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Palm measures and point stationarity

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joint work with

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Workshop on spatial stochastic modelling, Edinburgh

11.06.2004

Contents

1. Palm measures

Framework: We consider

(i) a measurable space $(\Omega, \mathcal{A})$ carrying all random elements;

(ii) a (simple) point process $N = \{T_n : n \in \mathbb{N}\}$ in $\mathbb{R}^d$ containing infinitely many points;

(iii) a measurable flow of bijections $\theta_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$, satisfying the *flow property*

$$\theta_x \circ \theta_y = \theta_{x+y}, \quad x, y \in \mathbb{R}^d;$$

and define

(iv)

$$N(B) := \text{card}\{n \in \mathbb{N} : T_n \in B\}, \quad B \subset \mathbb{R}^d.$$

Assumption: The point process N is *adapted* (to the flow), i.e.

$$N(\omega, B + x) = N(\theta_x \omega, B), \quad \omega \in \Omega, x \in \mathbb{R}^d, B \subset \mathbb{R}^d$$

Definition: A measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$ is called *stationary* if

$$\mathbb{P} \circ \theta_x = \mathbb{P}, \quad x \in \mathbb{R}^d.$$

In this case the (possibly infinite) number

$$\lambda_{\mathbb{P}} := \mathbb{E}_{\mathbb{P}}[N([0,1]^d)]$$

is the *intensity* of $\mathbb{P}$.

Theorem: *If* $\mathbb{P}$ *is a* σ*-finite and stationary measure on* $(\Omega, \mathcal{A})$*, then there is a unique* σ*-finite measure* $\mathbb{P}_N$ *on* $(\Omega, \mathcal{A})$ *satisfying the refined Campbell theorem*

$$\mathbb{E}_{\mathbb{P}}\left[\int f(\theta_x, x)\, N(dx)\right] = \mathbb{E}_{\mathbb{P}_N}\left[\int f(\theta_0, x)\, dx\right]$$

for all measurable $f : \Omega \times \mathbb{R}^d \to [0, \infty)$*.*

Definition: The measure $\mathbb{P}_N$ is called *Palm measure* of $\mathbb{P}$. If $\lambda_N < \infty$ then the normalized Palm measure $\lambda_N^{-1}\mathbb{P}_N$ is called Palm probability measure of $\mathbb{P}$.

2. Time- and cycle stationarity in one dimension

Assumption: Assume that $d = 1$ and

$$\ldots < T_{-1} < T_0 \leq 0 < T_1 < T_2 < \ldots.$$

Definition: A measure $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ is called *cycle-stationary* if $\mathbb{Q}(T_0 \neq 0) = 0$ and

$$\mathbb{Q} \circ \theta_{T_1} = \mathbb{Q}.$$

Theorem: *The Palm measure of a σ-finite and stationary measure on $(\Omega, \mathcal{A})$ is cycle-stationary. If, conversely, $\mathbb{Q}$ is a σ-finite and cycle-stationary measure, then there is a unique σ-finite stationary measure $\mathbb{P}$ such that $\mathbb{Q} = \mathbb{P}_N$.*

References:

E.L. Kaplan (1955) Transformations of stationary random sequences. *Math. Scand.* **3**, 127–149.

C. Ryll–Nardzewski (1961) Remarks on processes of calls. *Proc. 4th Berleley Symp. Math. Statist. Probab.* **2**, 455-465.

I.M. Slivnyak (1962) Some properties of stationary flows of homogeneous random events. *Th. Probab. Appl.* **7**, 336–341.

3. Invariance properties of Palm measures

Definition: A *point map* is a measurable mapping $\pi : \Omega \to \mathbb{R}^d$ such that $\pi \in N$ on the event $\{0 \in N\}$. A point map π is being called *bijective* if $x \mapsto \pi \circ \theta_x + x$ is a bijection on N.

Example: (Olle Häggström) Define a point map $\pi : \Omega \to \mathbb{R}^d$ by

$$\pi := \begin{cases} y, & \text{if } 0 \in N \text{ and } 0 \text{ and } y \in N \setminus \{0\} \text{ are} \\ & \text{mutual nearest neighbors in } N \\ 0, & \text{otherwise.} \end{cases}$$

This map is called *mutual nearest neighbor matching*.

Theorem: *Let $\mathbb{P}_N$ be the Palm measure of a stationary σ-finite measure $\mathbb{P}$ and $\pi : \Omega \to \mathbb{R}^d$ a bijective point map. Then*

$$\mathbb{P}_N(\theta_\pi \in \cdot) = \mathbb{P}_N.$$

References:

J. Mecke (1975) Invarianzeigenschaften allgemeiner Palmscher Maße. *Math. Nachr.* **65**, 335–344.

H. Thorisson (2000) *Coupling, Stationarity, and Regeneration.* Springer, New York.

3. Characterization of Palm measures

Definition: A measure $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ is called *point-stationary* (with respect to N) if $\mathbb{Q}(0 \notin N) = 0$ and $\mathbb{Q}$ is invariant under all $\sigma(N)$-measurable bijective point-shifts π, i.e.

$$\mathbb{Q}(\cdot) = \mathbb{Q}(\theta_\pi \in \cdot).$$

Theorem: (Heveling/L. 2004) *A measure $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ is the Palm measure of some stationary σ-finite measure $\mathbb{P}$ iff $\mathbb{Q}$ is σ-finite and point-stationary.*

Proof of the characterization:

- Mecke's intrinsic characterization of Palm measures
- construction of a nested family of bijective point maps exhausting the points of N

Theorem: (Mecke 1967) *A measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{A})$ *is the Palm measure of some stationary* σ*-finite measure* $\mathbb{P}$ *iff* $\mathbb{Q}(0 \notin N) = 0$ *and*

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_x, -x)\, N(dx)\right] = \mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_0, x)\, N(dx)\right]$$

for any measurable $f : \Omega \times \mathbb{R}^d \to [0, \infty)$.

Definition:

(i) The space of all *point configurations* in $\mathbb{R}^d$ is defined as

$$\mathbf{N} := \{\varphi \subset \mathbb{R}^d : \varphi \text{ is locally finite}\},$$

while $\mathbf{N}_0$ denotes the set of all $\varphi \in \mathbf{N}$ such that $0 \in \varphi$.

(ii) A (canoncial) *point map* is a mapping $\delta : \mathbf{N}_0 \to \mathbb{R}^d$ such that $\delta(\varphi) \in \varphi$ for any $\varphi \in \mathbf{N}_0$.

(iii) Any point map δ generates a *point shift*

$$x \mapsto \delta(\varphi - x) + x, \quad x \in \varphi,$$

on $\varphi \in \mathbf{N}$. If this mapping is bijective for any $\varphi \in \mathbf{N}$, then the point map δ is called *bijective*.

Proposition: *There exist bijective point maps* $\delta_{m,n} : \mathbf{N}_0 \to \mathbb{R}^d$, $m, n \in \mathbb{N}$, *such that the following two properties hold for any* $\varphi \in \mathbf{N}$ *and* $x \in \varphi$:

(i) *For all* $m \in \mathbb{N}$ *and all* $n, n' \in \mathbb{N}$ *with* $n \neq n'$,

$$\{\delta^i_{m,n}(\varphi, x) : i \in \mathbb{Z}\} \cap \{\delta^i_{m,n'}(\varphi, x) : i \in \mathbb{Z}\} = \{x\},$$

(ii) $$\bigcup_{m=1}^{\infty} \{\delta^i_{m,n}(\varphi, x) : n \in \mathbb{N}, i \in \mathbb{Z}\} = \varphi.$$

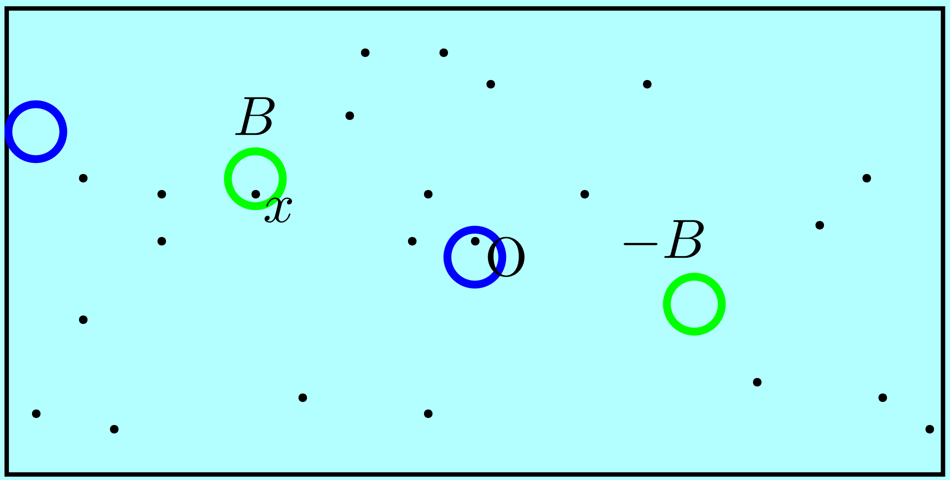
5. Construction of bijective point shifts

Symmetric area search based on a (small) set $B \subset \mathbb{R}^d \setminus \{0\}$:

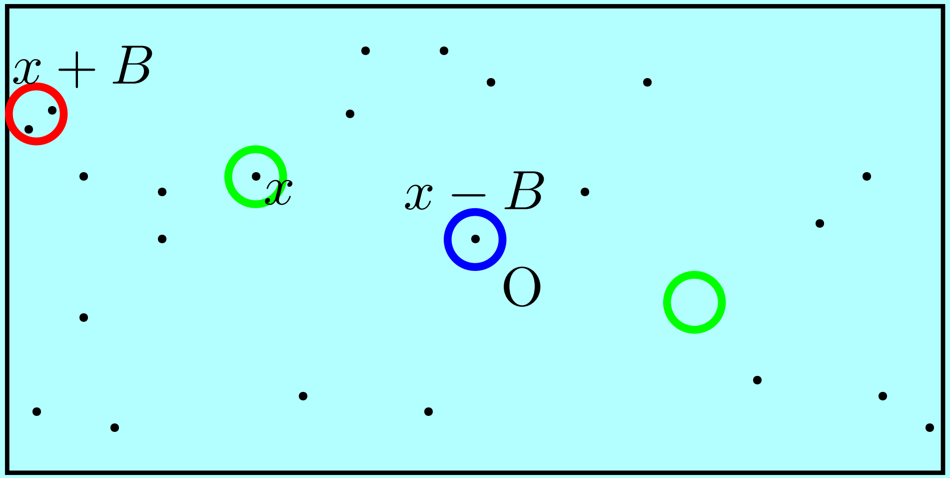
Take $\varphi \in \mathbf{N}_0$. If

$$\varphi \cap (B \cup -B) = \{x\},$$
$$\varphi \cap ((x + B) \cup (x - B)) = \{0\},$$

then we define $\delta(\varphi) := x$. Otherwise we let $\delta(\varphi) := 0$.

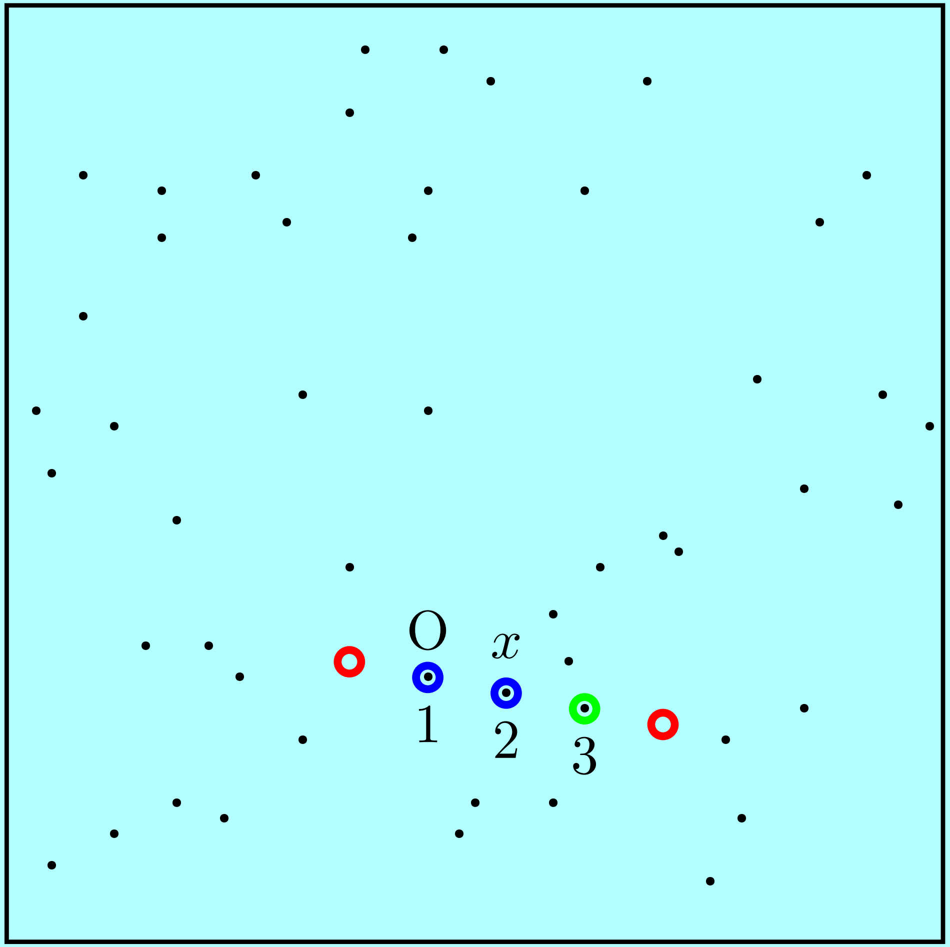


Symmetric area search that fails, i.e. $\delta(\varphi) = 0$:

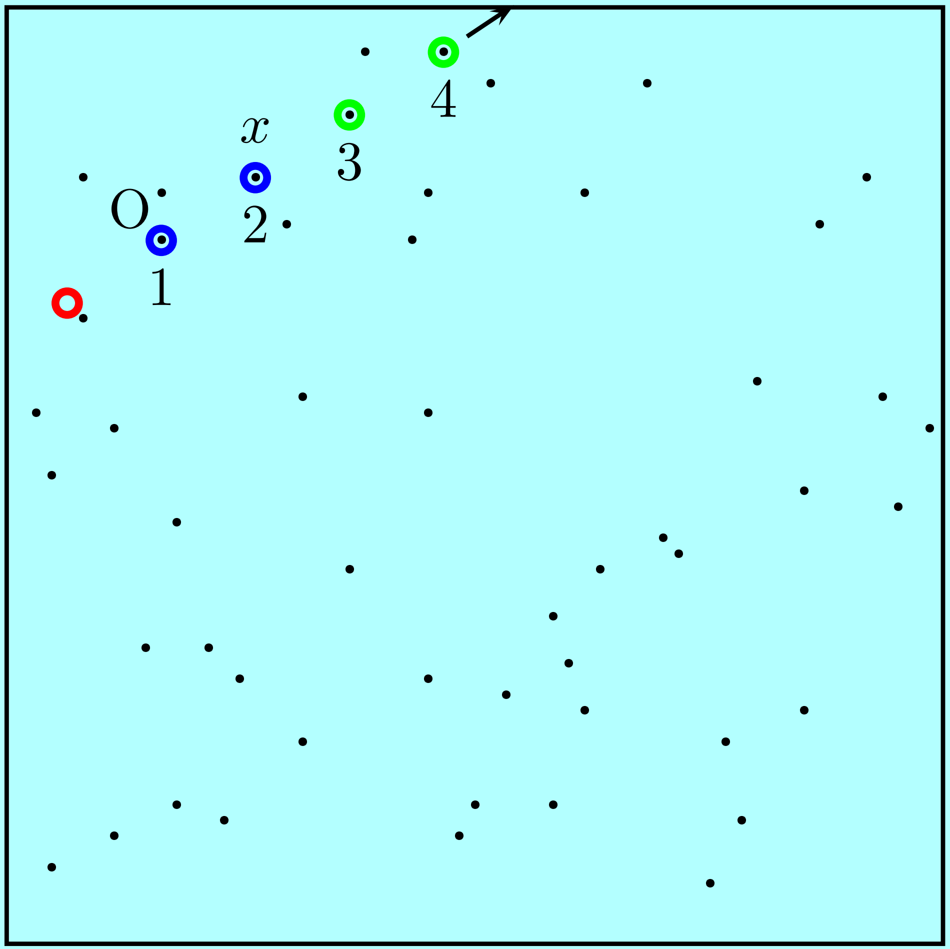


Conclusion: Symmetric area search detects all points in $\varphi \setminus \{0\}$, provided that there is no line passing through 0 that contains 3 or more points of φ.

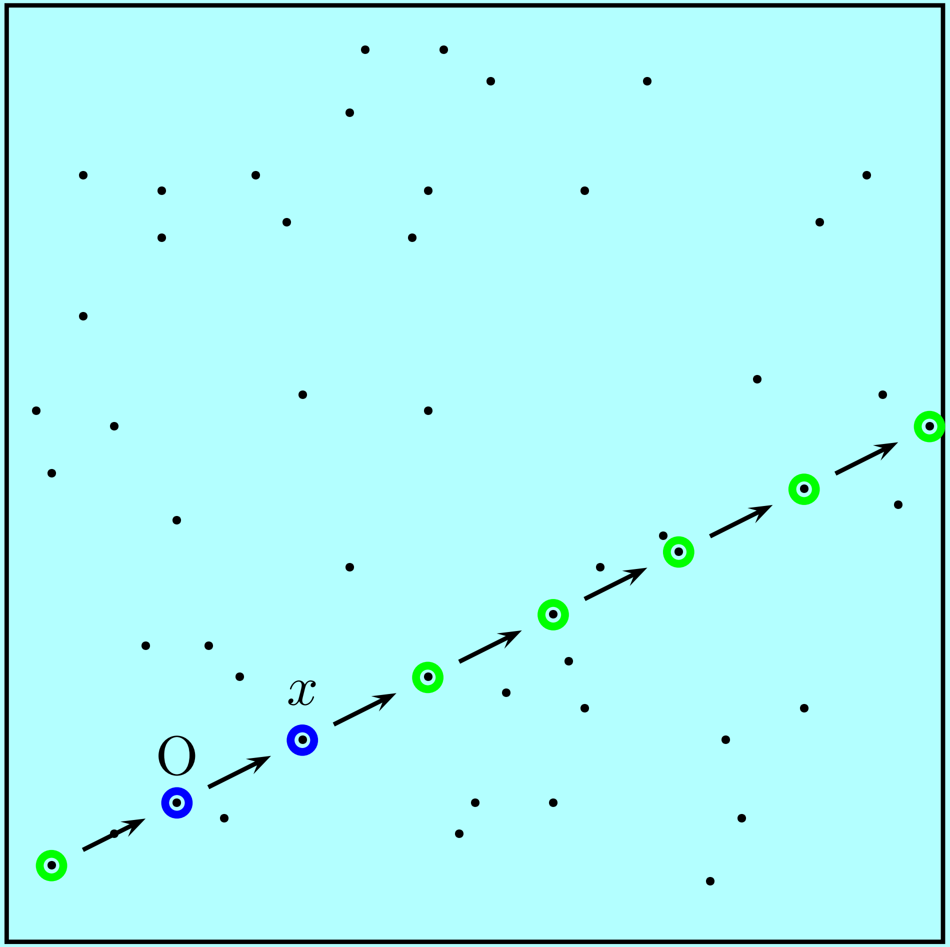
Finite chains:



One-sided infinite chain:



Two-sided infinite chain:



5. Translation invariant graphs

Definition: Let $\delta : \mathbf{N}_0 \to \mathbb{R}^d$ be a bijective point map. Drawing a directed line from any $x \in \varphi$ to $\delta(\varphi, x)$ equips $\varphi \in \mathbf{N}$ with the structure of a *directed graph* $G_\delta(\varphi)$ with vertex set φ. In case $\delta(\varphi, x) = x$ the point x is *isolated* in $G_\delta(\varphi)$.

Remark: The graph $G_\delta(\varphi)$ is constructed in a translation invariant way. If there is a directed line from x to x' in $G_\delta(\varphi)$ then there is a directed line from $x - y$ to $x' - y$ in $G_\delta(\varphi - y)$.

Theorem: (Heveling/L. 2004) *There is a bijective point map* $\delta : \mathbf{N}_0 \to \mathbb{R}^d$ *having the following property. For any stationary probability measure* $\mathbb{P}$ *on* $(\Omega, \mathcal{A})$ *the components of the directed graph* $G_\delta(N)$ *are almost everywhere with respect to both* $\mathbb{P}$ *and* $\mathbb{P}_N$ *doubly infinite paths.*

Theorem: (Holroyd/Peres 2003) *Let* $\mathbb{P}$ *be a stationary probability measure on* $(\Omega, \mathcal{A})$ *such that* N *is* $\mathbb{P}$*-a.s. non-equidistant, ergodic and of finite intensity. Then there is a directed graph* G *with vertex set* N *that is constructed in a deterministic isometry-invariant way and whose components are* $\mathbb{P}$*-a.s. doubly infinite paths.*

Theorem: (Timar 2004) *Let $\mathbb{P}$ be a stationary probability measure on $(\Omega, \mathcal{A})$ such that N is $\mathbb{P}$-a.s. non-equidistant, ergodic and of finite intensity. Then there is a bijective point shift $\delta : \mathbf{N}_0 \to \mathbb{R}^d$ such that*

$$N = \{\delta^n(N, 0) : n \in \mathbb{Z}\} \quad \mathbb{P}_N - a.s.$$

Further references:

P.A. Ferrari, C. Landim, H. Thorisson (2004). Poisson trees, succession lines and coalescing random walks. to appear in *Annales de l'Institut Henri Poincaré (B), Probability and Statistics.*

A.E. Holroyd, Y. Peres (2003). Trees and matchings from point processes. *Electronic Comm. Probab.* **8**, 17–27.