DETERMINATION OF A CONVEX BODY
FROM MINKOWSKI SUMS OF ITS PROJECTIONS

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Abstract

For a convex body $K$ in $\mathbb{R}^d$ and $1 \leq k \leq d - 1$, let $P_k(K)$ be the Minkowski sum (average) of all orthogonal projections of $K$ onto $k$-dimensional subspaces of $\mathbb{R}^d$. It is known that the operator $P_k$ is injective if $k \geq d/2$, $k = 3$ for all $d$, and if $k = 2, d \neq 14$.

We will show that $P_2(K)$ determines a convex body $K$ among all centrally symmetric convex bodies and $P_{k+1}(K)$ determines a convex body $K$ among all bodies of constant width. Corresponding stability results will also be given. Furthermore, it will be shown that any convex body $K$ is determined by the two sets $P_k(K)$ and $P_k'(K)$ if $1 < k < k'$. Concerning the range of $P_k$, $1 \leq k \leq d - 2$, we show that its closure (in the Hausdorff-metric) does not contain any polytopes other than singletons.

1. Introduction and main results

Let $K$ be a convex body (a non-empty, compact convex set) in $\mathbb{R}^d$. We consider the question whether $K$ is determined by certain means of its orthogonal projections on subspaces. To make this precise, let $L^k_d$, $1 \leq k \leq d$, denote the compact manifold of all $k$-dimensional linear subspaces of $\mathbb{R}^d$. The unique rotation invariant probability measure on $L^k_d$ is denoted by $\nu_k$. Let $K^d$ be the family of all convex bodies in $\mathbb{R}^d$, $h(K, \cdot)$ the support function of $K \in K^d$ and $K|L$ its orthogonal projection on $L \in L^k_d$. Throughout the following, we think of $h(K, \cdot)$ as a function on the unit sphere $S^{d-1}$ of $\mathbb{R}^d$. The mapping on $S^{d-1}$ given by

$$u \mapsto \int_{L^k_d} h(K|L, u) \nu_k(dL)$$

is a support function of a convex body $P_k(K)$. (We refer the reader to Schneider’s book [13] for details on support functions and convex geometry in general). Hence

$$h(P_k(K), u) = \int_{L^k_d} h(K|L, u) \nu_k(dL), \quad u \in S^{d-1}. \quad (1.1)$$

gives rise to a mapping $P_k : K^d \rightarrow K^d$, which will be called $k$-dimensional Minkowski average (or Minkowski sum) of $K$ in the following. Note that the case $k = d$ is only included for technical reasons; it leads trivially to $P_d(K) = K$. The Minkowski average $P_{d-1}$ was first defined and studied by Schneider [12]. The question whether all convex bodies $K$ in $K^d$ are uniquely determined by their Minkowski averages $P_k(K)$, is obviously an injectivity problem for $P_k$.

Spriestersbach [16] showed that $P_{d-1}$ is injective for all $d \geq 3$. Goodey [4] extended her arguments and proved that the members of $K^d$ are determined by any of their $k$-dimensional Minkowski averages, whenever $k \geq d/2$. He also discussed the cases $k = 1$ and $k = 2:

2000 Mathematics Subject Classification 52A20 (primary) 33C55, 33C90 (secondary).
For $k = 1$, a convex body $K \in \mathbb{K}^d$ is determined by $P_1(K)$ among all centrally symmetric convex bodies. If $K$ is not centrally symmetric, a translate of its central symmetral $\frac{1}{2}(K + (-K))$ has the same image under $P_1$, and hence $K$ is not determined by $P_1(K)$. For $k = 2$, all convex bodies $K \in \mathbb{K}^d$ are determined by $P_2(K)$ if $d \neq 14$. In $\mathbb{R}^{14}$, however, $P_2$ is not injective: if $K \in \mathbb{K}^{14}$ has sufficiently smooth boundary and positive radii of curvature at all boundary points, then there is a convex body $L \neq K$ such that $P_2(L) = P_2(K)$. The case $k = 3$ was treated by Goodey & Jiang [6]. They showed that $P_3$ is injective for all $d \geq 4$.

In the present paper we will show that the restrictions of $P_k$ to certain subclasses $\tilde{K}^d$ of $\mathbb{K}^d$ are injective. This injectivity means that a convex body $K \in \tilde{K}^d$ is uniquely determined by $P_k(K)$ among all bodies in $\tilde{K}^d$: if $P_k(K) = P_k(L)$, $L \in \tilde{K}^d$, then $K = L$ follows. Besides the class of centrally symmetric bodies, we will consider the class of bodies of constant width, i.e. those bodies whose central symmetral is a ball. We will also give a stability result in terms of the $L_2$-metric $\delta_2$ of $\mathbb{K}^d$ given by

$$\delta_2(K, M) := \| h(K; \cdot) - h(M, \cdot)\|_2, \quad L, M \in \mathbb{K}^d,$$

where $\| \cdot \|_2$ is the usual $L_2$-norm of square integrable functions on the sphere. This metric comes in naturally when working with spherical harmonic expansions of support functions. It is also possible to give a corresponding result in the Hausdorff metric, see section 4.

**Theorem 1.**

a) Assume that $2 \leq k \leq d - 1$ is even. Then $P_k$ is injective on the class of centrally symmetric convex bodies. For all $R > 0$ there is a constant $c > 0$ such that

$$\delta_2(K, M) \leq c \cdot \delta_2(P_k(K), P_k(M))^{\frac{1}{d+1}}$$

holds for all centrally symmetric convex bodies contained in $RB^d$ ($B^d$ being the unit ball in $\mathbb{R}^d$).

b) Assume that $3 \leq k \leq d - 1$ is odd. Then $P_k$ is injective on the class of bodies of constant width. For all $R > 0$ there is a constant $c > 0$ such that

$$\delta_2(K, M) \leq c \cdot \delta_2(P_k(K), P_k(M))^{\frac{1}{d+1}}$$

holds for all bodies of constant width contained in $RB^d$.

Besides injectivity, Sprîiestabalîs [16] has shown a stability result (on all of $\mathbb{K}^d$) for $P_k$ with $k = d - 1$. It appears that in the displayed formula of her Theorem 2, an exponent $1/2$ is missing on the right hand side. With this modification, her stability result coincides with the one above (for suitable classes of convex bodies depending on the parity of $d - 1$).

As the injectivity of $P_k$ in $\mathbb{R}^d$ is still an open problem for many parameter tuples $(k, d)$, it is of interest whether $K$ is determined (in $\mathbb{K}^d$) by a pair of Minkowski–averages $(P_k(K), P_{k'}(K))$ with $k \neq k'$. We will show that this is true if the cases $k = 1$ and $k' = 1$ are excluded.

**Theorem 2.** Any convex body $K$ is determined (among all convex bodies) by the two bodies $P_k(K)$ and $P_{k'}(K)$, if $k, k' \in \{2, \ldots, d - 1\}$, $k \neq k'$.

The proofs of Theorems 1 and 2 rely on the fact that $P_k : \mathbb{K}^d \to \mathbb{K}^d$ is a contin-
uous, positive homogeneous and Minkowski-additive mapping that commutes with rotations (fixing the origin 0). These properties allow the use of tools from harmonic analysis.

Goodey and Jiang [6] showed that the range $P_k(K^d)$ of $P_k$ is closed. They constructed convex bodies which are not in the range of $P_k$, $1 \leq k \leq d-1$, showing that the range is a proper subset of $K^d$. This can also be seen more directly: Assume that $0 \in K$ with $\{0\} \neq K$. Then there is an $x \in \mathbb{R}^d \setminus \{0\}$ such that the line segment $[0,x]$ with endpoints 0 and $x$ lies in $K$. Rescaling of $K$ allows us to assume that $\|x\| = 1$. By definition of $P_k$ we conclude

$$h(P_k(K), u) \geq h(P_k([0,x]), u) = \int_{\mathcal{L}_k^d} \max\{0, \langle x|L,u \rangle \} \nu_k(dL), \quad (1.2)$$

where $\langle \cdot , \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$. If $u \neq -x$ then the open set $\{L \in \mathcal{L}_k^d | \langle x|L,u \rangle > 0 \}$ is nonempty (it contains for example any $L \in \mathcal{L}_k^d$ with $(x+u)/2 \in L \subset (x-u)^\perp$). Hence this set has positive $\nu_k$-measure and the integral on the right hand side of (1.2) is positive for all $u \neq -x$. This implies that $P_k(K)$ has interior points. As translations and dilations of $K$ only affect the position and the size of $P_k(K)$, this argument shows that $P_k(K^d)$ only contains singletons and $d$-dimensional convex bodies. In fact, not all $d$-dimensional convex bodies are members of $P_k(K^d)$, which is our third result.

**Theorem 3.** For $k \in \{1, \ldots, d-2\}$ the range $P_k(K^d)$ of the Minkowski sum $P_k$ does not contain any polytopes (other than singletons).

In the next section, a more explicit representation of $P_k$ as an integral transform will be derived. This is used in section 3 to show that $P_{k-2}$, $P_k$ and $P_{k+2}$ are closely related, which can be made explicit with the use of the Laplace Beltrami operator. This connection is fundamental for the proofs of our main results. As a consequence of this connection, the first surface area measure of $P_k(K)$, $2 \leq k \leq d-2$, is shown to have a density which can be written as a linear combination of the support functions of $P_{k-2}$, $P_k$ and $P_{k+2}$, see Corollary 1. This will imply Theorem 3. Only from section 4 on, spherical harmonics are used as a tool. A combination of the results of section 3 with the (known) multiplier property of $P_k$ will show the injectivity and stability results described above. In section 5 we will give a proof of Theorem 2. Finally, section 6 discusses the possibility to show the injectivity of $P_k$ on $K^d$ for a given pair $(k,d)$ based on known results and methods as presented in this paper.

2. The operator $p_k$

Goodey [4] defines a continuous linear operator

$$p_k : C(S^{d-1}) \to C(S^{d-1})$$

on the Banach space $C(S^{d-1})$ of continuous functions on the unit sphere. This is the functional equivalent of $P_k$, defined in such a way that

$$h(P_k(K), \cdot) = p_k(h(K, \cdot)), \quad K \in K^d. \quad (2.1)$$

As differences of support functions lie dense in $C(S^{d-1})$, the endomorphism $p_k$ is uniquely defined by (2.1).

Goodey extends $p_k$ to a continuous linear operator on $L_2(S^{d-1})$. This extension
turns out to be an endomorphism of $L_2(S^{d-1})$. It intertwines the group action of $SO_d$ on $L_2(S^{d-1})$. (The action of the rotation group $SO_d$ on a rotation invariant space of spherical functions is given by $\vartheta f := f \circ \vartheta^{-1}$, $\vartheta \in SO_d$.) Therefore methods of harmonic analysis can be applied to tackle the injectivity problem. We will use spherical harmonics later to discuss injectivity. Before this, we show a more explicit of harmonic analysis can be applied to tackle the injectivity problem. We will use spherical harmonics later to discuss injectivity. Before this, we show a more explicit

representation of $p_k$ and derive properties of $P_k$ from it without using spherical harmonics. Some of the arguments below could be shortened using spherical harmonics. We prefer, however, to give elementary proofs without their use. The following result appeared implicitly in the above mentioned papers, see in particular [12] for the case $k = d - 1$.

If not stated otherwise, integration on $S^{d-1}$ is always understood with respect to the spherical Lebesgue-measure with total mass

$$\omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}.$$

**Proposition 1.** For all $k \in \{1, \ldots, d-1\}$ and $f \in C(S^{d-1})$ we have

$$(p_k f)(u) = c_{k,d} \int_{S^{d-1} \cap S^1_u} \left(\frac{\langle u, v \rangle}{1 - \langle u, v \rangle^2}\right)^{k/2} f(v) \, dv,$$

(2.2)

where $c_{k,d} := \frac{\omega_{d-k}}{\omega_d \omega_{d-1}}$.

**Proof.** For $f \in C(S^{d-1})$, we put

$$(q_k f)(u) := \int_{S^{d-1} \cap S^1_u} \left(\frac{\langle u, v \rangle}{1 - \langle u, v \rangle^2}\right)^{k/2} f(v) \, dv, \quad u \in S^{d-1}. $$

That the integral $(q_k f)(u)$ exists, can be seen as follows: Let $e_1, e_2, \ldots, e_d$ be the standard basis in $\mathbb{R}^d$. Recall that cylindrical coordinates of $u \in S^{d-1}$ with respect to $e_d$ read

$$u = xe_d + \sqrt{1 - x^2} v, \quad x \in [-1, 1], \quad v \in S^{d-1} \cap e_\perp^d.$$  

(2.3)

Then,

$$du = (1 - x^2)^{(d-3)/2} dx \, dv,$$

(2.4)

where integration of $v$ is understood with respect to the spherical Lebesgue-measure on the great circle $S^{d-1} \cap e_\perp^d$. The function

$$\Phi(x) = 1(0 \leq x \leq 1) x^k (1 - x^2)^{(1-k)/2}$$

(2.5)

is integrable in $x \in [-1, 1]$ with respect to the weight function $(1 - x^2)^{(d-3)/2}$. It follows that $\Phi(\langle \cdot, e_d \cdot \rangle)$, and hence $\Phi(\langle \cdot, \cdot \rangle)$, $u \in S^{d-1}$, is integrable with respect to spherical Lebesgue measure. Hence, $q_k f$ exists, and moreover, the operator $q_k : C(S^{d-1}) \rightarrow L_\infty(S^{d-1})$ is continuous. Obviously, $q_k$ is linear and commutes with rotations. A combination of these properties implies $q_k : C(S^{d-1}) \rightarrow C(S^{d-1})$.

We have to show

$$p_k f = c_{k,d} q_k f, \quad \text{for all } f \in C(S^{d-1}).$$

(2.6)

As differences of support functions lie dense in $C(S^{d-1})$, it is enough to consider support functions $f$ only. As $p_k$ and $q_k$ both commute with rotations, it is enough to show (2.6) for one single point of $S^{d-1}$, for example for $e_d$. Let $\nu_{e_d}$ be the invariant
probability measure on the compact group $SO_d(e_d)$ of all rotations at 0 fixing $e_d$. Then, for any $K \in K_d$,

$$h(K, \cdot) := \int_{SO_d(e_d)} h(\vartheta K, \cdot) \nu_{e_d}(d\vartheta)$$

defines a body of revolution $K$ for which the line $g_d$ passing through 0 and $e_d$ is the rotation axis. Furthermore,

$$(p_k h(K, \cdot))(e_d) = (p_k h(K, \cdot))(e_d), \quad (q_k h(K, \cdot))(e_d) = (q_k h(K, \cdot))(e_d)$$

due to linearity, continuity and the intertwining property of the operators $p_k$ and $q_k$. Hence it is enough to show

$$(p_k h(K, \cdot))(e_d) = c_{k,d}(q_k h(K, \cdot))(e_d) \quad (2.7)$$

for all bodies of revolution $K$ with axis $g_d$. This will be done by direct calculation. As $K$ is a body of revolution with axis $g_d$, we can write $h(K, \cdot) = h((e_d, \cdot))$ with a continuous function $h$ on the interval $[-1, 1]$.

For $p_k$, the calculations are based on an integral geometric decomposition of $\nu_k$, already used by Goodey [4], p. 258. For the reader’s convenience, we summarize his arguments here: If $K$ is the convex body with support function $h((e_d, \cdot))$, then

$$h(K|L, e_d) = h(K, e_d|L) = \langle e_d, e_L \rangle h(\langle e_d, e_L \rangle),$$

where the unit vector $e_L$ in the direction $e_d|L$ is defined for $\nu_k$-almost all $L \in L^d_k$. Thus,

$$(p_k h((e_d, \cdot)))(e_d) = \int_{L^d_k} \langle e_d, e_L \rangle h(\langle e_d, e_L \rangle) \nu_k(dL).$$

Inserting a suitable Jacobian, the integration with respect to $\nu_k$ can be decomposed as follows: We first integrate invariantly over all $k - 1$ dimensional subspaces $F$ in the hyperplane $e_d^\perp$ orthogonal to $e_d$ and then over all $k$-dimensional subspaces $L$ containing $F$. $L$ is the linear span of $F$ and the line $g = L \cap F^\perp$, so $\langle e_d, e_L \rangle = \langle e_d, e_g \rangle$ does not depend on $F$. This argument leads to

$$(p_k h((e_d, \cdot)))(e_d) = \frac{\omega_{k,d-1}}{\omega_d} \int_0^1 x^k (1 - x^2)^{(d-k-2)/2} h(x) \, dx. \quad (2.8)$$

Concerning $q_k$, cylindrical coordinates (2.4) yield

$$(q_k h((e_d, \cdot)))(e_d) = \frac{\omega_{d-1}}{\omega_d} \int_0^1 x^k (1 - x^2)^{(d-k-2)/2} h(x) \, dx.$$

Comparison with (2.8) gives (2.7) and hence the assertion. □

**Remark.** Proposition 1 shows that

$$(p_k f)(u) = c_{k,d} \int_{S^{d-1}u^\perp} \frac{\cos^k \varphi(u, v)}{\sin^{k-1} \varphi(u, v)} f(v) \, dv, \quad f \in C(S^{d-1}),$$

where $\varphi(u, v)$ is the angle between $u$ and $v$. Hence, $p_k$ is a spherical trigonometric cap-transform. In particular, we have for $P_1(K)$, $K \in K_0$,

$$h(P_1(K), u) = \frac{2}{\omega_d} \int_{S^{d-1}u^\perp} \cos \varphi(u, v) h(K, v) \, dv.$$
If we use the Steiner–point \( s(K) \in \mathbb{R}^d \) of \( K \) defined by
\[
s(K) := \frac{d}{|K|} \int_{S^{d-1}} uh(K, u) \, du,
\]
we obtain
\[
h(P_1(K), u) = \frac{1}{d} (s(K), u) + \frac{1}{|K|} \int_{S^{d-1}} |(u, v)| h(K, v) \, dv,
\]
which was already observed by Goodey \[4\].

Another interesting case occurs, if we formally put \( k = 0 \) and \( c_0 := \frac{1}{|K| d - 1} \) in (2.2) to define
\[
(p_0 f)(\cdot) := \frac{1}{|K| d - 1} \int_{S^{d-1} \cap (\cdot)^{\perp}} \sin \delta(\cdot, v) f(v) \, dv, \quad f \in C(S^{d-1}),
\]
which is the sine-cap transform. This transform appeared in Schneider \[14\] in the course of the investigation of a stereological problem. Schneider showed that this transform is injective for all \( d \geq 3 \). For later use we define \( P_0(K) \) for \( K \in \mathcal{K}^d \) to be the convex body with
\[
h(P_0(K), \cdot) = p_0(h(K, \cdot)).
\]
That \( p_0 \) maps the family of support functions of convex bodies into itself, follows from \[9\]. Note that the case \( k = 0 \) in (1.1) would lead to the singleton \( P_0(K) = \{0\} \) and hence is different from the above definition.

Proposition 1 gives a new proof of the well known fact that \( p_k \) is a self-adjoint operator:
\[
\int_{S^{d-1}} (p_k f)(u) g(u) \, du = \int_{S^{d-1}} f(u) (p_k g)(u) \, du
\]
for all \( f, g \in C(S^{d-1}) \). This follows from (2.2) and an application of Fubini’s theorem.

3. The Laplace–Beltrami operator and \( P_h(K) \)

We will now examine the first surface area measure of \( P_h(K) \). To do so, we recall the definition of the spherical Laplace–Beltrami operator \( \Delta \): Let \( f \) be a twice differentiable function on \( S^{d-1} \), \( f \) its positively homogeneous extension to \( \mathbb{R}^d \setminus \{0\} \) and \( \Delta_L \) the usual Laplace operator. Then \( \Delta f \) is the restriction of \( \Delta_L f \) to \( S^{d-1} \).

The Laplace–Beltrami operator can be extended to generalized functions \( F \) on the sphere by setting
\[
(\Delta F)(\varphi) := F(\Delta \varphi), \quad \varphi \in C^\infty(S^{d-1}),
\]
see e.g. Berg \[2\], where it is also shown that \( \Delta \) commutes with rotations.

If \( S_1(K, \cdot) \) denotes the first surface area measure of \( K \in \mathcal{K}^d \), and \( \Box := \frac{1}{d - 1} \Delta + 1 \), we have
\[
\Box h(K, \cdot) = S_1(K, \cdot),
\]
in the sense of generalized functions (see Goodey & Weil \[7\]). In particular, if \( h(K, \cdot) \) is twice differentiable, \( S_1(K, \cdot) \) is absolutely continuous with respect to spherical Lebesgue measure and the left hand side of (3.2) is a density. The following Proposition implies an explicit representation of \( \Delta h(P_h(K), \cdot) \). As a corollary, a connection
with (3.2) gives a representation of \( S_1(P_k(K), \cdot) \). We will make use of the following generalised block operators

\[
\Box^d_k := \frac{\Delta + k(d - 2k) + 1}{k(k - 1)}, \quad k = 2, \ldots, d - 2,
\]

(3.3)

which are, like \( \Box \), linear combinations of \( \Delta \) and the identity.

**Proposition 2.** For \( 3 \leq k \leq d - 3 \) we have

\[
\Box^d_k(p_k f) = \frac{d - k}{k - 2} p_{k - 2} f - p_{k + 2} f, \quad f \in C(S^{d - 1}),
\]

(3.4)

in the sense of generalized functions.

For \( k = 2 \) and \( k = d - 2, d \geq 5 \), the corresponding formulae read as follows:

\[
\Box^d_2(p_2 f) = (d - 2)p_0 f - p_4 f, \quad f \in C(S^{d - 1}),
\]

where \( p_0 \) is given by (2.10), and

\[
\Box^d_{d - 2}(p_{d - 2} f) = \frac{2}{d - 4} p_{d - 4} f - \frac{1}{\omega_{d - 1}} f, \quad f \in C(S^{d - 1}).
\]

(3.6)

**Proof.** Fix \( k \in \{2, \ldots, d - 2\} \). For \( \varphi \in C^\infty(S^{d - 1}) \) we have

\[
(\Delta p_k f)(\varphi) = \int_{S^{d - 1}} (p_k f)(u) (\Delta \varphi)(u) du = \int_{S^{d - 1}} f(u) (p_k(\Delta \varphi))(u) du,
\]

due to (3.1) and (2.11). If we can show

\[
(p_k(\Delta \varphi))(u) = (k - 1) k d - k \frac{1}{k - 2} (p_{k - 2} \varphi)(u)
\]

(3.7)

\[
- (k(d - 2k) + 1)(p_k \varphi)(u)
\]

\[
- (k - 1) k (p_{k + 2} \varphi)(u)
\]

for \( k \in \{3, \ldots, d - 3\} \) and all \( u \in S^{d - 1} \), then (3.4) follows with (2.11). As both \( p_k \) and \( \Delta \) commute with rotations, it is enough to show (3.7) for \( u = e_d \) and all \( \varphi \in C^\infty(S^{d - 1}) \). This will be done in the following.

For a continuous spherical function \( g \) define the symmetrization \( \overline{g} \) with respect to \( e_d \) by

\[
\overline{g}(u) := \int_{SO_d(e_d)} g(\vartheta u) \nu_{e_d}(d\vartheta), \quad u \in S^{d - 1}.
\]

As \( \Delta \) commutes with rotations, it can be concluded that

\[
p_k(\Delta \overline{g})(e_d) = p_k(\Delta \overline{g})(e_d) = p_k(\Delta \varphi)(e_d) = p_k(\Delta \varphi)(e_d).
\]

(3.8)

The function \( \overline{g} \) is rotationally symmetric, so there is a function \( \psi : [-1, 1] \rightarrow \mathbb{R} \) with \( \overline{g} = \psi(\langle e_d, \cdot \rangle) \). Using cylindrical coordinates (2.3) on the sphere, we have

\[
\Delta = D + \frac{1}{1 - x^2} \Delta_{d - 1},
\]

where \( D := (1 - x^2) \frac{d^2}{dx^2} - (d - 1) x \frac{d}{dx} \) is a differential operator (in \( x \in \mathbb{R} \)) and \( \Delta_{d - 1} \) is the Laplace–Beltrami operator on the great circle \( S^{d - 1} \cap e_d^\perp \), see [2]. Combined with (3.8), (2.2) and (2.4), this gives

\[
p_k(\Delta \varphi)(e_d) = p_k [(D \psi)((e_d, \cdot))]}(e_d)
\]

\[
= c_{k,d} \int_0^1 x^k (1 - x^2)^{(d - k - 2)/2} (D \psi)(x) dx.
\]
The function
\[ F_k(x) = c_{k,d} 1(0 \leq x \leq 1) x^k (1 - x^2)^{(d-k-2)/2} \]
can be interpreted as a generalized function on \( \mathbb{R} \), hence
\[ p_k(\Delta \varphi)(e_d) = F_k(D\psi) = (D^* F_k)(\psi), \]
with
\[ D^* F_k = \frac{d^2}{dx^2} (1-x^2 F_k) + (d-1) \frac{d}{dx} (xF_k). \]

Explicit calculation gives
\[ c_{k,d}^{-1} \cdot D^* F_k = k \frac{d}{dx} \left[ 1(0 \leq x \leq 1) x^{k-1} (1-x^2)^{(d-k)/2} \right] + (k-1) \frac{d}{dx} \left[ 1(0 \leq x \leq 1) x^{k+1} (1-x^2)^{(d-k-2)/2} \right]. \]

This yields
\[ D^* F_k = k(k-1) c_{k,d,k-2,d} F_{k-2} - (k(d-2k) + 1) F_k \]
\[- (k-1)(d-k-2) c_{k,d,k+2,d} F_{k+2}, \quad k \neq d-2 \]
and
\[ D^* F_{d-2} = (d-3)(d-2) c_{d-2,d,d-4,d} F_{d-4} - ((d-2)(4-d) + 1) F_{d-2} \]
\[- (d-3) c_{d-2,d,d-1} \delta_1, \]
where \( \delta_1 \) is the probability measure supported by \( \{1\} \). Together with (3.9) and
\[ F_j(\psi) = (p_j \varphi)(e_d), \quad j = 0, \ldots, d-1, \]
we obtain (3.7) with \( u = e_d \) and (3.4) is shown.

To prove (3.5) and (3.6), the same line of arguments together with \( \delta_1(\psi) = \varphi(e_d) \) can be used.

We remark that (3.6) implies an explicit formula to reconstruct the support function of \( K \in \mathcal{K}^d \) from \( P_{d-4}(K) \) and \( P_{d-2}(K) \), namely
\[ \frac{1}{\omega_{d-1}} h(K, \cdot) = \frac{2}{d-4} h(P_{d-4}(K), \cdot) - \Delta_{d-2}^{d} h(P_{d-2}(K), \cdot). \]

**Corollary 1.** For \( 2 \leq k \leq d-2 \) we have
\[ S_k(P_k(K), \cdot) = a_k \cdot h(P_{k+2}(K), \cdot) + b_k \cdot h(P_k(K), \cdot) + c_k \cdot h(P_{k-2}(K), \cdot). \]
Here, \( a_2 = \frac{d-2}{d-4}, \quad c_{d-2} = -\frac{(d-2)(d-3)}{(d-1)\omega_{d-1}} \) and the other coefficients are given by
\[ a_k = \frac{(d-k)(k-1)}{(d-1)(k-2)} k, \quad b_k = \frac{k-1}{d-1} (2k - (d-2)), \quad c_k = -\frac{k-1}{d-1} k \]
for \( 2 \leq k \leq d-2 \).

This result allows us to prove Theorem 3: As the surface area measures in Corollary 1 are all absolutely continuous with respect to spherical Lebesgue measure, the bodies \( P_k(K) \) cannot be polytopes other than singletons for \( k \in \{2, \ldots, d-2\} \).
Concerning the case $k=1$: (2.9) shows that $h(P_1(K), \cdot)$ is up to a linear function the cosine transform of an absolutely continuous measure and hence cannot be a polytope other than a singleton. This shows Theorem 3.

4. The multipliers of $p_k$

We now recall some facts about spherical harmonics which are needed in the sequel. As a general reference, the book of Groemer [8] is recommended, as it describes also convex geometric implications. We assume $d \geq 2$.

Spherical harmonics of degree $n$ are the restrictions to $S^{d-1}$ of harmonic polynomials on $\mathbb{R}^d$, homogeneous of degree $n$. Let $\mathcal{H}_n^d$ be the space of all spherical harmonics of degree $n$ in $d$–dimensional space. There is only one rotationally symmetric function $f \in \mathcal{H}_n^d$ with axis $g_a = \text{span} \{e_d\}$, normalized such that $f(e_d) = 1$: This function is $f = P_n^d((e_d, \cdot))$, where $P_n^d(x)$ is the Legendre polynomial of dimension $d$ and degree $n$. $P_n^d(x)$ can for example be defined with Rodrigues’ formula:

$$
(1 - x^2)^{(d-3)/2}P_n^d(x) = e_{n,d} \left( \frac{d}{dx} \right)^n (1 - x^2)^{(d-3)/2+n}
$$

with the constant

$$
e_{n,d} = (-2)^n \frac{(d-3)!}{(d+2n-3)!} \left( \frac{d-2}{2} \right)_n.
$$

Here we used the shifted factorial notation $(a)_n := a(a + 1) \cdots (a + n - 1)$ for $n > 0$ and $(a)_0 := 1$, $a \in \mathbb{R}$. It should be noted that Gegenbauer polynomials coincide (up to normalization) with corresponding Legendre polynomials. It is therefore straightforward to transfer results about Gegenbauer polynomials to those about Legendre polynomials and vice versa. In the present paper, we will only use Legendre polynomials.

The space $\bigoplus_{n=0}^{\infty} \mathcal{H}_n^d$ is dense in $C(S^{d-1})$. Consider $f \in C(S^{d-1})$. Its (condensed) harmonic expansion is given by

$$
f \sim \sum_{n=0}^{\infty} Q_n
$$

where $Q_n \in \mathcal{H}_n^d$ is defined by

$$
Q_n(\cdot) := \dim \mathcal{H}_n^d \int_{S^{d-1}} P_n^d((v, \cdot))f(v) \, dv.
$$

The harmonic expansion of $f$ converges in quadratic mean to $f$. Hence $f$ is uniquely determined by its harmonic expansion.

If $f = h(K, \cdot)$ is the support function of a convex body $K \subset \mathbb{R}^d$, with harmonic expansion (4.2), then

$$
Q_0(\cdot) = \frac{1}{2} \mathfrak{v}(K),
$$

where $\mathfrak{v}(K)$ is the mean width of $K$, and

$$
Q_1(\cdot) = \langle s(K), \cdot \rangle.
$$

Furthermore, $K$ is centrally symmetric if and only if $Q_{2n+1} \equiv 0$ for all $n = 1, 2, \ldots$. $K$ is a body of constant width if and only if $Q_{2n} \equiv 0$ for all $n = 1, 2, \ldots$ (this fact was already observed by Minkowski in [10]).
It is well known that the operator $p_k$ is a multiplier transformation: if $f$ is continuous with spherical expansion (4.2), then

$$p_k f \sim c_{k,d} \sum_{n=0}^{\infty} a_{n,k,d} Q_n$$

with the multipliers

$$a_{n,k,d} = \int_0^1 x^k (1 - x^2)^{(d-k-2)/2} P_n^d(x) \, dx.$$  \hspace{1cm} (4.5)

Due to Proposition 1, this is also an immediate consequence (c.f. \cite{8}, Proposition 3.4.3) of (2.11) and the Funk-Hecke theorem (in the form stated by Seeley \cite{15}). Hence $p_k$ is injective on $C(S^{d-1})$ if and only if all the numbers $a_{n,k,d}$, $n = 0, 1, 2, 3, \ldots$ are nonzero. As for every $n = 1, 2, \ldots$ there is $Q_n \in \mathcal{H}^d_2$, $Q_n \neq 0$, such that $1 + Q_n$ is the support function of a convex body, this implies that $p_k$ is injective on $K^d$ if and only if $a_{n,k,d} = 0$ for all $n = 0, 1, 2, \ldots$. Furthermore, $p_k$ is injective on the subclass of all centrally symmetric convex bodies if and only if all $a_{n,k,d} = 0$ for all $n = 0, 1, 2, \ldots$ and $a_{1,k,d} \neq 0$. (The condition $a_{1,k,d} \neq 0$ means that the Steiner point of $K$ is determined by $p_k(K)$.) Finally, $p_k$ is injective on the subclass of all bodies of constant width, if $a_{2n+1,k,d} = 0$ for all $n = 0, 1, 2, \ldots$ and $a_{0,k,d} \neq 0$.

The injectivity results shown in \cite{4} and \cite{6} are based on these facts: It is shown with the help of recursion formulae for Gegenbauer polynomials that $a_{n,k,d} \neq 0$ for all $n$ and suitable pairs $(k, d)$. We will avoid these recursion formulae and show a representation of $a_{n,k,d}$ as explicit finite sum. The main tool for this approach is the fact that the Legendre polynomial $P_n^d(x)$ of dimension $d$ can be written as a linear combination of Legendre polynomials of dimension $d'$. The coefficients of this linear combination, the so called connection coefficients, can be given explicitly (see \cite{1}, Theorem 7.1.4'): We have for $d, d' \geq 3$

$$P_n^d(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} c_{n,j}^{d,d'} \cdot P_{n-2j}^{d'}(x),$$  \hspace{1cm} (4.6)

with

$$c_{n,j}^{d,d'} = \frac{n!}{j!(n-2j)!} \cdot \frac{(\frac{d-2}{2} - j)(\frac{d-d'}{2} + j)(d' - 2)n - 2j}{(\frac{d}{2} - j)(d - 2)n} \cdot \frac{(2n - 2j + d' - 2)}{(d' - 2)}.$$  

(4.6) also holds for $d' = 2$ if we put

$$c_{n,j}^{d,2} := \lim_{d' \to 2} c_{n,j}^{d,d'} = \begin{cases} 2 \binom{n}{j} & \text{for } n - 2j > 0, \\ 1 & \text{for } n - 2j = 0. \end{cases}$$

As Legendre polynomials $P_n^d(x)$ are orthogonal polynomials on $[-1, 1]$ with weight function $(1 - x^2)^{(d-3)/2}$, the definition (4.5) of $a_{n,k,d}$ suggests to set $d' := d - k + 1$. This approach leads to the following result.

**Proposition 3.** For $2 \leq k \leq d - 1$, we have

$$a_{n,k,d} = \sum_{j=0}^{\lfloor n/2 \rfloor} c_{n,j}^{d,d-k+1} \cdot \gamma_{n-2j,k,d-k+1}, \quad n = 0, 1, 2, \ldots,$$

\hspace{1cm} (4.7)
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\[
\gamma_{n,k,d} = \begin{cases} 
  \frac{k!\Gamma((k-n+1)/2)\Gamma((d+2n-1)/2)}{2^{n+1}(k-n)!\Gamma((d+k+n)/2)(\frac{d-1}{2})_n}, & 0 \leq n \leq k \\
  \frac{k!}{2^{k+1}(\frac{d-1}{2})_{k+1}}P_{n-k-1}^{d+2k+2}(0), & k < n.
\end{cases}
\]

Proof. We put \(d' = d - k + 1\) in (4.6), multiply both sides with \(x^k\) and integrate with respect to \(x \in [0,1]\). We obtain
\[
a_{n,k,d} = \sum_{j=0}^{[n/2]} c_{n,j} a_{n-j,k,d-k+1} \tilde{\gamma}_{n-2j,k,d-k+1}
\]
with
\[
\tilde{\gamma}_{n,k,d} := \int_0^1 x^k(1-x^2)^{(d-3)/2}P_n^d(x) \, dx.
\]

We will now show that \(\tilde{\gamma}_{n,k,d} = \gamma_{n,k,d}\) holds. Applying Rodrigues’ formula (4.1) two times yields
\[
\int (1-x^2)^{(d-3)/2}P_n^d(x) \, dx = -\frac{1}{d-1}(1-x^2)^{(d+2)/2}P_{n+2}^{d+2}(x).
\]

This and partial integration imply \(\tilde{\gamma}_{n,0,d} = \frac{1}{d-1}P_{n+2}^{d+2}(0)\) and for \(k \geq 1\)
\[
\tilde{\gamma}_{n,k,d} = \frac{k}{d-1} \tilde{\gamma}_{n-1,k-1,d+2}.
\]
Repeated application gives for \(n > k\)
\[
\tilde{\gamma}_{n,k,d} = \frac{k!}{2^{k+1}(\frac{d-1}{2})_{k+1}}P_{n-k-1}^{d+2k+2}(0).
\]

For \(n \leq k\) we obtain
\[
\tilde{\gamma}_{n,k,d} = \frac{k!}{2^n(k-n)!(\frac{d-1}{2})_n} \tilde{\gamma}_{0,k-n,d+2n}.
\]

This corresponds to the definition of \(\gamma_{n,k,d}\) and the assertion is shown. \(\square\)

We note that \(P_{2n+1}^d(0) = 0\), whereas
\[
P_{2n}^d(0) = (-1)^n \frac{(2n)!}{n!} \frac{(\frac{d-2}{2})_n}{(d-2)_{2n}}, \quad n = 0, 1, 2, \ldots.
\]

Hence \(\gamma_{n,k,d}\) is explicitly known. It can also be seen that for certain indices \(n, k, d\), the sum in (4.7) becomes alternating and hence the positivity of the result is difficult to show. On the other hand, the positivity of certain numbers \(a_{n,k,d}\) can be shown easily.

Corollary 2. Assume \(2 \leq k \leq d - 1\). Then
a) \(a_{n,k,d} > 0\) for all \(n = 0, 1, 2, \ldots, k + 2\).
b) If \(k\) is even, then \(a_{2n,k,d} > 0\) for all \(n = 0, 1, 2, \ldots\) and there is a constant \(b > 0\) such that
\[
a_{2n,k,d}^{-1} \leq b(2n)^{d-k}, \quad \text{for all } n = 1, 2, \ldots.
\]
c) If $k$ is odd, then $a_{2n+1,k,d} > 0$ for all $n = 0, 1, 2, \ldots$ and there is a constant $b > 0$ such that

$$a_{2n+1,k,d}^{-1} \leq b (2n+1)^{d-k}, \quad \text{for all } n = 0, 1, 2, \ldots \quad (4.11)$$

**Proof.** The main observation is that the constants $c_{n,j}^{d,d-k+1}$ in the previous proposition are all positive, as $d - k + 1 < d$.

To prove a) consider first the case of an even $k$. If $n \leq k + 2$, then $a_{n,k,d}$ is a positively weighted sum of $\gamma_{0,k,d-k+1}, \gamma_{2,k,d-k+1}, \ldots, \gamma_{k+2,k,d-k+1}$ due to Proposition 3. The last of these numbers vanishes, the others are positive. This shows a) in the case of even $k$. For odd $k$ the arguments are similar.

We prove b) and assume that $k$ is even. A combination of Proposition 3 and the fact that $P_j^d(0) = 0$ for odd $j$ shows that the representation for $a_{2n,k,d}$ has only $k/2 + 1$ positive summands:

$$a_{2n,k,d} = \sum_{j=0}^{k/2} c_{2n-n-j}^{d,d-k+1} \cdot \gamma_{2j,k,d-k+1} > 0.$$  

This proves the positivity of the multipliers. To determine the asymptotic behavior of the multipliers, recall that

$$\frac{(a)_n}{(a')_n} \sim n^{a-a'} \quad \text{as } n \to \infty,$$  

holds for arbitrary positive numbers $a$ and $a'$. Hence, there is a constant $b > 0$ such that

$$a_{2n,k,d} \geq \frac{1}{b} (2n)^{k-d}$$  

for all $n$. The exponent of $2n$ is best possible. This shows assertion b).

To prove c), the same line of arguments as in b) can be used. \hfill \Box

Goodey [4] mentions a similar estimate for the case $k = 2$ and arbitrary $n$. If (5.1) in [4] is corrected for a missing term $\binom{n+d-3}{n}$, it coincides with (4.10) for even $n$.

It is well known that the polynomial estimates (4.10) and (4.11) are the basis for stability results like those in Theorem 1. Similar arguments have been used for particular transforms before (see e.g. Campi [3], Goodey & Groemer [5] and Spriestersbach [16]). The next proposition gives a rather general result of this type using an established method of proof. In particular, together with Corollary 2 b) and c), it provides Theorem 1.

**Proposition 4.** Let $p : C(S^{d-1}) \to C(S^{d-1})$, $d \geq 3$, be an endomorphism that maps the family of support functions of convex bodies into itself, and define

$$P : \mathcal{K}^d \to \mathcal{K}^d \quad \text{by} \quad h(P(K), \cdot) := p(h(K, \cdot)).$$  

Furthermore, assume that $p$ is a multiplier transformation with multipliers $a_0, a_1, \ldots$. Then $P$ is injective on

$$\mathcal{K}^d_p := \{ K \in \mathcal{K}^d | h(K, \cdot) \sim \sum_{a \neq 0} Q_a \}.$$  

The following stability result holds: Assume in addition the existence of constants
b, \beta > 0 \text{ such that }
|a_n|^{-1} \leq bn^{\beta} \quad \text{for all } n \in \{2, 3, 4 \ldots \} \text{ with } a_n \neq 0. \quad (4.12)

Then, for all \( R > 0 \) there is a constant \( c = c(d, R, P) \) with
\[
\delta_2(K, M) \leq c \cdot \delta_2(P(K), P(M))^{1/(\beta+1)}
\]
for all sets \( K, M \in \mathbb{K}^d \) contained in \( RB^d \).

**Proof.** The injectivity statement follows from the fact that a continuous function is uniquely determined by its spherical harmonic expansion.

To prove the stability statement, we set \( F = h(K, \cdot) - h(M, \cdot) \) and \( \gamma = 2/\beta \) in Lemma 3.4.13 in [8] and obtain
\[
\delta_2(K, M) \leq \Gamma(K, M, (a_n))^{3/(\beta+2)} \cdot \delta_2(P(K), P(M))^{1/(\beta+1)}
\]

The expression
\[
\Gamma(K, M, (a_n)) = \sum_{a_n \neq 0} |a_n|^{-2/\beta} \|Q_n - R_n\|_2^2
\]
depends on the expansions \( b(K, \cdot) \sim \sum_{n=0}^\infty Q_n \) and \( h(M, \cdot) \sim \sum_{n=0}^\infty R_n \). Due to (4.3), (4.4), the sums for \( n = 0 \) and \( n = 1 \) can be estimated by a constant \( c_1 > 0 \) depending on \( d, R \) and \( a_0, a_1 \), but not on \( K \) or \( M \). The assumption (4.12) combined with \( n^\beta \leq (n - 1)(n + d - 1) \) for \( n \geq 2 \) yields
\[
\Gamma(K, M, (a_n)) \leq c_1 + b^{-2/\beta} \sum_{n=2}^\infty (n - 1)(n + d - 1) \|Q_n - R_n\|_2^2.
\]

As \( \|Q_n - R_n\|_2^2 \leq 2(||Q_n||_2^2 + ||R_n||_2^2) \), Lemma 3 in [5] implies
\[
\sum_{n=2}^\infty (n - 1)(n + d - 1) \|Q_n - R_n\|_2^2 \leq d(d-1) \left( \delta(K) + \delta(M) \right),
\]
where
\[
\delta(\cdot) = \frac{d}{\omega_d} W_{d-1}(\cdot) - W_{d-2}(\cdot)
\]
depends on the Minkowski-functionals \( W_{d-1}(\cdot) \) and \( W_{d-2}(\cdot) \). As these functionals are nonnegative and monotonic with respect to set inclusion, we conclude that \( \Gamma(K, M, (a_n)) \) can be estimated from above by a constant \( c_2 \) that depends on \( (a_n) \) and \( R \) but not on \( K \) or \( M \). Summarizing we have shown that (4.13) holds with \( c := c_2^{\beta/(\beta+2)} \). \( \square \)

The result of the foregoing proposition can easily be rewritten in terms of the Hausdorff metric \( \delta \), see Vitale [17]: With the notation and assumptions as above, there is a constant \( c' = c'(d, R, P) \) such that
\[
\delta(K, M) \leq c' \cdot \delta(P(K), P(M))^{1/(\beta+1)}.
\]

5. **Determination of a convex body from two different Minkowski averages**

It remains to show Theorem 2. For this, fix \( k, k' \in \{2, \ldots, d-1\} \), \( k \neq k' \) and a convex body \( K \). The convex body \( K \) is determined by the bodies \( P_k(K) \) and \( P_{k'}(K) \) if and only if for all \( n = 0, 1, 2, \ldots \) at least one of the numbers \( a_{n,k,d} \) and
$a_{n,k,d}$ is nonzero. If $k$ and $k'$ are of different parity, Corollary 2 guarantees this condition. Therefore, Theorem 2 follows from the following final result.

**Lemma 1.** For all numbers $k,k' \in \{2, \ldots, d-1\}$ of the same parity and all $n = 0, 1, 2, \ldots$ at least one of the numbers $a_{n,k,d}$ and $a_{n,k',d}$ is nonzero.

**Proof.** Assume $k < k'$ and that there is a number $n$ such that $a_{n,k,d} = a_{n,k',d} = 0$. Due to Corollary 2 a) we have $n > k' + 2$.

Spherical harmonics are eigenfunctions of the Laplace Beltrami operator, see [2]:

$$
\Delta Q_n = -n(n+d-2)Q_n \quad \text{for } Q_n \in \mathcal{H}_n^d, \ n = 0, 1, 2, \ldots.
$$

Hence, the block operator defined in (3.3) fulfills

$$
\Box^d_j Q_n = -b_{n,j,d} Q_n \quad \text{for } Q_n \in \mathcal{H}_n^d,
$$

with

$$
b_{n,j,d} = \frac{n(n+d-2) - j(d-2j) - 1}{j(j-1)}
$$

for $j = 3, 4, \ldots, d-3$, $n = 0, 1, 2, \ldots$. (4.4) implies

$$
\frac{d-j-2}{j} a_{n,j+2,d} = a_{n,j,d} + a_{n,j-2,d}, \quad j = 3, 4, \ldots, d-3. \quad (5.1)
$$

An easy calculation shows $b_{n,j,d} > 0$ for all $j < n$, hence for all $j \leq k'$. Formula (5.1) shows that a positive combination of two consecutive members of $(a_{n,k,d}, a_{n,k+2,d}, a_{n,k+4,d}, \ldots, a_{n,k',d})$ equals the successor of them. As $a_{n,k,d} = a_{n,k',d} = 0$ we conclude $a_{n,j,d} = 0$ for all $j = k, k+2, k+4, \ldots, k'$. Repeated application of (5.1), starting with $j = k'$, now yields $a_{n,d-2,d} = 0$ or $a_{n,d-1,d} = 0$, which contradicts the known injectivity of $P_{d-2}$ and $P_{d-1}$. Note that for the last argument the positivity of $b_{n,j,d}$ is not needed anymore.

6. Injectivity of $P_k$: a discussion

The original question whether or not $P_k$ is injective on $K^d$ is still open for many pairs $(k,d)$. In the present paper we could show that the multipliers $a_{n,k,d}$ are positive, whenever $n \leq k + 2$ or $n \equiv k \mod 2$. Injectivity of $P_k$ would follow, if the remaining multipliers all were nonzero. The following result is a step in this direction. As an induction argument is applied here, it is convenient to include the cases $k = 0$ and $k = 1$ in the statement.

**Lemma 2.** Assume $0 \leq k \leq d - 1$. Then $a_{n,k,d} > 0$ for all $n \equiv k + 1 \mod 4$.

**Proof.** We first consider the case where $k$ is even. The lemma is true for $k = 0$: The numbers $a_{n,0,d}$ are multipliers of the sine-cap transform. Hence, $a_{4n+1,0,d} > 0$, $n = 0, 1, 2, \ldots$ follows from [14], [4], Lemma 4.e) and f) shows the assertion for $k = 2$.

Now fix $n \in \mathbb{N}_0$ with $n \equiv k + 1 \mod 4$. The statement will follow by induction, if the assumption $a_{n,k,d} > 0$ implies $a_{n,k+4,d} > 0$. This will be shown with arguments similar to those in the proof of Lemma 1. If we assume $a_{n,k,d} > 0$ but $a_{n,k+4,d} \leq 0$ then $n \geq k + 7$ due to Corollary 2 a). A combination of (5.1) and Corollary 2 a)
contradiction to the results in \([1,2]\) as long as \(m\) induction on \(a\) shows that members of \(A\) are nonzero. This is nontrivial, however: Simulations and known cases show that \(k\) successors are negative, too. For the known cases \(n\) starts with positive values and, at a certain transition point, becomes non-positive, members of \(A\) property could be shown for general \(a\) We have For example, for the known case \((k,d)\) proven by showing that \(P\) to show injectivity of \(K\) as \(k,d\) is the sequence of multipliers of the cosine transform in \(\mathbb{R}^d\). It is well known that \((-1)^{n+1}\tau_{2n,d} > 0\) for all \(n = 0, 1, 2\ldots\), see e.g. [13]. This completes the proof.

To show injectivity of \(P_k\) on \(K^d\) for a given pair \((k,d)\) with 2 \(\leq k \leq d - 1\), it remains to show that all members of 

\[ A_{k,d} := \{a_{k+4n+3,k,d} \mid n = 0, 1, 2, \ldots\} \]

are nonzero. This is nontrivial, however: Simulations and known cases show that members of \(A_{k,d}\) may become negative; there are even pairs \((k,d)\) for which all members of \(A_{k,d}\) are negative. The typical behavior seems to be that \(a_{k+4n-1,k,d}\) starts with positive values and, at a certain transition point, becomes non-positive, as \(n\) increases. Furthermore, if one of the members of \(a_{k+4n-1,k,d}\) is negative, then all successors are negative, too. For the known cases \(k = 2\) and \(k = 3\), this property is proven by showing that \((n+d-3)a_{k+4n-1,k,d}\) is typically decreasing. If the mentioned property could be shown for general \(k\), the point of transition ought to be checked. For example, for the known case \((k,d) = (2,14)\) the transition occurs for \(n = 5\). We have \(a_{5,2,14} = 0\). As the point of transition may occur for large \(n\), the check is tedious and probably not possible for all pairs \((k,d)\).

References


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