Günter Last
Institut für Mathematische Stochastik
Universität Karlsruhe (TH)

On Boolean models with general compact particles

Günter Last

joint work with

Daniel Hug (Freiburg) and Wolfgang Weil (Karlsruhe)

Gaussian Measure and Geometric Convexity, Snowbird

18.07.2004

Contents

2. Steiner-formulae for convex bodies

Definition: Let $A \subset \mathbb{R}^d$ be a convex body and consider its *parallel sets*

$$A_{\oplus r} := \{x \in \mathbb{R}^d : d(x, A) \leq r\}, \quad r \geq 0.$$

Then

$$V_d(A_{\oplus r}) = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(A),$$

where $V_0(A), \ldots, V_d(A)$ are the *intrinsic volumes* of A. Here, κ_j is the (j-dimensional) volume of the Euclidean unit ball in $\mathbb{R}^j$.

Definition: For a closed set $A \subset \mathbb{R}^d$ we write $p(A, z) := y$ whenever y is a uniquely determined point in A with $d(A, z) = |y - z|$. This is the *metric projection* of z on to A. If $d(A, z) > 0$ we define

$$u(A, z) := \frac{z - p(A, z)}{d(A, z)} \in S^{d-1}.$$

Definition: Let $A \subset \mathbb{R}^d$ be a convex body. Then

$$N(A) := \{(p(A, z), u(A, z)) : z \notin A\}$$

is called *normal bundle* of A.

Theorem: *Let $A \subset \mathbb{R}^d$ be a convex body. There exist uniquely determined finite measures $C_0(A;\cdot),\ldots,C_{d-1}(A;\cdot)$ on $N(A)$ satisfying the local Steiner-formula*

$$\int_{\mathbb{R}^d\setminus A} f(x)dx = \sum_{i=0}^{d-1}(d-i)\kappa_{d-i}\int_0^\infty\int_{N(A)} t^{d-1-i} \times f(x+tu)C_i(A;d(x,u))dt$$

for any measurable function $f:\mathbb{R}^d\to[0,\infty)$.

Definition: The measures $C_0(A;\cdot),\ldots,C_{d-1}(A;\cdot)$ are called (generalized) *curvature measures* of A.

2. Support measures of a general closed set $A \subset \mathbb{R}^d$

Prerequisites:

- exoskeleton

$$\operatorname{exo}(A) := \{z \in \mathbb{R}^d \setminus A : p(A, z) \text{ does not exist}\},$$

- normal bundle

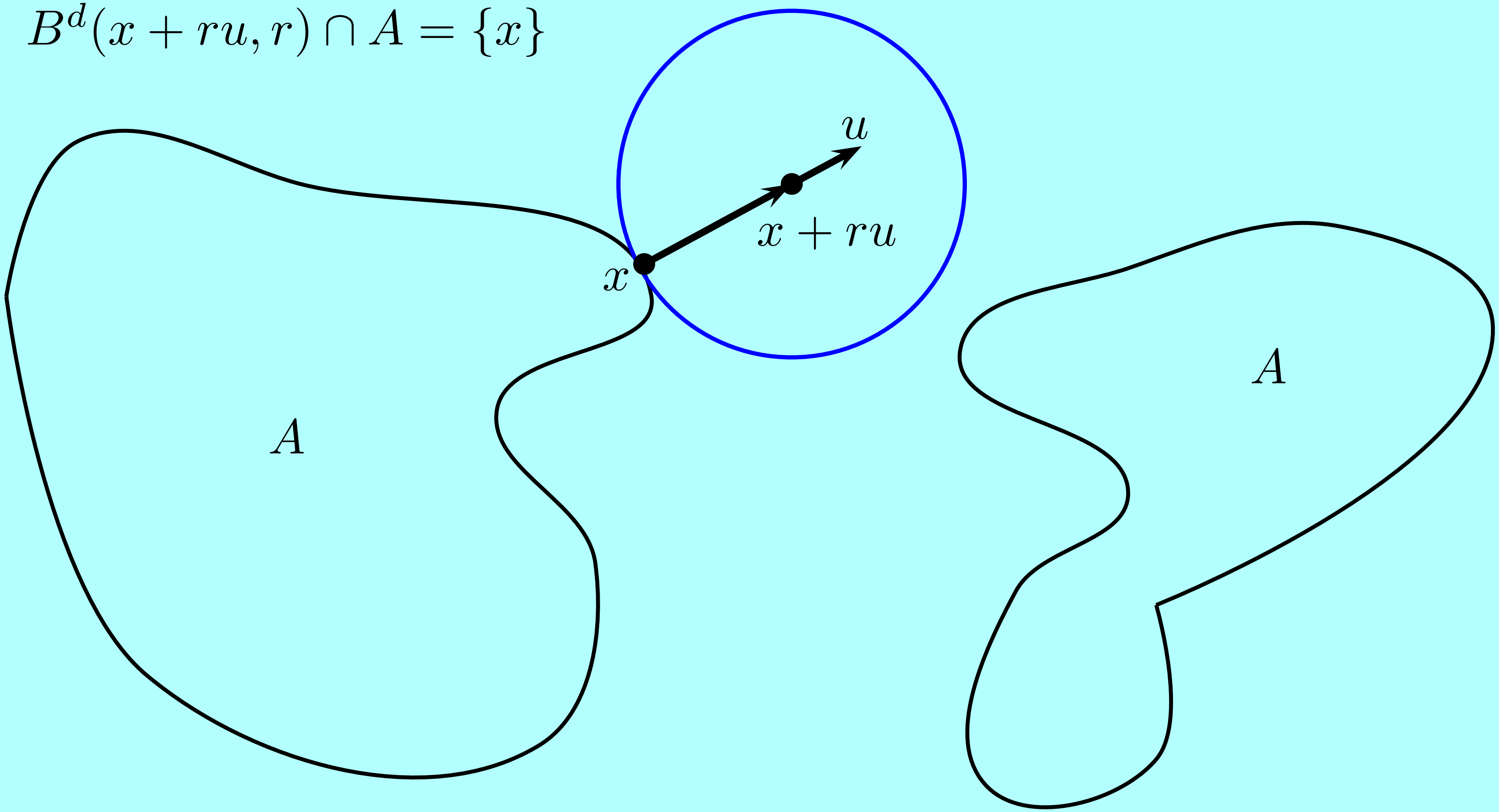
$$N(A) := \{(p(A, z), u(A, z)) : z \notin (A \cup \operatorname{exo}(A))\},$$

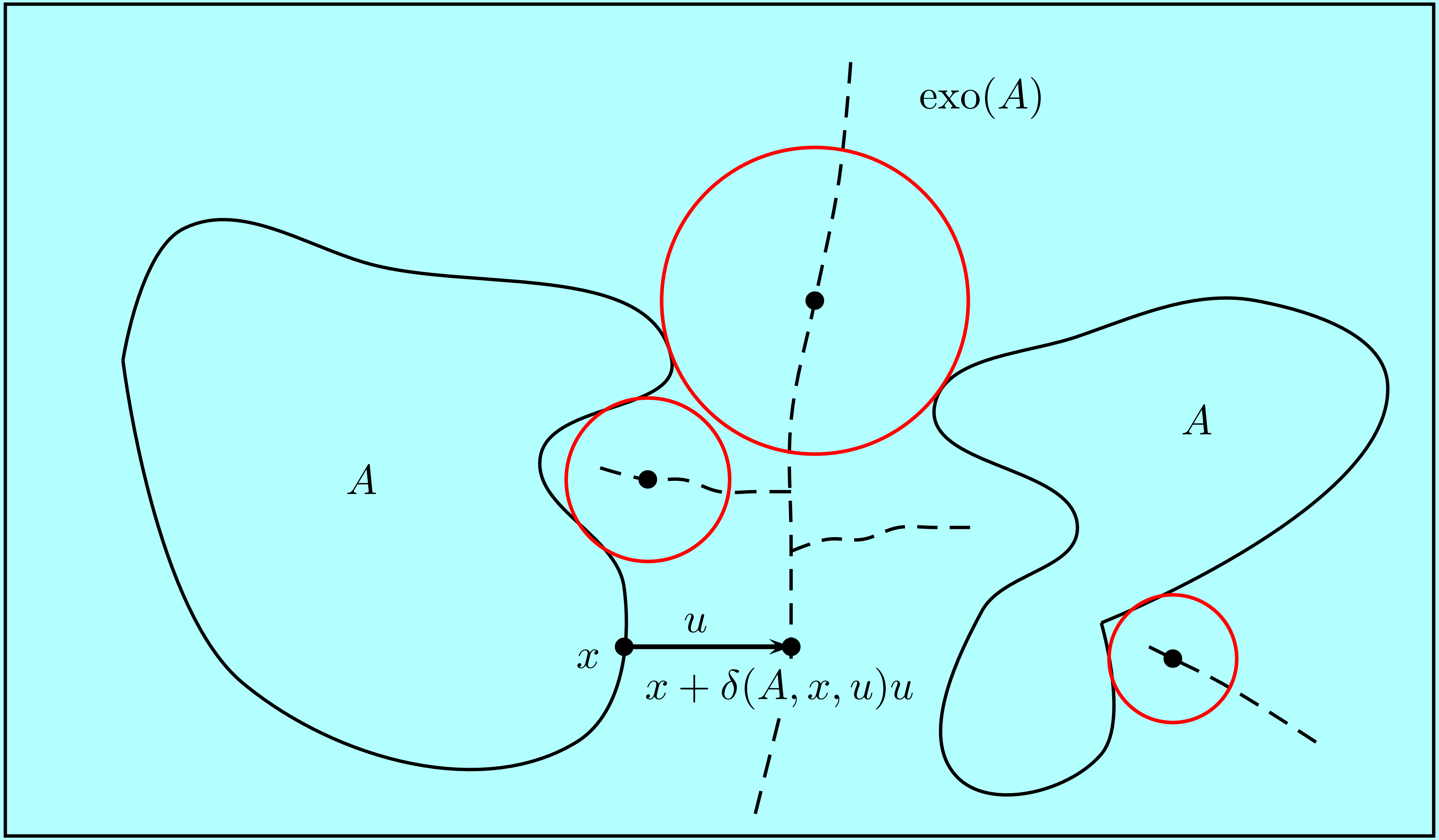
- reach function $\delta(A, \cdot) : N(A) \to [0, \infty]$.

Proposition: *The exoskeleton of A has Lebesgue measure* 0.

$(x, u) \in N(A)$=normal bundle of $A \subset \mathbb{R}^d$

$B^d(x + ru, r) \cap A = \{x\}$




exo(A)
A
A
u
x
$x + \delta(A, x, u)u$

Theorem: *There exist uniquely determined signed measures $\mu_0(A;\cdot),\ldots,\mu_{d-1}(A;\cdot)$ on $N(A)$ satisfying*

$$\int_{N(A)} \mathbf{1}\{x \in B\}(\delta(A,x,u) \wedge r)^{d-j}|\mu_j|(A;d(x,u)) < \infty,$$

$j = 0,\ldots,d-1$, for all compact sets $B \subset \mathbb{R}^d$ and all $r > 0$, such that, for any measurable bounded function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support,

$$\int_{\mathbb{R}^d \setminus A} f(x)dx = \sum_{i=0}^{d-1}(d-i)\kappa_{d-i}\int_0^\infty \int_{N(A)} t^{d-1-i}\mathbf{1}\{t < \delta(A,x,u)\} f(x+tu)\mu_i(A;d(x,u))dt.$$

Definition: The measures $\mu_0(A;\cdot),\dots,\mu_{d-1}(A;\cdot)$ are called *support measures* of A. They are extended to $\mathbb{R}^d \times S^{d-1}$ by letting $\mu_i(\mathbb{R}^d \times S^{d-1} \setminus N(A)) = 0$.

Fact: *Assume that A is a convex body. Then* $\operatorname{exo}(A) = \emptyset$, $\delta(A;\cdot) \equiv \infty$ *and* $\mu_0(A;\cdot),\dots,\mu_{d-1}(A;\cdot)$ *are the curvature measures of A.*

Proposition: *Assume that A a (locally) finite union of convex bodies. Let $C_i(A;\cdot)$ be the* additive *extension of the i-th curvature measure. Then $\mu_i(A;\cdot)$ is the restriction of $C_i(A;\cdot)$ to the normal bundle $N(A)$.*

Some references:

L.L. Stachó, On curvature measures, *Acta Sci. Math.* **41** (1979), 191–207.

M. Zähle, Integral and current representation of Federer's curvature measures, *Arch. Math.* **46** (1986), 557–567.

D. Hug, G. Last, and W. Weil, A local Steiner–type formula for general closed sets and applications. *Mathematische Zeitschrift* **246**, (2004), 237–272.

Generalized principal curvatures:

$k_1(A,x,u),\dots,k_{d-1}(A,x,u) \in (-\infty,\infty]$, $(x,u) \in N(A)$.

Define

$$H_j(A,x,u) := \prod_{i=1}^{d-1}(1+k_i(A,x,u)^2)^{-1/2} \sum_{|I|=j} \prod_{l\in I} k_l(A,x,u),$$

where the summation extends over all subsets $I \subset \{1,\dots,d-1\}$ of cardinality j. Then

$$\mu_i(A;\cdot) = \frac{1}{(d-i)\kappa_{d-i}} \int_{N(A)} \mathbf{1}\{(x,u)\in\cdot\} H_{d-1-i}(A,x,u)\mathcal{H}^{d-1}(d(x,u)).$$

3. The Boolean model of Stochastic Geometry

Definition: Assume that the random points ξ_n, $n \in \mathbb{N}$, form a stationary Poisson process Φ in $\mathbb{R}^d$ with positive and finite intensity γ. Assume further that $Z_1, Z_2, \ldots$ is a sequence of independent, identically distributed random compact subsets of $\mathbb{R}^d$, independent of Φ. Then

$$Z := \bigcup_{n \in \mathbb{N}} (Z_n + \xi_n),$$

is called *Boolean model*. If

$$\mathbb{E}[V_d(Z_1 + B^d)] < \infty$$

then Z is (almost surely) a closed set. This assumption will be made throughout the talk.

Fact: *The* capacity functional *of a Boolean model Z is given by*

$$\mathbb{P}(Z \cap B \neq \emptyset) = 1 - \exp\left[-\gamma \mathbb{E}[V_d(Z_1 + (-B))]\right], \qquad (1)$$

for all Borel sets $B \subset \mathbb{R}^d$. In particular, the volume fraction $p := \mathbb{P}(0 \in Z)$ *of Z is given by*

$$p = 1 - \exp\left[-\gamma \mathbb{E}[V_d(Z_1)]\right] < 1.$$

Fact: *The Boolean model Z is* stationary, *i.e.*

$$Z + x \stackrel{d}{=} Z, \quad x \in \mathbb{R}^d.$$

Definition: Let $Z \subset \mathbb{R}^d$ be a stationary random closed set. Then

$$H(t, C) := \mathbb{P}(d(Z, 0) \le t, u(Z, 0) \in C \,|\, 0 \notin Z), \quad C \subset S^{d-1},$$

is called *direction dependent contact distribution* of Z.

Theorem: *The direction dependent contact distribution function $H(\cdot, C)$ of a Boolean model Z is absolutely continuous with density*

$$t \mapsto (1 - H(t, S^{d-1}))\gamma \sum_{i=0}^{d-1} (d-i)\kappa_{d-i} t^{d-1-i}$$
$$\mathbb{E}\left[\int \mathbf{1}\{t < \delta(Z_1, x, u)\}\mathbf{1}\{u \in C\}\mu_i(Z_1; d(x, u))\right].$$

4. Intensities of the Boolean model

Definition: For any closed set $A \subset \mathbb{R}^d$, $i \in \{0, \ldots, d-1\}$ and $t \geq 0$ we define a signed Radon measure $\mu_{i,t}(A; \cdot)$ on $\mathbb{R}^d \times S^{d-1}$ by

$$\mu_{i,t}(A; \cdot) := \int \mathbf{1}\{(x,u) \in \cdot\}\mathbf{1}\{\delta(A,x,u) > t\}\mu_i(A; d(x,u)).$$

Proposition: *For any random closed set $X \subset \mathbb{R}^d$ the mappings*

$$\mathbb{E}[\mu_{(i,t)}(X; \cdot)], \quad i = 0, \ldots, d-1, \, t > 0,$$

are signed Radon measures on $\mathbb{R}^d \times S^{d-1}$.

Fact: *Assume that $Z \subset \mathbb{R}^d$ is a stationary random closed set. Let $i \in \{0, \dots, d-1\}$ and $t > 0$. Then there is a finite signed measure $\lambda_{i,t,Z}$ on S^{d-1} such that*

$$\mathbb{E}[\mu_{i,t}(Z; B \times \cdot)] = V_d(B)\lambda_{i,t,Z}(\cdot)$$

for any Borel set $B \subset \mathbb{R}^d$. If $\mathbb{E}[\mu_i(Z; \cdot)]$ is of locally finite variation then this result extends to $t = 0$ and we write $\lambda_{i,Z} := \lambda_{i,0,Z}$

Definition: The number $\lambda_{i,t,Z}(S^{d-1})$ is the *intensity* of the random signed Radon measure $\mu_{i,t}(Z; \cdot \times S^{d-1})$ while $\lambda_{i,t,Z}(C)$ is the intensity of the random signed Radon measure $\mu_{i,t}(Z; \cdot \times C)$.

The following theorem generalizes a classical result by Matheron (1975) on Boolean models with convex grains:

Theorem: *Let Z be a Boolean model. Then we have for any $i \in \{0, \dots, d-1\}$ and any $t > 0$ that*

$$\lambda_{i,t,Z}(\cdot) = \gamma \mathbb{P}(d(0, Z) > t) \mathbb{E}[\mu_{(i,t)}(Z_1; \mathbb{R}^d \times \cdot)].$$

If $\mathbb{E}[\mu_i(Z_1; \cdot)]$ is for any $i \in \{0, \dots, d-1\}$ of locally finite variation, then this equation extends to $t = 0$, i.e.

$$\lambda_{i,Z}(\cdot) = \gamma(1 - p)\mathbb{E}[\mu_i(Z_1; \mathbb{R}^d \times \cdot)].$$

5. The Wiener sausage

$(B_t)_{t\geq 0}$ = standard Brownian motion in $\mathbb{R}^3$

$Z := \{B_t : 0 \leq t \leq 1\}$ = path of the motion between 0 and 1.

Facts: *The random closed set Z has almost surely the Hausdorff dimension 2. The two-dimensional Hausdorff measure of Z is almost surely* 0.

Definition: Let $r > 0$. The parallel set

$$Z_{\oplus r} = \{x \in \mathbb{R}^d : d(x, Z) \leq r\}$$

is called *Wiener sausage.*

Theorem: (Spitzer, 1964) *The expected volume of the Wiener sausage is given by*

$$\mathbb{E}[V_3(Z_{\oplus r})] = 4\pi r + 4\sqrt{2\pi}r^2 + \frac{4}{3}\pi r^3.$$

By the general Steiner formula we have

$$\mathbb{E}[V_3(Z_{\oplus r})] = \sum_{i=0}^{2} \kappa_{3-i}\mathbb{E}\left[\int (\delta(Z,x,u)^{3-i} \wedge r^{3-i})\mu_i(Z;d(x,u))\right].$$

Since almost surely

$$\mu_2(Z;\cdot) = 0.$$

we have

$$\mathbb{E}[V_3(Z_{\oplus r})] = \sum_{i=0}^{1} \kappa_{3-i}\mathbb{E}\left[\int (\delta(Z,x,u)^{3-i} \wedge r^{3-i})\mu_i(Z;d(x,u))\right].$$

and using the fact (to be proved) that

$$\int \mathbf{1}\{\delta(Z,x,u)=\infty\}\mu_0(Z;d(x,u)) = 1$$

we obtain that

$$\begin{aligned}\mathbb{E}[V_3(Z_{\oplus r})] =& \pi r^2 \mathbb{E}\left[\int \mathbf{1}\{\delta(Z,x,u)=\infty\}\mu_1(Z;d(x,u))\right]\\ &+ \pi\mathbb{E}\left[\int \mathbf{1}\{\delta(Z,x,u)<\infty\}(\delta(Z,x,u)^2\wedge r^2)\mu_1(Z;d(x,u))\right]\\ &+ \frac{4}{3}\pi r^3\\ &+ \frac{4}{3}\pi\mathbb{E}\left[\int \mathbf{1}\{\delta(Z,x,u)<\infty\}(\delta(Z,x,u)^3\wedge r^3)\mu_0(Z;d(x,u))\right].\end{aligned}$$

It might be conjectured that

$$\mathbb{E}\left[\int \mathbf{1}\{\delta(Z,x,u)<\infty\}(\delta(Z,x,u)^2\wedge r^2)\mu_1(Z;d(x,u))\right] = 0.$$

Conjecture: *We have that*

$$\mathbb{E}\left[\int \mathbf{1}\{\delta(Z,x,u)=\infty\}\mu_1(Z;d(x,u))\right]=\frac{4\sqrt{2}}{\sqrt{\pi}}$$

and

$$\mathbb{E}\left[\int \mathbf{1}\{s<\delta(Z,x,u)<\infty\}\mu_0(Z;d(x,u))\right]=s^{-2},\quad s>0.$$