

## A NOTE ON COMMUTING POWERS IN BANACH ALGEBRAS

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Throughout  $\mathcal{A}$  is a complex unital Banach algebra with unit  $\mathbf{1}$ . For  $a \in \mathcal{A}$  the spectrum and the spectral radius of  $a$  are denoted by  $\sigma(a)$  and  $r(a)$ , respectively.

Let  $m$  be a positive integer and  $a \in \mathcal{A}$ . We say that  $\sigma(a)$  is *irrotational* ( $\text{mod } 2\pi/m$ ) (see [1]) if  $\lambda, \mu \in \sigma(a)$  and  $\lambda^m = \mu^m$  imply that  $\lambda = \mu$ .

The main result of this paper reads as follows:

**Theorem.** *Let  $a, b \in \mathcal{A}$  be invertible and let  $m$  be a positive integer.*

- (1) *If  $a^m b^m = b^m a^m$  and if  $\sigma(a)$  is irrotational ( $\text{mod } 2\pi/m$ ), then  $ab^m = b^m a$ .*
- (2) *If  $a^m b^m = b^m a^m$  and if  $\sigma(a)$  and  $\sigma(b)$  are irrotational ( $\text{mod } 2\pi/m$ ), then  $ab = ba$ .*
- (3) *If  $a^m b^m = (ab)^m = b^m a^m$  and if  $\sigma(ab)$  is irrotational ( $\text{mod } 2\pi/m$ ), then  $ab = ba$ .*

For the proof of the above result we need some preparations.

If  $m$  is a positive integer, let  $\epsilon_k = e^{2k\pi i/m}$  for  $k = 1, \dots, m$ . Then

$$\epsilon_k^m = 1 \quad (k = 1, \dots, m), \quad \epsilon_k \neq 1 \quad (k = 1, \dots, m-1)$$

and  $\epsilon_m = 1$ .

If  $a \in \mathcal{A}$  is invertible, define the bounded linear operator  $T_a : \mathcal{A} \rightarrow \mathcal{A}$  by

$$T_a x = a^{-1} x a \quad (x \in \mathcal{A}).$$

**Proposition 1.** *Suppose that  $a \in \mathcal{A}$  is invertible,  $m$  is a positive integer and that  $\sigma(a)$  is irrotational ( $\text{mod } 2\pi/m$ ). Let the bounded linear operator  $S : \mathcal{A} \rightarrow \mathcal{A}$  be given by*

$$S = \prod_{k=1}^{m-1} (T_a - \epsilon_k I),$$

where  $I$  denotes the identity operator on  $\mathcal{A}$ . Then:

- (1)  $S$  is invertible;
- (2)  $T_a^m - I = (T_a - I)S = S(T_a - I)$ .

*Proof.* (1) We show that  $T_a - \epsilon_k I$  is invertible for  $k = 1, \dots, m-1$ . To this end suppose that  $T_a - \epsilon_k I$  is not invertible for some  $k \in \{1, \dots, m-1\}$ . It follows from [1, Proposition 18.9] that there are  $\lambda, \mu \in \sigma(a)$  such that  $\lambda = \epsilon_k \mu$ , hence  $\lambda^m = \epsilon_k^m \mu^m = \mu^m$ . Consequently  $\lambda = \mu$  and therefore  $\epsilon_k = 1$ , a contradiction.

(2) follows from the identity

$$\lambda^m - 1 = (\lambda - 1) \prod_{k=1}^{m-1} (\lambda - \epsilon_k) \quad (\lambda \in \mathbb{C}).$$

□

**Proposition 2.** *Let  $a$  and  $m$  be as in Proposition 1. If  $x \in \mathcal{A}$  and  $a^m x = x a^m$ , then  $ax = xa$ .*

*Proof.* From  $a^m x = x a^m$  we get  $(T_a^m - I)x = 0$ . By Proposition 1 we have  $S(T_a - I)x = 0$ . Since  $S$  is invertible,  $T_a x = x$  and so  $ax = xa$ . □

*Proof of the Theorem.*

- (1) is a consequence of Proposition 2.  
(2) By (1),  $b^m a = ab^m$ . Now apply Proposition 2 to  $b$ .  
(3) We have

$$a^m (ab)^m = a^m b^m a^m = b^m a^m a^m = (ab)^m a^m.$$

Thus, by Proposition 2,  $aba^m = a^m ab$ , therefore

$$ba^m = a^{-1} aba^m = a^{-1} a^m ab = a^m b,$$

hence

$$(ab)^m b = a^m b^m b = ba^m b^m = b(ab)^m.$$

Now use Proposition 2 to see that  $abb = bab$ . Since  $b$  is invertible we derive  $ab = ba$ .  $\square$

**Proposition 3.** *Let  $a, b \in \mathcal{A}$  and  $m$  a positive integer.*

- (1) *If  $\sigma(a) \subseteq [0, \infty)$ , then  $\sigma(a)$  is irrotational (mod  $2\pi/m$ ).*  
(2) *If  $m \geq 2$  and  $(1 + r(a))^{m-1} < 2$ , then  $\mathbf{1} - a$  is invertible and  $\sigma(\mathbf{1} - a)$  is irrotational (mod  $2\pi/m$ ).*  
(3) *Suppose that  $b \in \mathcal{A}$  is invertible,  $a$  is invertible,  $\sigma(a)$  is irrotational (mod  $2\pi/m$ ) and that  $a^m = b^m$ . Then  $ab = ba$ .*

*Proof.* (1) Clear.

(2) We have  $r(a) < 1$ , hence  $\mathbf{1} - a$  is invertible. Now let  $\lambda, \mu \in \sigma(\mathbf{1} - a)$  and  $\lambda^m = \mu^m$ . There are  $\alpha, \beta \in \sigma(a)$  such that  $\lambda = 1 - \alpha$  and  $\mu = 1 - \beta$ . Then

$$\begin{aligned} 0 &= (1 - \alpha)^m - (1 - \beta)^m = \sum_{k=0}^{m-1} \binom{m}{k} (-1)^k (\alpha^k - \beta^k) \\ &= -m(\alpha - \beta) + \sum_{k=2}^{m-1} \binom{m}{k} (\alpha - \beta) h_k(\alpha, \beta) \end{aligned}$$

where  $h_k(\alpha, \beta) = (-1)^k (\alpha^{k-1} + \alpha^{k-2}\beta + \dots + \alpha\beta^{k-2} + \beta^{k-1})$ .

Hence  $|h_k(\alpha, \beta)| \leq kr(a)^{k-1}$ . Therefore

$$m|\alpha - \beta| \leq |\alpha - \beta| \sum_{k=2}^{m-1} \binom{m}{k} kr(a)^{k-1} = |\alpha - \beta| \left( \sum_{k=1}^{m-1} \binom{m}{k} kr(a)^{k-1} - m \right).$$

If the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = (1 + x)^m$ , then  $f(x) = \sum_{k=0}^m \binom{m}{k} x^k$ , thus

$$f'(x) = \sum_{k=1}^m \binom{m}{k} kx^{k-1}, \text{ hence } \sum_{k=1}^m \binom{m}{k} kx^{k-1} = m(1 + x)^{m-1}.$$

It follows that

$$m|\alpha - \beta| \leq |\alpha - \beta| (m(1 + r(a))^{m-1} - m)$$

and so

$$|\alpha - \beta| \leq |\alpha - \beta| ((1 + r(a))^{m-1} - 1)$$

Now suppose that  $\alpha \neq \beta$ . Then

$$1 \leq (1 + r(a))^{m-1} - 1 < 2 - 1 = 1,$$

a contradiction. This gives  $\lambda = \mu$ .

(3) We have

$$a^m b = b^m b = bb^m = ba^m.$$

Now use Proposition 2. □

**Examples.**

- (1) If  $r(a) < 1$ , then  $\sigma(\mathbf{1} - a)$  is irrotational (mod  $2\pi/2$ ).
- (2) If  $r(a) < \sqrt{2} - 1$ , then  $\sigma(\mathbf{1} - a)$  is irrotational (mod  $2\pi/3$ ).

**Corollary 1.** Suppose that  $a, b \in \mathcal{A}$ ,  $\sigma(a) \subseteq (0, \infty)$ ,  $\sigma(b) \subseteq (0, \infty)$  and that  $a^m b^m = b^m a^m$  for some positive integer  $m$ . Then  $ab = ba$ .

*Proof.* Proposition 3 (1) and Theorem (2). □

**Corollary 2.** Let  $\mathcal{A}$  be the Banach algebra of all bounded linear operators on a complex Hilbert space, let  $A \in \mathcal{A}$  be invertible and let  $m$  be a positive integer.

- (1) If  $\sigma(A)$  is irrotational (mod  $2\pi/m$ ) and if  $A^m$  is normal, then  $A$  is normal.
- (2) If  $A^m (A^*)^m = (AA^*)^m = (A^*)^m A^m$ , then  $A$  is normal.

*Proof.* (1)  $A^*$  is invertible and  $\sigma(A^*) = \{\lambda \in \mathbb{C}, \bar{\lambda} \in \sigma(A)\}$ , thus  $\sigma(A^*)$  is irrotational (mod  $2\pi/m$ ). Now use part (2) of the Theorem.

(2) Since  $\sigma(AA^*) \subseteq (0, \infty)$ , the result follows from Proposition 3 (1) and part (3) of the Theorem. □

**Corollary 3.** Suppose that  $a, b \in \mathcal{A}$ ,  $r(a) < 1$ ,  $r(b) < 1$  and

$$(\mathbf{1} - a)^2 (\mathbf{1} - b)^2 = (\mathbf{1} - b)^2 (\mathbf{1} - a)^2.$$

Then  $ab = ba$ .

*Proof.* Example (1) and part (1) of the Theorem. □

The quasi-product  $x \circ y$  of  $x, y \in \mathcal{A}$  is defined by

$$x \circ y = x + y - xy.$$

Given  $z \in \mathcal{A}$ , a quasi-square-root of  $z$  is an element  $x \in \mathcal{A}$  with

$$x \circ x = z.$$

**Corollary 4.** Let  $a, b \in \mathcal{A}$  and  $a \circ a = b \circ b$ .

- (1) If  $r(a) < 1$ , then  $ab = ba$ .
- (2) If  $r(a) < 1$  and  $r(b) < 1$ , then  $a = b$ .

*Proof.* (1) Since  $a \circ a = b \circ b$ , we have  $(\mathbf{1} - a)^2 = (\mathbf{1} - b)^2$ . Now  $\mathbf{1} - a$  is invertible, hence  $\mathbf{1} - b$  is invertible. The result follows from Example (1) and Proposition 3 (3).

(2) By (1),  $ab = ba$ . From  $a \circ a = b \circ b$  we see that

$$(a - b)(a + b) = -2(a - b),$$

hence

$$(a - b)(a + b + 2\mathbf{1}) = 0.$$

Corollary 4.3 in [1] gives  $r(a + b) \leq r(a) + r(b) < 2$ , thus  $-2 \notin \sigma(a + b)$  and so  $a = b$ . □

**Corollary 5.** Let  $a, b \in \mathcal{A}$  and  $a^2 = b^2$ .

- (1) If  $r(\mathbf{1} - a) < 1$ , then  $ab = ba$ .
- (2) If  $r(\mathbf{1} - a) < 1$  and  $r(\mathbf{1} - b) < 1$ , then  $a = b$ .

*Proof.* Let  $\tilde{a} = \mathbf{1} - a$ ,  $\tilde{b} = \mathbf{1} - b$ . Then

$$\tilde{a} \circ \tilde{a} = \mathbf{1} - a^2 \quad \text{and} \quad \tilde{b} \circ \tilde{b} = \mathbf{1} - b^2,$$

thus  $\tilde{a} \circ \tilde{a} = \tilde{b} \circ \tilde{b}$ . Now use Corollary 4. □

**Corollary 6.** *Suppose that  $a, b \in \mathcal{A}$ ,  $\sigma(a) \subseteq (0, \infty)$ ,  $\sigma(b) \subseteq (0, \infty)$  and  $a^m = b^m$  for some positive integer  $m$ . Then  $a = b$ .*

*Proof.* By Proposition 3 (3),  $ab = ba$ . Let  $c = ab^{-1}$ . Then  $c^m = \mathbf{1}$ . Let  $\lambda \in \sigma(c)$ . Corollary 4.3 in [1] gives  $\lambda = \alpha/\beta$  with  $\alpha \in \sigma(a)$  and  $\beta \in \sigma(b)$ , hence  $\lambda > 0$ . Since  $\lambda^m = 1$ , it follows that  $\lambda = 1$ . Thus  $\sigma(c) = \{1\}$ . We have

$$\lambda^m - 1 = (\lambda - 1)h(\lambda)$$

with some entire function  $h$  such that  $h(1) \neq 0$ . Therefore

$$0 = c^m - \mathbf{1} = (c - \mathbf{1})h(c)$$

and  $h(c)$  is invertible. Hence  $c = \mathbf{1}$  and so  $a = b$ . □

## References

- [1] F. F. Bonsall, J. Duncan, *Complete Normed Algebras*, Springer (1973).

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