_s + \mathrm{d}M_s)$$
$$+ \int_0^1 Df(Z_{s-})^{\mathrm{T}}\,\mathrm{d}(RL)_s + \sum_0^1 Df(Z_{s-})^{\mathrm{T}}\Delta X_s \mathbb{1}\big\{|X_s| > 1\big\}.$$

The last term of (14) can be written as

$$J_3 = -\sum_0^1 Df(Z_{s-})^{\mathrm{T}}R\Delta L_s + \sum_0^1\big[\Delta f(Z_s) - Df(Z_{s-})^{\mathrm{T}}\Delta X_s\big].$$

The last two terms of J_1 can be combined with the two terms of J_3 and give

$$J_1 + J_3 = \int_0^1 a^{\mathrm{T}} Df(Z_{s-})\,\mathrm{d}s + \int_0^1 Df(Z_{s-})^{\mathrm{T}}(\sigma\,\mathrm{d}B_s + \mathrm{d}M_s) \\ + \int_0^1 Df(Z_{s-})^{\mathrm{T}}\,\mathrm{d}\big(RL^{\mathrm{c}}\big)_s + \sum_0^1 \big[\Delta f(Z_s) - Df(Z_{s-})^{\mathrm{T}} \Delta X_s \mathbb{1}\big\{|X_s| \leqslant 1\big\}\big].$$

Now substitute $J_1 + J_3$ and J_2 in (14) and take expectations on both sides, to arrive at (13). To obtain the last term of (14), first observe that

$$Z_s = \pi_R(Z_{s-} + \Delta X_s),$$

where π_R is defined in theorem 2, for all times s at which $\Delta X_s > 0$. Insert this in the last term of $J_1 + J_3$ and use the refined Campbell theorem (or, compensation formula for Poisson processes) to obtain the last term of (13). Assumption (12) guarantees that all terms in (13) are finite, justifying the above calculation. □

4.2. Spectrally positive jump part

We now consider the case when the components X^i, $i = 1, \ldots, d$, of the Lévy process X have only non-negative jumps; equivalently, the Lévy measure ν is supported on $\mathbb{R}_+^d$:

$$\nu\big((-\infty, 0)^d\big) = 0.$$

We may call such a process "spectrally positive", in analogy to the one-dimensional terminology. This ensures that (if there is a jump at time s) $Z_{s-} + \Delta X_s \in \mathbb{R}_+^d$, and so $\pi_R(Z_{s-} + \Delta X_s) = Z_{s-} + \Delta X_s$. So the last term of (13) is simplified by the removal of π_R.

Further structure of the marginal distribution of the stationary process Z can be obtained, by manipulating (13), and this is what we do next. To start with, note that a spectrally positive Lévy process in $\mathbb{R}^d$ has a Laplace transform

$$\mathbb{E}\mathrm{e}^{-\theta X_t} = \mathrm{e}^{-t\Phi(\theta)}$$

which is defined for all $\theta \geqslant 0$. We will work with this Laplace transform, instead of the characteristic function. The Laplace exponent $\Phi(\theta)$ can be obtained from (1):

$$\Phi(\theta) = -\log \mathbb{E}\mathrm{e}^{-\theta^{\mathrm{T}} X_1} \\ = a^{\mathrm{T}}\theta - \frac{1}{2}\theta^{\mathrm{T}} C\theta + \int_{\mathbb{R}_+^d} \big(1 - \mathrm{e}^{-\theta^{\mathrm{T}} x} - \mathbb{1}\big\{|x| \leqslant 1\big\}\theta^{\mathrm{T}} x\big)\nu(\mathrm{d}x), \quad \theta > 0. \quad (15)$$

There is another thing that is ensured by spectral positivity: the reflectors L^j are continuous. This simplifies (13) further.

Define next the "boundary measures" by

$$\beta_j(A) := \frac{1}{t}\mathbb{E}\int_0^t \mathbb{1}\{Z_{s-} \in A\}\,\mathrm{d}L_s^j, \quad j = 1, \ldots, d, \quad (16)$$

i.e. the expected fraction of time, between 0 and t, spent in the Borel set A by the process Z, when the "time clock" runs according to the jth reflector. These quantities correspond to the third term of (13). Since L^j is supported on the face F_j, it follows that β_j is also supported on the same face. Also, by stationarity, β_j does not depend on t. Getting a handle on the β_j's is the main difficulty in obtaining information about the distribution of Z_t.

An interpretation of $\beta_j(A)$ is worth-mentioning. Note that $A \mapsto \int_A \mathrm{d}L_s^j$ defines a stationary random measure on the real line with intensity

$$\lambda_j := \mathbb{E}L_1^j$$

(which is shown to be finite in theorem 5). Moreover, letting $t = 1$ in (7) and taking expectations yields

$$(\lambda_1, \ldots, \lambda_d)^{\mathrm{T}} = -R^{-1}\mu, \tag{17}$$

and, of course, $\lambda_j > 0$, for all j. It makes sense to introduce the *Palm probability measure* $\mathbb{P}_j$ of $\mathbb{P}$ with respect to L^j (see, for instance, the original paper by Mecke [28]); for convenience we can assume that $\mathbb{P}_j$ is given on our basic sample space supporting the process X. Thus, the quantity

$$\mathbb{P}_j(Z_0 \in A) := \frac{\beta_j(A)}{\lambda_j}$$

is the Palm probability, with respect to L^j, of the event that $Z_0 \in A$. Roughly speaking, this is the probability that the state of the network is in A, at a typical departure point of node j, given that this node is empty.

Define also the Laplace transform of β_j:

$$\hat{\beta}_j(\theta) := \int_{\mathbb{R}_+^d} \mathrm{e}^{-\theta^{\mathrm{T}} z}\beta_j(\mathrm{d}z) = \int_{F_j} \mathrm{e}^{-\theta^{\mathrm{T}} z}\beta_j(\mathrm{d}z) = \mathbb{E}\int_0^1 \mathrm{e}^{-\theta^{\mathrm{T}} Z_s}\,\mathrm{d}L_s^j \tag{18}$$

(and note that this does not depend on θ_j), and the Laplace transform of the distribution of Z_0:

$$\varphi(\theta) := \mathbb{E}\,\mathrm{e}^{-\theta^{\mathrm{T}} Z_0}. \tag{19}$$

Our next result relates the Laplace exponent (15), with the Laplace transforms (18), (19).

Theorem 5. Assume that X is spectrally positive and let Z be a stationary solution to the Skorokhod reflection problem with reflection matrix R. Then the boundary measures β_j, $j = 1, \ldots, d$, are finite and their Laplace transforms are related to the Laplace transform φ of Z_0 by

$$\Phi(\theta)\varphi(\theta) = -\sum_j \hat{\beta}_j(\theta)\theta^{\mathrm{T}} r_j. \tag{20}$$

Proof. With

$$f(z) = \mathrm{e}^{-\theta^{\mathrm{T}} z},$$

we have

$$Df(z) = -\theta f(z), \qquad D^2 f(z) = \theta\theta^{\mathrm{T}} f(z).$$

Since ν is a Lévy measure and $D^2 f(z)$ is bounded the condition (12) of theorem 4 is clearly satisfied. Hence (13) gives

$$\begin{aligned} 0 &= -a^{\mathrm{T}}\theta\varphi(\theta) + \frac{1}{2}\big(\sigma^{\mathrm{T}}\theta\theta^{\mathrm{T}}\sigma\big)\varphi(\theta) - \mathbb{E}\sum_j \int_0^1 \mathrm{e}^{-\theta^{\mathrm{T}} Z_{s-}}\theta^{\mathrm{T}} r_j \,\mathrm{d}L_s^j \\ &\quad + \varphi(\theta)\int_{\mathbb{R}_+^d} \big(\mathrm{e}^{-\theta x} - 1 + \theta^{\mathrm{T}} x \mathbb{1}\{|x| \leqslant 1\}\big)\nu(\mathrm{d}x) \\ &= -\mathbb{E}\sum_j \int_0^1 \mathrm{e}^{-\theta^{\mathrm{T}} Z_{s-}}\theta^{\mathrm{T}} r_j \,\mathrm{d}L_s^j - \varphi(\theta)\Phi(\theta), \end{aligned} \tag{21}$$

where we used the fact that $\mathrm{tr}(\sigma^{\mathrm{T}}\theta\theta\sigma) = \theta^{\mathrm{T}}\sigma\sigma^{\mathrm{T}}\theta$. By definition we have

$$\mathbb{E}\sum_j \int_0^1 \mathrm{e}^{-\theta^{\mathrm{T}} Z_{s-}}\theta^{\mathrm{T}} r_j \,\mathrm{d}L_s^j = \sum_j \hat{\beta}_j(\theta)\theta^{\mathrm{T}} r_j$$

provided we can show that the components of L_1 are integrable. To this end we take in (21) $\theta_i = 1$ and $\theta_j = 0$ for $j \neq i$. It follows that

$$\mathbb{E}\left|\sum_j r_{ij}\int_0^1 \mathrm{e}^{-Z_s^i}\,\mathrm{d}L_s^j\right| < \infty, \quad i = 1, \ldots, d.$$

The diagonal elements in the above sum equal $r_{ii}L_1^i$. Since $R = I - P^{\mathrm{T}}$ and P^{T} is non-negative, the off-diagonal elements are greater or equal than $r_{ij}L_1^j$. It follows that the (componentwise) positive part $(RL_1)^+$ of RL_1 is integrable and so is $R^{-1}(RL_1)^+$. However, the inverse R^{-1} of R is non-negative implying that $R^{-1}x^+ \geqslant (R^{-1}x)^+$ for all vectors $x \in \mathbb{R}^d$. Hence we can conclude that $R^{-1}(RL_1)^+ \geqslant L_1$ proving that L_1 is integrable. □

4.3. Dimension one

When $d = 1$, and the Lévy process has finite mean, theorem 5 provides a solution for φ. Here $\Phi(\theta)$, given by (15), takes the form

$$\Phi(\theta) = \mu\theta - \frac{1}{2}C\theta^2 + \int_0^\infty \big(1 - \mathrm{e}^{-\theta x} - \theta x\big)\nu(\mathrm{d}x),$$

possible because $\mathbb{E}X_1 = \mu$ is a finite number. (This can also be derived from (5).) Now, R is a positive number which, without loss of generality, may be taken to be 1. The stability condition $R^{-1}\mu < 0$ gives $\mu < 0$. Also, $\hat{\beta}_1$ is just a constant, equal to $\mathbb{E}L_1$. Hence (20) gives $\Phi(\theta)\varphi(\theta) = -\theta\mathbb{E}L_1$. Dividing by θ, and letting $\theta \to 0$, we obtain $\mathbb{E}L_1 = -\mu$, and so

$$\varphi(\theta) = \frac{\mu\theta}{\Phi(\theta)}.$$

In a personal discussion Offer Kella pointed out to us that an alternative proof of theorem 5 can be based on a suitable special case of the martingales discussed in [20]. As a historical note, we mention that the formula has been derived by different methods by various authors: Zolotarev [42] used an analytical approach; Takács [37] used exchangeability; Bingham [3] used Wiener–Hopf factorization; and Harrison [9] used Markov process theory.

Three well-known examples are contained in the previous formula.

Example 1. Brownian motion with drift. Let $\nu = 0$, $\mu < 0$, so that X_t is the sum of a linear term μt and a Brownian motion with zero mean and variance C. Then $\Phi(\theta) = \mu\theta - \frac{1}{2}C\theta^2$. The Laplace transform of the marginal distribution of the stationary process Z is given by

$$\varphi(\theta) = \frac{\mu\theta}{\mu\theta - (1/2)C\theta^2} = \frac{1}{1 + (C/(2|\mu|)\theta)}.$$

This is the Laplace transform of an exponential random variable with mean $C/2|\mu|$.

Example 2. Compound Poisson process with drift. Let $C = 0$, $\mu < 0$, and let $\nu(\mathrm{d}x) = \lambda F(\mathrm{d}x)$, where $\lambda > 0$, and F a probability measure on $\mathbb{R}_+$ with no atom at zero. Let $m = \int_0^\infty xF(\mathrm{d}x)$, and $\widehat{F}(\theta) = \int_0^\infty \mathrm{e}^{-\theta x} F(\mathrm{d}x)$. Then $\Phi(\theta) = -(|\mu| + \lambda m)\theta + \lambda(1 - \widehat{F}(\theta))$. Thus, $X_t = -(|\mu| + \lambda m)t + \sum_{j=1}^{N_t} \xi_j$, where N is a Poisson(λ) process, and the $\{\xi_j\}$ are i.i.d. random variables, independent of N, with common distribution F. The compound Poisson process $\sum_{j=1}^{N_t} \xi_j$ represents cumulative work arriving in an $M/GI/\infty$ queue at a rate equal to λm. This is served at a rate $|\mu| + \lambda m$, so that $|\mu|$ represents the excess capacity (positive for stability purposes). The formula for the Laplace transform of Z_0,

$$\varphi(\theta) = \frac{|\mu|\theta}{(|\mu| + \lambda m)\theta - \lambda + \lambda\widehat{F}(\theta)},$$

is the well-known Pollaczek–Khinchine formula, which can be inverted by means of a series in the convolutions of the integrated tails of F.

Example 3. Stable Lévy process with drift. Let $C = 0$, $\mu < 0$, and $\nu(dx) = (c/x^{\alpha+1})\mathrm{d}x$, $x > 0$, $c > 0$, $1 < \alpha < 2$. Thus the jump part is α-stable and spectrally positive (see [1]). The Laplace exponent is given by

$$\Phi(\theta) = \mu\theta + \int_0^\infty \left(1 - \theta x - \mathrm{e}^{-\theta x}\right)\frac{c\mathrm{d}x}{x^{\alpha+1}} = \mu\theta - c'\theta^\alpha,$$

where $c' = c\Gamma(2-\alpha)/((\alpha-1)\alpha)$. Hence,

$$\varphi(\theta) = \frac{1}{1 + (c'/|\mu|)\theta^{\alpha-1}}, \tag{22}$$

which is the Laplace transform of the Mittag-Leffler distribution. Asymptotics and other properties of this distribution can be found in [43]. See also [34] for the derivation of this distribution in a related stochastic model. This example is of special interest in modeling high-speed network traffic. It arises as a limit of a special kind of traffic models.

The next section deals with a multi-dimensional generalization of this.

4.4. Spectrally positive and stable jump part

We are particularly interested in processes X whose jump part is α-stable for some $\alpha \in (1, 2)$, and spectrally positive. We let $\mathbb{E}X_t = \mu t$, and assume, as in theorem 3, that

$$R^{-1}\mu < 0.$$

We recall that such a process has the representation $X_t = \mu t + \sigma B_t + M_t$ (see (3), (5)), where M is a zero-mean martingale satisfying the self-similarity property $\{M_{\lambda t}\} \stackrel{d}{=} \{\lambda^{1/\alpha} M_t\}$, for all $\lambda > 0$. One way to build an α-stable Lévy process M in d dimensions is to let the components $M^1, \ldots, M^d$ of X to be independent α-stable Lévy processes with Lévy measure proportional to $\mathrm{d}x/x^{\alpha+1}$, $x > 0$; the parameter α has to be taken larger than 1 to ensure finite first moment. More generally, we may consider that each component M^i is a positive linear combination of a collection of "background" processes $W^1, \ldots, W^\ell$:

$$M^i = \sum_{j=1}^{\ell} b_{ij} W^j, \quad i = 1, \ldots, d.$$

The processes $W^1, \ldots, W^\ell$ are taken i.i.d. α-stable Lévy processes. This is a reasonable model for a stochastic communication network, if we assume that independent traffic processes are being multiplexed by switches before injected into the network. A simple calculation shows that, in this case, the Laplace exponent of M is given by

$$\sum_j \biggl(\sum_i \theta_i b_{ij}\biggr)^{\alpha} \int_0^\infty \bigl(1 - \mathrm{e}^{-r} - r\bigr) \frac{\mathrm{d}r}{r^{\alpha+1}}.$$

More generally, requiring that M be α-stable implies that its Lévy measure is given, in polar coordinates, by

$$\nu(\mathrm{d}r, \mathrm{d}u) = \frac{\mathrm{d}r}{r^{\alpha+1}} \tau(\mathrm{d}u), \quad r > 0,\ u \in S_+^{d-1}, \tag{23}$$

where S_+^{d-1} is the positive part of the unit $(d-1)$-dimensional sphere,

$$S_+^{d-1} := \bigl\{x \in \mathbb{R}_+^d\colon |x| = 1\bigr\},$$

and τ is a finite measure on S_+^{d-1}. With (23) as Lévy measure, we calculate the Laplace exponent M as follows:

$$\int_{S_+^{d-1}} \int_0^\infty \left(1 - \mathrm{e}^{-r(\theta^{\mathrm{T}} u)} - r\left(\theta^{\mathrm{T}} u\right)\right) r^{-\alpha-1}\, \mathrm{d}r\, \tau(\mathrm{d}u) = -\gamma \int_{S_+^{d-1}} \left(\theta^{\mathrm{T}} u\right)^\alpha \tau(\mathrm{d}u), \qquad (24)$$

where γ is just a constant, given by

$$\gamma = \int_0^\infty \left(\mathrm{e}^{-r} - 1 + r\right) \frac{\mathrm{d}r}{r^{\alpha+1}} = \frac{\Gamma(2-\alpha)}{(\alpha-1)\alpha}.$$

We are now in position to define our model. It is going to be a Lévy process with centered jump part having Laplace exponent (24), together with a zero-mean Brownian part σB_t and a drift μt, so that

$$\Phi(\theta) := -\log \mathbb{E}\, \mathrm{e}^{-\theta^{\mathrm{T}} X_1} = \mu^{\mathrm{T}} \theta - \frac{1}{2} \theta^{\mathrm{T}} C \theta - \int \left(\theta^{\mathrm{T}} u\right)^\alpha \tau(\mathrm{d}u), \qquad (25)$$

where

$$C = \sigma \sigma^{\mathrm{T}} = [c_{ij}]$$

is the covariance matrix of the Brownian motion. The constant γ has been absorbed by the measure τ. Thus, (25) gives the Laplace exponent of the process we are interested in. In the sequel, (25) will be used in formula (20), for the Laplace transform of the marginal of the stationary process Z. The goal is to derive some of its properties. Recall that such a Z exists, provided that $R^{-1}\mu < 0$ (theorem 3).

We now recall the definition (16) of the boundary measures β_j. We define β_j^i be the restriction of β_j on the face F_i:

$$\beta_j^i(A) := \beta_j(A \cap F_i). \qquad (26)$$

This makes sense for all $i, j = 1, \ldots, d$; for $i = j$, we have $\beta_j^j = \beta_j$, because β_j is supported on F_j. The Laplace transform of β_j^i is

$$\hat{\beta}_j^i(\theta) := \int_{F_i \cap F_j} \mathrm{e}^{-\theta^{\mathrm{T}} z} \beta_j(\mathrm{d}z) = \mathbb{E} \int_0^1 \mathrm{e}^{-\theta^{\mathrm{T}} Z_s} \mathbb{1}\left(Z_s^i = 0\right) \mathrm{d}L_s^j.$$

We also define the constant

$$\delta_i := \sum_j r_{ij} \hat{\beta}_j^i(0). \qquad (27)$$

The following lemma is useful in the sequel. It basically derives some information about the Laplace transforms of the components of X and Z, where Z is a stationary reflection of X along R. We let $\Phi_i(\theta_i)$ be the Laplace exponent of the component X_t^i. It is defined by

$$\Phi_i(\theta_i) = -\log \mathbb{E}\, \mathrm{e}^{-\theta_i X_t^i}, \quad \theta_i \geqslant 0,$$

and is obtained by setting $\theta_j = 0$ for all $j \neq i$, in the expression (25) for $\Phi(\theta) = \Phi(\theta_1, \dots, \theta_d)$:

$$\Phi_i(\theta_i) = \mu_i\theta_i - \frac{1}{2}c_{ii}\theta_i^2 - \theta_i^\alpha \int_{S_+^{d-1}} u_i^\alpha \tau(\,\mathrm{d}u). \tag{28}$$

If $c_{ii} = 0$ then the component X^i does not contain a Brownian motion. Let

$$\kappa_i := \int_{S_+^{d-1}} u_i \tau(\mathrm{d}u). \tag{29}$$

If $\kappa_i = 0$ then the component X^i has no jump part. Recall, from (19), that $\varphi(\theta)$ denotes the Laplace transform of Z_0, and let $\varphi_i(\theta_i)$ be the Laplace transform of the component Z_0^i:

$$\varphi_i(\theta_i) = \mathbb{E}\,\mathrm{e}^{-\theta_i Z_0^i}, \quad \theta_i \geqslant 0;$$

it is obtained by setting $\theta_j = 0$ for all $j \neq i$, in $\varphi(\theta)$.

Lemma 1. Let X be a Lévy process defined by its Laplace exponent $\Phi(\theta)$ as in (25). Let Z be a stationary solution to the Skorokhod reflection problem with reflection matrix R. Then, for all $i = 1, \dots, d$:

(i)

$$\lim_{\theta_i\to\infty} \frac{1}{\theta_i}\varphi(\theta)\Phi(\theta) = -\sum_j r_{ij}\hat{\beta}_j^i(\theta),$$

(ii)

$$\lim_{\theta_i\to\infty} \frac{1}{\theta_i}\varphi_i(\theta_i)\Phi_i(\theta_i) = -\sum_j r_{ij}\hat{\beta}_j^i(0) = -\delta_i.$$

(iii) If $c_{ii} > 0$ or $\kappa_i > 0$, then

$$\lim_{\theta_i\to\infty} \varphi_i(\theta_i) = 0 \quad \text{and} \quad \mathbb{P}\big(Z_0^i = 0\big) = 0,$$

$$\lim_{\theta_i\to\infty} \frac{1}{\theta_i}\varphi_i(\theta_i)\Phi(\theta) = -\delta_i.$$

Proof. From equation (20) we have

$$-\frac{1}{\theta_i}\varphi(\theta)\Phi(\theta) = \frac{1}{\theta_i}\sum_k\sum_\ell \hat{\beta}_\ell(\theta)\theta_k r_{k\ell} = \sum_\ell \hat{\beta}_\ell(\theta) r_{i\ell} + \frac{1}{\theta_i}\sum_{k\neq i}\sum_\ell \hat{\beta}_\ell(\theta)\theta_k r_{k\ell}. \tag{30}$$

Using bounded convergence we have

$$\lim_{\theta_i\to\infty} \hat{\beta}_\ell(\theta) = \lim_{\theta_i\to\infty} \int \mathrm{e}^{-\theta^{\mathrm{T}} z}\beta_\ell(\mathrm{d}z) = \int \mathbb{1}(z_i = 0)\mathrm{e}^{-\theta^{\mathrm{T}} z}\beta_\ell(\mathrm{d}z) = \hat{\beta}_\ell^i(\theta),$$

where the last equality is the definition of $\hat{\beta}^i_\ell(\theta)$. This is true for all ℓ, including $\ell = i$. But observe that $\hat{\beta}_i(\theta)$ does not depend on θ_i, hence, trivially, $\hat{\beta}^i_i(\theta) = \hat{\beta}_i(\theta)$. The second term of (30) clearly tends to zero, and this proves the first part of the lemma. Set now $\theta_j = 0$ for all $j \neq i$ in (30). This gives

$$-\frac{1}{\theta_i}\varphi_i(\theta_i)\Phi_i(\theta_i) = \sum_\ell r_{i\ell} \int \mathrm{e}^{-\theta_i z_i}\beta_\ell(\mathrm{d}z)$$
$$\to \sum_\ell \int \mathbb{1}(z_i = 0)\beta_\ell(\mathrm{d}z) = \sum_\ell r_{i\ell}\hat{\beta}^i_\ell(0), \quad \text{as } \theta_i \to \infty,$$

proving the second part of the lemma. The above limit was denoted by δ_i in (27). Consider the expression (28) for $\Phi_i(\theta_i)$ and write:

$$-\delta_i = \lim_{\theta_i\to\infty} \frac{1}{\theta_i}\varphi_i(\theta_i)\Phi_i(\theta_i) = \lim_{\theta_i\to\infty}\left[\mu_i - \frac{1}{2}c_{ii}\theta_i - \theta_i^{\alpha-1}\int u_i^\alpha \tau(\mathrm{d}u)\right]\varphi_i(\theta_i). \tag{31}$$

Assume now that $c_{ii} > 0$ or $\kappa_i > 0$. Then the term in brackets of (31) tends to $-\infty$. Hence $\varphi_i(\theta_i) \to 0$, as $\theta_i \to \infty$. This clearly implies $\mathbb{P}(Z^i_0 = 0) = 0$. Moreover, a straightforward computation involving $\Phi(\theta)$ (use expression (25), separate the dependence on θ_i from the remaining variables, and use the fact that $\varphi(\theta_i) \to 0$) shows that $\theta_i^{-1}\varphi_i(\theta_i)\Phi(\theta)$ has the same limit as (31). □

5. Absence of product form

The question in this section is motivated by a classical problem in stochastic network models. Namely, under what conditions is it true that the stationary distribution of Z_t is a product of the distributions of its marginals $Z^1_t, \dots, Z^d_t$? The question originates in the performance evaluation of networks. A classical example of such a case is the so-called Jackson network (see, e.g., [38]). Other examples are also in [38] and in [22]. Harrison and Williams [12] deal with the product-form question for Brownian networks, deriving a necessary and sufficient condition. Kella [19, and references therein] has examined this question for some classes of Lévy networks.

In the remaining of the section, we will proceed as follows: we consider a stochastic network driven by Lévy processes consisting of a Brownian part with drift and spectrally positive self-similar centered jump part. In other words, let X be a Lévy process with characteristic exponent (25). Assume that (9) holds, and let Z be a stationary solution to the Skorokhod reflection problem (7), (8). To see what kind of structure product form networks may have we shall need to examine the properties of the boundary measures β_j (16) more closely. Recall the notation β^i_j for the restriction of β_j on the face F_i and generalize it: Let I be a subset of $\{1, \dots, d\}$ of size $|I|$, and let F_I denote the $(d - |I|)$-dimensional face

$$F_I := \{x \in \mathbb{R}^d_+ \colon x_i = 0, \text{ for all } i \in I\}.$$

Let β_j^I denote the restriction of β_j on F_J:

$$\beta_j^I(A) := \beta_j(A \cap F_I).$$

Note that when $I = \{j\}$, a singleton, $\beta_i^I = \beta_i^j$, as in (26). The notation makes sense for all sets I, including $I = \emptyset$, for which $\beta_j^\emptyset = \beta_j$. Notice also the following obvious properties:

$$\begin{aligned} \beta_j^I &= \beta_j^{I\cup\{j\}}, && \text{for all } j, I, \\ \beta_j^I &= \beta_j^{I\setminus\{j\}}, && \text{if } j \in I. \end{aligned}$$

The Laplace transform of β_j^I is

$$\hat{\beta}_j^I(\theta) := \int_{F_I \cap F_j} \mathrm{e}^{-\theta^{\mathrm{T}} z} \beta_j(\mathrm{d}z) = \mathbb{E} \int_0^1 \mathrm{e}^{-\theta^{\mathrm{T}} Z_s} \mathbb{1}(Z_s \in F_I) \mathrm{d}L_s^j.$$

Note that $\hat{\beta}_j^I(\theta)$ does not depend on the variables $\{\theta_k, k \in F_I \cap \{j\}\}$. In particular, $\hat{\beta}_j^{1,\ldots,d}$ is a constant.

In what follows, we work with the stationary process $\{Z_t\}$, and assume that $Z_t^1, \ldots, Z_t^d$ are independent random variables for some (and hence for all) t. Equivalently, we assume that

$$\varphi(\theta) = \prod_{i=1}^d \varphi_i(\theta_i). \tag{32}$$

We then have:

Lemma 2. Assume for all $i = 1, \ldots, d$ that $c_{ii} > 0$ or $\kappa_i > 0$ (as defined in (29)), and that (32) holds. Then, for each $j = 1, \ldots, d$, the boundary measure β_j defined in (16) is supported on the relative interior of the face F_j.

Proof. The assertion is equivalent to

$$i \neq j \quad \Rightarrow \quad \beta_j(F_i) = \int_{F_i} \beta_j(\mathrm{d}z) = \hat{\beta}_j^i(0) = 0.$$

Combining (i) and (iii) of lemma 1, with the assumption (32), we obtain

$$\delta_i \prod_{j \neq i} \varphi_j(\theta_j) = \sum_j r_{ij} \hat{\beta}_j^i(\theta). \tag{33}$$

We are going to prove that (omit the variable θ to ease notation)

$$\hat{\beta}_j^I(\theta) \equiv 0, \tag{34}$$

for all $j = 1, \ldots, d$ and all sets $I \subseteq \{1, \ldots, d\}$ of size 2 or larger. We will do this by induction on the size of $I^c = \{1, \ldots, d\} - I$. First we treat the case $I = \{1, \ldots, d\}$.

Let, in (33), all variables θ_k, $k = 1, \dots, d$, tend to ∞. For the right-hand side of (33) we have

$$\lim_{\theta_1 \to \infty, \dots, \theta_d \to \infty} \hat{\beta}^i_j(\theta) = \hat{\beta}^{1,\dots,d}_j, \quad i, j = 1, \dots, d. \tag{35}$$

For the left-hand side of (33), recall, from lemma 1, that

$$\lim_{\theta_j \to \infty} \varphi_j(\theta_j) = 0, \quad j = 1, \dots, d. \tag{36}$$

Using (35), (36) in (33), we obtain

$$\sum_{j=1}^{d} r_{ij} \hat{\beta}^{1,\dots,d}_j = 0, \quad i = 1, \dots, d.$$

But the matrix R is invertible. So $\hat{\beta}^{1,\dots,d}_j = 0$ for all $j = 1, \dots, d$. If $d = 2$ then the proof is complete. Otherwise, assume $d \geqslant 3$. For the induction step we assume that $\hat{\beta}^I_j(\theta) \equiv 0$, for all j and all sets I of size s (where s is an integer > 2). We are going to prove the same thing for all sets I of size $s - 1$. Let then I be a set of size $s - 1$. Consider again (33) and let $\theta_k \to \infty$ for all $k \in I$. Let $i \in I$. Notice that

$$\lim_{\theta_k \to \infty, k \in I} \hat{\beta}^i_j(\theta) = \hat{\beta}^{I \cup \{i\}}_j(\theta) = \hat{\beta}^J_j(\theta), \quad \text{for all } i \in I, \; j = 1, \dots, d.$$

The left-hand side of (33) tends to zero if one variable tends to infinity. Hence,

$$\sum_{j=1}^{d} r_{ij} \hat{\beta}^I_j(\theta) \equiv 0, \quad \text{for all } i \in I. \tag{37}$$

Split the sum into a sum over $j \in I$ and one over $j \notin I$. For $j \notin I$, $\hat{\beta}^I_j = \hat{\beta}^{I \cup \{j\}}_j$, owing to an earlier observation. But the set $I \cup \{j\}$ has size s, and so, by the induction hypothesis, $\hat{\beta}^{I \cup \{j\}}_j(\theta) \equiv 0$. Thus (37) becomes

$$\sum_{j \in I} r_{ij} \hat{\beta}^I_j(\theta) \equiv 0, \quad \text{for all } i \in I.$$

These are $s-1$ equations with the same number of unknowns. Notice that the coefficients form a matrix which is a principal submatrix of R. The properties of R imply that all its principal submatrices have spectral radius strictly smaller than one as well, and are thus invertible. This yields that $\hat{\beta}^I_j(\theta) \equiv 0$ for all $j \in I$, and, as shown above, $\hat{\beta}^I_j = 0$, for all $j \notin I$, too. Thus (34) is proved, and so is the assertion of the lemma. □

According to what we proved, if the Lévy network has product form, and if none of the coordinates is deterministic (which amounts to $c_{ii} + \kappa_i > 0$ for all i) then $\hat{\beta}^i_j(\theta) \equiv 0$ for all $i \neq j$. This allows the algebraic identity (33) to be simplified further:

$$\delta_i \prod_{j \neq i} \varphi_j(\theta_j) = r_{ii} \hat{\beta}_i(\theta). \tag{38}$$

Since $r_{ii} > 0$ for all i, this allows to express $\hat{\beta}_i(\theta)$, for all i, in terms of the $\varphi_j(\theta)$. The following result proves that this can be done only in trivial networks. We shall assume that each component has a nontrivial jump part.

Theorem 6. Assume that $\kappa_i > 0$, for all $i = 1, \dots, d$. and that the distribution of Z_0 has product form, i.e., that (32) holds. Then $c_{ij} = r_{ij} = 0$ for all $i \neq j$, and $\tau = \sum_{r=1}^{d} \kappa_r \delta_{e_r}$, where e_r is the unit vector in the rth direction and δ_{e_r} is the Dirac measure at e_r, whereas the κ_r are the constants of (29).

Proof. Substitute (38) into (20) to obtain

$$\Phi(\theta) = -\sum_i \theta_i \sum_j \frac{r_{ij}\delta_j}{r_{jj}} \frac{1}{\varphi_j(\theta_j)}. \tag{39}$$

Fix i and set $\theta_j = 0$ for all $j \neq i$ in the above:

$$\Phi_i(\theta_i) = -\theta_i \frac{\delta_i}{\varphi_i(\theta_i)} - \theta_i \sum_{j \neq i} \frac{r_{ij}\delta_j}{r_{jj}},$$

and solve for $\varphi_i(\theta_i)$:

$$\begin{aligned}\varphi_i(\theta_i) &= \delta_i \left(-\frac{\Phi_i(\theta_i)}{\theta_i} - \sum_{j \neq i} \frac{r_{ij}\delta_j}{r_{jj}} \right)^{-1} \\ &= \delta_i \left(-\mu_i + \frac{1}{2} c_{ii}\theta_i + \theta_i^{\alpha-1} \int u_i^{\alpha} \tau(\mathrm{d}u) - \sum_{j \neq i} \frac{r_{ij}\delta_j}{r_{jj}} \right)^{-1},\end{aligned}$$

where the latter equality follows from (28). Some simplification is possible, because, setting $\theta_i = 0$ we obtain

$$1 = \delta_i \left(-\mu_i - \sum_{j \neq i} \frac{r_{ij}\delta_j}{r_{jj}} \right)^{-1}. \tag{40}$$

Thus

$$\varphi_i(\theta_i) = \delta_i \left(\delta_i + \frac{1}{2} c_{ii}\theta_i + \theta_i^{\alpha-1} \int u_i^{\alpha} \tau(\mathrm{d}u) \right)^{-1}.$$

This can be substituted back into (39) to yield:

$$\Phi(\theta) = -\sum_i \sum_j \frac{\theta_i r_{ij}}{r_{jj}} \left(\delta_j + \frac{1}{2} c_{jj}\theta_j + \theta_j^{\alpha-1} \int u_j^{\alpha} \tau(\mathrm{d}u) \right).$$

Now this must equal the original expression (25) for $\Phi(\theta)$. Using (40) and equating the two expressions we obtain

$$\frac{1}{2}\sum_{i,j} c_{ij}\theta_i\theta_j + \int\left(\sum_i \theta_i u_i\right)^{\alpha}\tau(\mathrm{d}u) = \frac{1}{2}\sum_{i,j} c_{jj}\theta_i\theta_j\frac{r_{ij}}{r_{jj}} + \sum_{i,j}\frac{\theta_i r_{ij}}{r_{jj}}\theta_j^{\alpha-1}\int u_i^{\alpha}\tau(\mathrm{d}u).$$

This is a tautology in θ. Hence the quadratic terms must be equal:

$$\sum_i\sum_j c_{ij}\theta_i\theta_j = \sum_i\sum_j c_{jj}\theta_i\theta_j\frac{r_{ij}}{r_{jj}}.$$

Recall that $c_{ij} = c_{ji}$ and write the right-hand side as

$$\frac{1}{2}\sum_i\sum_j c_{jj}\theta_i\theta_j\frac{r_{ij}}{r_{jj}} + \frac{1}{2}\sum_i\sum_j c_{ii}\theta_i\theta_j\frac{r_{ji}}{r_{ii}}.$$

We thus need to have

$$c_{ij} = \frac{r_{ij}c_{jj}}{2r_{jj}} + \frac{r_{ji}c_{ii}}{2r_{ii}}. \tag{41}$$

As for the remaining terms, we can easily conclude that the identity

$$\int\left(\sum_i \theta_i u_i\right)^{\alpha}\tau(\mathrm{d}u) = \sum_i\sum_j\frac{\theta_i r_{ij}}{r_{jj}}\theta_j^{\alpha-1}\int u_j^{\alpha}\tau(\mathrm{d}u) \tag{42}$$

is an impossibility unless τ has the trivial form:

$$\tau = \sum_{r=1}^{d}\kappa_r\delta_{e_r},$$

where e_r is the unit vector in the rth direction and δ_{e_r} is the Dirac measure at e_r, whereas the κ_r are constants, necessarily the constants defined in (29). Substituting this into (42) yields

$$\sum_{i\neq j}\frac{r_{ij}}{r_{jj}}\kappa_j\theta_i\theta_j^{\alpha-1} \equiv 0.$$

This holds for all θ. Hence the coefficients must all be zero. But $\kappa_j > 0$, by assumption, for all j. Thus $r_{ij} = 0$ for all $i \neq j$. Equation (41) then gives that $c_{ij} = 0$ for all $i \neq j$. □

We note at this point that (41) is precisely the necessary and sufficient condition of Harrison and Williams [12] under which the Brownian network (i.e. – in our notation – the network for which $\kappa_i = 0$ for all i) has product-form stationary distribution.

6. Tail asymptotics

In this section, we work again with Lévy processes with spectrally positive and stable jump part, as in section 4.4. Our goal is to derive asymptotic bounds for $\mathbb{P}(|Z_0| > x)$, as $x \to \infty$, where Z is the stationary reflection of X.

Recall the notation $\mathbb{P}_j$ for the Palm probability of $\mathbb{P}$ with respect to L_j, as discussed in section 4.2. The following statement relates the Laplace transform of Z_0, under $\mathbb{P}$, to the Laplace transforms of Z_0 under $\mathbb{P}_j$, for $j = 1, \dots, d$, and is merely an alternative way of writing the balance relation (20). It holds for Lévy process defined through (15). Although interesting in its own right, the following statement will be used in the sequel for obtaining tail asymptotics for Z_0 under $\mathbb{P}$. It should be kept in mind that $\mathbb{P}_j(A) = \mathbb{P}_j(A, Z_0^j = 0)$ is used in many places below.

Theorem 7. Assume that X has spectrally positive jump part (i.e., its Laplace exponent is given by (15)), and let Z be a stationary solution to the Skorokhod reflection problem with reflection matrix R. Then, for any $s > 0$ and any $\theta \in \mathbb{R}_+^d$

$$\begin{aligned} &s\Phi(s\theta)\int_0^\infty \mathrm{e}^{-st}\mathbb{P}\big(\theta^{\mathrm{T}} Z_0 \geqslant t\big)\,\mathrm{d}t \\ &\quad = \Phi(s\theta) - s\theta^{\mathrm{T}}\mu - s^2\sum_{j=1}^d \lambda_j\big(\theta^{\mathrm{T}} r_j\big)\int_0^\infty \mathrm{e}^{-st}\mathbb{P}_j\big(\theta^{\mathrm{T}} Z_0 \geqslant t\big)\,\mathrm{d}t, \end{aligned} \tag{43}$$

where $\lambda_j = \mathbb{E}L_1^j$, and $\mathbb{P}_j$ is the Palm probability of $\mathbb{P}$ with respect to L_j, $j = 1, \dots, d$.

Proof. From the definition of $\mathbb{P}_j$ and $\hat{\beta}_j$ (see (18)) we have

$$\hat{\beta}_j(\theta) = \mathbb{E}\int_0^1 \mathrm{e}^{-\theta^{\mathrm{T}} Z_s}\,\mathrm{d}L_s^j = \lambda_j \mathbb{E}_j \mathrm{e}^{-\theta^{\mathrm{T}} Z_0},$$

where $\mathbb{E}_j$ denotes expectation with respect to $\mathbb{P}_j$. We also have, for any $s > 0$, $\theta \in \mathbb{R}_+^d$,

$$\begin{aligned} \varphi(s\theta) &= \mathbb{E}\,\mathrm{e}^{-s\theta^{\mathrm{T}} Z_0} = 1 - s\int_0^\infty \mathrm{e}^{-st}\mathbb{P}\big(\theta^{\mathrm{T}} Z_0 \geqslant t\big)\,\mathrm{d}t, \\ \hat{\beta}_j(s\theta) &= \mathbb{E}_j \mathrm{e}^{-s\theta^{\mathrm{T}} Z_0} = 1 - s\int_0^\infty \mathrm{e}^{-st}\mathbb{P}_j\big(\theta^{\mathrm{T}} Z_0 \geqslant t\big)\,\mathrm{d}t. \end{aligned}$$

We substitute these in the expression

$$\Phi(s\theta)\varphi(s\theta) = -\sum_j \hat{\beta}_j(s\theta)s\theta^{\mathrm{T}} r_j,$$

(proved in theorem 5) to obtain:

$$s\Phi(s\theta)\int_0^\infty \mathrm{e}^{-st}\mathbb{P}\big(\theta^{\mathrm{T}}Z_0 \geqslant t\big)\,\mathrm{d}t$$
$$= \Phi(s\theta) + s\sum_{j=1}^{d}\lambda_j\big(\theta^{\mathrm{T}}r_j\big)\bigg(1 - s\int_0^\infty \mathrm{e}^{-st}\mathbb{P}_j\big(\theta^{\mathrm{T}}Z_0 \geqslant t\big)\,\mathrm{d}t\bigg).$$

On the other hand, we have from (17) that

$$\sum_{j=1}^{d}\lambda_j r_j = -\mu.$$

Combining the above we obtain the desired result (43). □

Note. We use the contents of section 4.3 about the one-dimensional Lévy network to obtain a different interpretation of (43). Fix $\theta \in \mathbb{R}_+^d$ and let $\{\widetilde{Z}_t,\ t \in \mathbb{R}\}$ be the stationary reflection of the one-dimensional process $\{\theta^{\mathrm{T}}X_t, t \in \mathbb{R}\}$, i.e., $\widetilde{Z}_t = \sup_{-\infty<u\leqslant t}(\theta^{\mathrm{T}}X_t - \theta^{\mathrm{T}}X_u)$. Let $\tilde{\psi}(\theta, s) = \int_0^\infty \mathrm{e}^{-st}\mathbb{P}(\widetilde{Z}_0 \geqslant t)\,\mathrm{d}t$ be the Laplace transform of the tail $\mathbb{P}(\widetilde{Z}_0 \geqslant t)$. Then the first two terms of the right-hand side of (43) satisfy $\Phi(s\theta) - s\theta^{\mathrm{T}}\mu = \tilde{\psi}(\theta, s)$. With $\psi(\theta, s)$, $\psi^j(\theta, s)$ being the Laplace transforms of $\mathbb{P}(\theta^{\mathrm{T}}Z_0 \geqslant t)$, $\mathbb{P}_j(\theta^{\mathrm{T}}Z_0 \geqslant t)$, respectively, (43) can be written as

$$\Phi(s\theta)\big(\psi(\theta, s) - \tilde{\psi}(\theta, s)\big) = -s\sum_j \lambda_j\big(\theta^{\mathrm{T}}r_j\big)\psi^j(\theta, s).$$

All the above hold for Φ of the form (15). Assuming now that, in addition, we have self-similarity, i.e., that Φ is of the form (25) we can obtain asymptotics for the tail probabilities $\mathbb{P}(|Z_0| \geqslant t)$ as $t \to \infty$. In the one-dimensional α-stable case ($1 < \alpha < 2$) we get from (22) and a rather direct application of a Tauberian theorem that

$$\mathbb{P}(Z_0 \geqslant t) \sim \frac{c}{a(\alpha - 1)\alpha}t^{1-\alpha} \quad \text{as } t \to \infty.$$

(In particular, we have $\mathbb{E}\,Z_0 = \infty$.) We extend this result to the case $d = 2$.

Theorem 8. Assume that $d = 2$ and let X be a Lévy process defined by its Laplace exponent $\Phi(\theta)$ as in (25) with $\sigma \equiv 0$ and $\tau \neq 0$. Let Z be a stationary solution to the Skorokhod reflection problem with reflection matrix R, assuming that (9) holds. Then there exist constants $c_1, c_2 > 0$, depending only on the parameters α, μ, R, τ of the model, such that

$$c_1 \leqslant \liminf_{x\to\infty} x^{\alpha-1}\mathbb{P}\big(|Z_0| \geqslant x\big) \leqslant \limsup_{x\to\infty} x^{\alpha-1}\mathbb{P}\big(|Z_0| \geqslant x\big) \leqslant c_2. \tag{44}$$

Proof. For $\theta \in \mathbb{R}_+^2$ and $s > 0$ we define

$$\psi(\theta, s) := \int_0^\infty \mathrm{e}^{-st}\mathbb{P}\big(\theta^{\mathrm{T}}Z_0 \geqslant t\big)\,\mathrm{d}t,$$

$$\psi_j(\theta_j, s) := \psi\big(e_j^{\mathrm{T}}\theta, s\big), \quad j = 1, 2,$$

where e_j^{T} is the unit row vector with 1 in the jth position. For simplicity we assume that $r_{12} < 0$ and $r_{21} < 0$. Put $\theta_2 = 0$ in (43). Noting that $\mathbb{P}_1(\theta_1 Z_0^1 \geqslant t) = 0$ for $t > 0$ we can solve for $\int_0^\infty \mathrm{e}^{-st}\mathbb{P}_2(\theta_1 Z_0^1 \geqslant t)\,\mathrm{d}t$ to obtain for $\theta_1 > 0$ that

$$\int_0^\infty \mathrm{e}^{-st}\mathbb{P}_2\big(\theta_1 Z_0^1 \geqslant t\big)\,\mathrm{d}t = \frac{\Phi_1(s\theta_1) - s\theta_1\mu_1}{s^2\lambda_2\theta_1 r_{12}} - \frac{\Phi_1(s\theta_1)}{s\lambda_2\theta_1 r_{12}}\psi_1(\theta_1, s), \tag{45}$$

where $\Phi_j(\theta_j) := \Phi(e_j^{\mathrm{T}}\theta)$, $j = 1, 2$. Similarly we obtain we obtain for $\theta_2 > 0$ that

$$\int_0^\infty \mathrm{e}^{-st}\mathbb{P}_1\big(\theta_2 Z_0^2 \geqslant t\big)\,\mathrm{d}t = \frac{\Phi_2(s\theta_2) - s\theta_2\mu_2}{s^2\lambda_1\theta_2 r_{21}} - \frac{\Phi_2(s\theta_2)}{s\lambda_1\theta_2 r_{21}}\psi_2(\theta_2, s). \tag{46}$$

We now take a $\theta = (\theta_1, \theta_2)$ with $\theta_1, \theta_2 > 0$, and substitute (45) and (46) in (43). We thus arrive at:

$$\psi(\theta, s) + g_1(\theta, s)\psi_1(\theta_1, s) + g_2(\theta, s)\psi_2(\theta_2, s) = g(\theta, s), \tag{47}$$

where

$$\begin{aligned}
g_1(\theta, s) &:= -\frac{\Phi_1(s\theta_1)}{\Phi(s\theta)}\frac{\theta^{\mathrm{T}} r_2}{\theta_1 r_{12}},\\
g_2(\theta, s) &:= -\frac{\Phi_2(s\theta_2)}{\Phi(s\theta)}\frac{\theta^{\mathrm{T}} r_1}{\theta_2 r_{21}},\\
g(\theta, s) &:= \frac{\Phi(s\theta) - s\theta^{\mathrm{T}}\mu}{s\Phi(s\theta)} - \frac{s\theta^{\mathrm{T}} r_2}{\Phi(s\theta)}\frac{(\Phi_1(s\theta_1) - s\theta_1\mu_1)}{s^2\theta_1 r_{12}} - \frac{s\theta^{\mathrm{T}} r_1}{\Phi(s\theta)}\frac{(\Phi_2(s\theta_2) - s\theta_2\mu_2)}{s^2\theta_2 r_{21}}.
\end{aligned}$$

From the expression (25) for Φ with $C = 0$, we easily obtain that

$$\begin{aligned}
g_1(\theta) &:= \lim_{s\to 0} g_1(\theta, s) = -\frac{\mu_1(\theta^{\mathrm{T}} r_2)}{r_{12}(\theta^{\mathrm{T}}\mu)},\\
g_2(\theta) &:= \lim_{s\to 0} g_2(\theta, s) = -\frac{\mu_2(\theta^{\mathrm{T}} r_1)}{r_{21}(\theta^{\mathrm{T}}\mu)},\\
g(\theta) &:= \lim_{s\to 0} s^{2-\alpha} g(\theta, s)\\
&= -\frac{\int(\theta^{\mathrm{T}} u)^\alpha \tau(\mathrm{d}u)}{\theta^{\mathrm{T}}\mu} + \frac{\theta^{\mathrm{T}} r_2}{\theta^{\mathrm{T}}\mu}\frac{\int(\theta_1 u_1)^\alpha \tau(\mathrm{d}u)}{\theta_1 r_{12}} + \frac{\theta^{\mathrm{T}} r_1}{\theta^{\mathrm{T}}\mu}\frac{\int(\theta_2 u_2)^\alpha \tau(\mathrm{d}u)}{\theta_2 r_{21}}
\end{aligned} \tag{48}$$

provided that $\theta^{\mathrm{T}}\mu \neq 0$. Making use of our flexibility in choosing θ we now let

$$\theta := \big(R^{\mathrm{T}}\big)^{-1} x,$$

where x has strictly positive components. Clearly, θ is strictly positive, and $\theta^{\mathrm{T}} r_j = x_j > 0$, $j = 1, 2$. On the other hand, $\theta^{\mathrm{T}}\mu = x^{\mathrm{T}} R^{-1}\mu < 0$, by stability condition (9). Since $\tau \neq 0$ we conclude that $g(\theta) > 0$. For the remainder of the proof we assume that $\mu_1 < 0$

and $\mu_2 \geqslant 0$. (The case $\mu_1 < 0$, $\mu_2 < 0$ is easier while $\mu_1 \geqslant 0$, $\mu_2 \geqslant 0$ is of course impossible.) Then $g_1(\theta) > 0$, $g_2(\theta) \leqslant 0$ and there is a constant $g_2 < 1$ such that

$$\left|g_2(\theta)\right| = \frac{x_1\mu_2}{r_{21}x^{\mathrm{T}}R^{-1}\mu} \leqslant g_2,$$

for sufficiently large x_2 (or sufficiently small x_1). We now fix such a θ. Combining the above inequalities with the obvious estimates

$$\mathbb{P}\big(\theta_i Z_0^i \geqslant x\big) \leqslant \mathbb{P}\big(\theta^{\mathrm{T}} Z_0 \geqslant x\big), \quad i = 1, 2,$$

we obtain from (47) and (48) that

$$\limsup_{s\to 0} s^{2-\alpha}\psi_i(\theta_i, s) < \infty.$$

This in turn allows to replace the coefficients $g_i(\theta, s)$ in (47) by their limits $g_i(\theta)$ and to conclude from (48) that

$$\lim_{s\to 0} s^{2-\alpha}\big[\psi(\theta, s) + g_1(\theta)\psi_1(\theta_1, s) + g_2(\theta)\psi_2(\theta_2, s)\big] = g(\theta). \tag{49}$$

The expression appearing in the brackets of (49) is the Laplace transform of the (non-negative) measure U with density

$$u(x) := \mathbb{P}\big(\theta^{\mathrm{T}} Z_0 \geqslant x\big) + g_1(\theta)\mathbb{P}\big(\theta_1 Z_0^1 \geqslant x\big) + g_2(\theta)\mathbb{P}\big(\theta_2 Z_0^2 \geqslant x\big).$$

Using a Tauberian theorem (see, e.g., [7, theorem XIII.5.2]), we obtain

$$\lim_{x\to\infty} x^{\alpha-2} U(0, x] = \frac{g(\theta)}{\Gamma(3-\alpha)}. \tag{50}$$

Observing that

$$(1 - g_2)\mathbb{P}\big(\theta^{\mathrm{T}} Z_0 \geqslant x\big) \leqslant u(x) \leqslant \big(1 + g_1(\theta)\big)\mathbb{P}\big(\theta^{\mathrm{T}} Z_0 \geqslant x\big)$$

it is now straightforward to modify the proof of the monotone density theorem (see [2, theorem 1.7.2]) to infer (44) with $|Z_0|$ replaced with $\theta^{\mathrm{T}} Z_0$, and

$$c_1 = \frac{g(\theta)}{(1 + g_1(\theta))\Gamma(2-\alpha)}, \qquad c_2 = \frac{g(\theta)}{(1 - g_2)\Gamma(2-\alpha)}.$$

Since the components of θ are strictly positive, this is enough to conclude the original assertion. □

As a side remark, it could be observed that the asymptotics of theorem 8 are extensions, to the Lévy case, of the ones for a Brownian network. In this more classical situation, the tails of the stationary distributions are exponential.

7. Remarks and open problems

The result of section 3 shows that under the natural stability condition $R^{-1}\mu < 0$, there is a stationary solution for the Skorokhod reflection problem when X is a Lévy process with $\mathbb{E}X_1 = \mu$. When X is a Brownian motion it is known [12] that the stationary solution is unique (at least in distribution). We have no such result for general Lévy processes. The methods developed by Kella and Whitt [21] for the reflection of a process X with stationary increments also do not yield uniqueness. Although we believe that uniqueness of Z (when X is a Lévy process) holds, it is still an open problem.

The non-product form result of section 5 has been shown under the assumption that each component X^i has a nontrivial jump part ($\kappa_i > 0$ for all i). We believe that a more general result holds, but have no proof of it: suppose that some components are purely Brownian. Then, in order to have product form, it is necessary and sufficient that the network splits into two disjoint parts, a part containing Brownian inputs only, and a part containing general Lévy inputs; the two parts do not communicate; the Brownian part must have a covariance structure related to the transition probabilities by means of (41), while the Lévy part is a collection of independent non-communicating nodes. It would be interesting to show that such a structure is, indeed, the case, in order that product form be obtained.

Getting hold of information on the boundary measures β_i is a key to obtaining further results on the stationary distribution. It may be possible to do so by analytical methods, and it is a challenging problem.

Our tail estimates of section 6 are rough. First, they are asymptotics on the norm of Z_0 and not on individual components; second, there is a gap between the upper and lower bound, which should be closable. We have only derived the bounds for a 2-dimensional network. It is desirable to generalize to arbitrary d. Finally, as pointed out to us by Gennady Samorodnitsky, it might be possible to derive the asymptotics for more general processes whose Lévy measures have regularly-varying tails.

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