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Operator Exponentials on Hilbert Spaces

Christoph Schmoeger

Abstract

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$. In this paper we consider the following class of operators:

$$\hat{\Sigma}(\mathcal{H}) = \{S \in \mathcal{L}(\mathcal{H}) \colon S \text{ is a scalar type operator and } \sigma(S) \cap \sigma(S + 2k\pi i) \subseteq \{k\pi i\} \text{ for } k = 1, 2, \dots\}.$$

The main results of this paper read as follows:

1. If $T, S \in \hat{\Sigma}(\mathcal{H})$ and $e^T e^S = e^S e^T$ then $T^2 S^2 = S^2 T^2$.
2. If $S \in \hat{\Sigma}(\mathcal{H})$, $T \in \mathcal{L}(\mathcal{H})$ and $e^T = e^S$ then $TS^2 = S^2 T$.

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1 Terminology and results

Throughout this paper let $\mathcal{H}$ denote a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{L}(\mathcal{H})$ the spectrum and the spectral radius of A are denoted by $\sigma(A)$ and $r(A)$, respectively. The set of eigenvalues of A is denoted by $\sigma_p(A)$. For the resolvent set of A we write $\rho(A)$. We use $N(A)$ and $A(\mathcal{H})$ to denote the kernel and the range of A, respectively.

An operator $S \in \mathcal{L}(\mathcal{H})$ is called a *scalar type operator* if S admits a representation

$$S = \int_{\sigma(S)} \lambda E(d\lambda),$$

where $E(d\lambda)$ denotes integration with respect to a spectral measure $E(\cdot)$ on $\mathcal{H}$. See [1], [2] and [14] for properties of spectral measures and scalar type operators.

If $A \in \mathcal{L}(\mathcal{H})$ is *normal* ($AA^* = A^*A$) then A is a scalar type operator and the values of the spectral measure of A are selfadjoint projections (see [1], Theorem 7.18).
J. Wermer [14] has shown that the scalar type operators on $\mathcal{H}$ are those operators which are similar to normal operators. More precisely, Wermer has shown that for every finite set $S_1, \dots, S_n$ of commuting scalar type operators on $\mathcal{H}$ there is a selfadjoint operator

$B \in \mathcal{L}(\mathcal{H})$ with a bounded everywhere defined inverse such that the operators BS_iB^{-1}, $i = 1, \ldots, n$, are all normal.

We write $\Sigma(\mathcal{H})$ for the class of all scalar type operators on $\mathcal{H}$. In the present paper we consider the following class of operators:

$$\hat{\Sigma}(\mathcal{H}) = \{S \in \Sigma(\mathcal{H}) : \sigma(S) \cap \sigma(S + 2k\pi i) \subseteq \{k\pi i\} \text{ for } k = 1, 2, \ldots\}.$$

Now we state the main results. Proofs will be given in Section 3, in Section 4 we present some corollaries.

Theorem 1.1 *If $T \in \hat{\Sigma}(\mathcal{H})$, $S \in \mathcal{L}(\mathcal{H})$ and $e^T e^S = e^S e^T$ then $e^S T^2 = T^2 e^S$. If in addition $\sigma_p(T) \cap \{k\pi i : \ k = 1, 2, \ldots\} = \emptyset$ then $e^S T = T e^S$.*

Theorem 1.2 *If $T, S \in \hat{\Sigma}(\mathcal{H})$ and $e^T e^S = e^S e^T$ then $T^2 S^2 = S^2 T^2$.*

Theorem 1.3 *Suppose that $T, S \in \hat{\Sigma}(\mathcal{H})$ and that $e^T e^S = e^S e^T$.*

(a) *If $\sigma_p(T) \cap \{k\pi i : \ k = 1, 2, \ldots\} = \emptyset$ then $TS^2 = S^2 T$.*

(b) *If $\sigma_p(T) \cap \{k\pi i : \ k = 1, 2, \ldots\} = \sigma_p(S) \cap \{k\pi i : \ k = 1, 2, \ldots\} = \emptyset$ then $TS = ST$.*

For related results concerning the equation $e^A e^B = e^B e^A$ see [10], [11], [12] and [15].

Theorem 1.4 *Suppose that $T, S \in \mathcal{L}(\mathcal{H})$, $T + S \in \hat{\Sigma}(\mathcal{H})$ and that*

$$e^{T+S} = e^T e^S = e^S e^T.$$

If $\sigma_p(T + S) \cap \{k\pi i : \ k = 1, 2, \ldots\} = \emptyset$ then $TS = ST$.

Theorem 1.5 *If $S \in \hat{\Sigma}(\mathcal{H})$, $T \in \mathcal{L}(\mathcal{H})$ and $e^T = e^S$ then $TS^2 = S^2 T$.*
If in addition $\sigma_p(S) \cap \{k\pi i : \ k = 1, 2, \ldots\} = \emptyset$ then $TS = ST$.

For related results concerning the equation $e^A = e^B$ see [3], [9] and [11].

2 Preparations

In this section we collect some results which we need for the proofs of the theorems in Section 1.

Proposition 2.1 *Suppose that $A \in \mathcal{L}(\mathcal{H})$ is normal.*

(a) *If $\mu \in \mathbb{C}$ then $(A - \mu)(\mathcal{H}) = (A^* - \overline{\mu})(\mathcal{H})$.*

(b) *If* $B \in \mathcal{L}(\mathcal{H})$ *then*

$$E(\sigma(A) \cap \sigma(B))(\mathcal{H}) = \bigcap_{\lambda \in \rho(B)} (A - \lambda)(\mathcal{H}),$$

where $E(\cdot)$ *denotes the spectral measure of* A.

Proof. (a) Since A is normal, $A - \mu$ is normal. Exercise 12.36 in [8] gives the result.
(b) is shown in [7, Theorem 1], see also [6]. ■

Let $A \in \mathcal{L}(\mathcal{H})$. The map $\delta_A : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, defined by

$$\delta_A(C) = CA - AC \quad (C \in \mathcal{L}(\mathcal{H}))$$

is called the *inner derivation* determined by A. It is clear that δ_A is a bounded linear operator on $\mathcal{L}(\mathcal{H})$ with $\|\delta_A\| \leq 2\|A\|$.

Throughout this paper let f denote the entire function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \begin{cases} z^{-1}(e^z - 1), & \text{if } z \neq 0, \\ 1, & \text{if } z = 0. \end{cases}$$

Let $M_A = \{\lambda \in \sigma(\delta_A) : f(\lambda) = 0\}$.

Proposition 2.2 *Let* $A \in \mathcal{L}(\mathcal{H})$.

(a) *If* $M_A = \emptyset$, *then* $f(\delta_A)$ *is an invertible operator on* $\mathcal{L}(\mathcal{H})$.

(b) *If* $\lambda \in M_A$ *then* λ *is a simple zero of* f *and there is* $j \in \mathbb{Z} \setminus \{0\}$ *with* $\lambda = 2j\pi i$.

(c) M_A *is a finite set,* $M_A \subseteq \{\pm 2\pi i, \pm 4\pi i, \ldots\}$.

(d) *If* $M_A \neq \emptyset$ *and* $M_A = \{\lambda_1, \ldots, \lambda_p\}$ *with* $\lambda_j \neq \lambda_k$ *for* $j \neq k$ *then*

$$N(f(\delta_A)) = N(\delta_A - \lambda_1) \oplus \ldots \oplus N(\delta_A - \lambda_p).$$

(e) $\sigma(\delta_A) = \{\lambda - \mu : \ \lambda, \mu \in \sigma(A)\}$.

(f) $e^{\delta_A}(C) = e^{-A} C e^A$ *for all* $C \in \mathcal{L}(\mathcal{H})$.

(g) $f(\delta_A)(\delta_A(C)) = e^{-A} C e^A - C$ *for all* $C \in \mathcal{L}(\mathcal{H})$.

Proof. (a) If $M_A = \emptyset$, then $f(\lambda) \neq 0$ for all $\lambda \in \sigma(\delta_A)$, thus $f(\delta_A)$ is invertible.
(b), (c) and (d) are shown in [11].
(e) follows from [4], and Proposition 6.4.8 in [5] shows that (f) holds.
(g) follows from (f) and $zf(z) = f(z)z = e^z - 1$. ■

Proposition 2.3 *Let A be a normal operator in $\mathcal{L}(\mathcal{H})$ and let $E(\cdot)$ be its spectral measure. If $\lambda_0 \in \mathbb{C}$, $C \in N(\delta_A - \lambda_0)$, $D \in N(\delta_A + \lambda_0)$ then*

$$C(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H})$$

and

$$D^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H}).$$

Proof. From $CA - AC = \lambda_0 C$ we get $AC = C(A - \lambda_0)$. Put $B = A - \lambda_0$. Now take $\mu \in \rho(B)$. Then

$$\begin{aligned}(A-\mu)C(B-\mu)^{-1} &= AC(B-\mu)^{-1} - \mu C(B-\mu)^{-1} \\ &= CB(B-\mu)^{-1} - \mu C(B-\mu)^{-1} \\ &= C(B-\mu)(B-\mu)^{-1} = C,\end{aligned}$$

thus $C(\mathcal{H}) \subseteq (A - \mu)(\mathcal{H})$. Since $\mu \in \rho(B)$ was arbitrary, we derive

$$C(\mathcal{H}) \subseteq \bigcap_{\mu \in \rho(B)} (A - \mu)(\mathcal{H}).$$

Proposition 2.1(b) implies now that

$$C(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(B))(\mathcal{H}) = E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H}).$$

Now suppose that $D \in N(\delta_A + \lambda_0)$, hence $DA = (A - \lambda_0)D = BD$. Therefore $D^*B^* = A^*D^*$. A similar computation as above shows that for $\mu \in \rho(B^*)$ we have

$$(A^* - \mu)D^*(B^* - \mu)^{-1} = D^*,$$

thus

$$D^*(\mathcal{H}) \subseteq \bigcap_{\mu \in \rho(B^*)} (A^* - \mu)(\mathcal{H}).$$

Since $\rho(B^*) = \{\overline{\lambda} : \lambda \in \rho(B)\}$, we get from Proposition 2.1 that

$$\begin{aligned}D^*(\mathcal{H}) &\subseteq \bigcap_{\lambda \in \rho(B)} (A - \lambda)^*(\mathcal{H}) = \bigcap_{\lambda \in \rho(B)} (A - \lambda)(\mathcal{H}) \\ &= E(\sigma(A) \cap \sigma(B))(\mathcal{H}) = E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H}).\end{aligned}$$

■

The following propositions are of central importance for our investigations.

Proposition 2.4 *Let A be a normal operator in $\mathcal{L}(\mathcal{H})$ and suppose that*

$$\sigma(A) \cap \sigma(A + 2k\pi i) \subseteq \{k\pi i\} \quad \text{for} \quad k = 1, 2, \ldots .$$

If $k \in \mathbb{N} \setminus \{0\}$, $C \in N(\delta_A + 2k\pi i)$ and $D \in N(\delta_A - 2k\pi i)$ then

$$AC = k\pi i C = -CA$$

and

$$DA = k\pi i D = -AD.$$

Proof. Put $\lambda_0 = -2k\pi i$. From $C \in N(\delta_A - \lambda_0)$ we get from Proposition 2.3 that

$$C(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A + 2k\pi i))(\mathcal{H}).$$

Since $\sigma(A) \cap \sigma(A + 2k\pi i) \subseteq \{k\pi i\}$,

$$E(\sigma(A) \cap \sigma(A + 2k\pi i))(\mathcal{H}) \subseteq E(\{k\pi i\}).$$

From Theorem 12.29 in [8] it follows that $E(\{k\pi i\}) = N(A - k\pi i)$. Thus

$$C(\mathcal{H}) \subseteq N(A - k\pi i),$$

hence $AC = k\pi i C$. From $CA - AC = -2k\pi i C$ we conclude that $CA = -k\pi i C = -AC$. For $D \in N(\delta_A - 2k\pi i) = N(\delta_A + \lambda_0)$ we get from Proposition 2.3 that

$$D^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A + 2k\pi i))(\mathcal{H}) \subseteq N(A - k\pi i).$$

Thus $AD^* = k\pi i D^*$. Therefore $AD^*x = k\pi i D^* x$ for each $x \in \mathcal{H}$. The normality of A gives $A^*D^*x = -k\pi i D^* x$, hence $A^*D^* = -k\pi i D^*$, thus $DA = k\pi i D$. From $DA - AD = 2k\pi i$ we derive

$$AD = DA - 2k\pi i = -k\pi i D = -DA. \qquad \blacksquare$$

Proposition 2.5 *Suppose that $S \in \hat{\Sigma}(\mathcal{H})$ and $k \in \mathbb{N} \setminus \{0\}$.*

(a) *If $C \in N(\delta_S + 2k\pi i)$ then $SC = k\pi i C = -CS$.*

(b) *If $D \in N(\delta_S - 2k\pi i)$ then $DS = k\pi i D = -SD$.*

(c) *If $U \in N(f(\delta_S))$ then $SU + US = 0$.*

(d) *If $\sigma_p(S) \cap \{n\pi i : \ n = 1, 2, \ldots\} = \emptyset$ then $N(f(\delta_S)) = \{0\}$.*

Proof. We know that there are operators X and A in $\mathcal{L}(\mathcal{H})$ such that X is invertible in $\mathcal{L}(\mathcal{H})$, A is normal and

$$S = X^{-1}AX.$$

Therefore we have $S-\lambda = X^{-1}(A-\lambda)X$ for each $\lambda \in \mathbb{C}$ and $\sigma(S) = \sigma(A)$ and $\sigma(S-\lambda) = \sigma(A-\lambda)$. Since $S \in \hat{\Sigma}(\mathcal{H})$, we derive that

$$(*) \qquad \sigma(A) \cap \sigma(A + 2n\pi i) \subseteq \{n\pi i\}$$

for $n = 1, 2, \ldots$.

(a) From $CS - SC = -2k\pi i C$, we get

$$CX^{-1}AX - X^{-1}AXC = -2k\pi i C,$$

therefore $(XCX^{-1})A - A(XCX^{-1}) = -2k\pi i(XCX^{-1})$. This shows that $XCX^{-1} \in N(\delta_A + 2k\pi i)$. From $(*)$ and Proposition 2.4 we see that

$$AXCX^{-1} = k\pi i XCX^{-1} = -XCX^{-1}A,$$

hence $SC = k\pi i C = -CS$.

(b) Similar.

(c) Follows from (a), (b) and Proposition 2.2(d).

(d) Let $n \in \mathbb{N}\backslash\{0\}$. Since $n\pi i \notin \sigma_p(S)$, we see from (a) that $N(\delta_S + 2n\pi i) = \{0\}$. In view of Proposition 2.2(d) it remains to show that $N(\delta_S - 2n\pi i) = \{0\}$. Take $D \in N(\delta_S - 2n\pi i)$ and put $\tilde{D} = XDX^{-1}$. As in the proof of (a) we see that $\tilde{D} \in N(\delta_A - 2n\pi i)$. From Proposition 2.3 it follows that

$$\tilde{D}^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A + 2n\pi i))(\mathcal{H}).$$

By $(*)$ we get $\tilde{D}^*(\mathcal{H}) \subseteq E(\{n\pi i\}) = N(A - n\pi i)$. Since $\sigma_p(A) = \sigma_p(S)$ and $n\pi i \notin \sigma_p(S)$, it follows that $N(A - n\pi i) = \{0\}$. Thus $\tilde{D}^* = 0$, hence $D = 0$. ■

3 Proofs

Proof of Theorem 1.1. Use Proposition 2.2(g) to see that

$$f(\delta_T)(\delta_T(e^S)) = e^{-T}e^S e^T - e^S = 0,$$

hence $V = \delta_T(e^S) = e^S T - Te^S \in N(f(\delta_T))$. Proposition 2.5(c) shows that

$$0 = TV + VT = Te^S T - T^2 e^S + e^S T^2 - Te^S T = e^S T^2 - T^2 e^S.$$

If $\sigma_p(T) \cap \{k\pi i : \; k = 1, 2, \ldots\} = \emptyset$, then by Proposition 2.5(d), $V = 0$, thus $e^S T = Te^S$. ■

Proof of Theorem 1.2. It follows from Theorem 1.1 that $T^2e^S = e^ST^2$. By Proposition 2.2(g) we derive

$$f(\delta_S)(\delta_S(T^2)) = e^{-S}T^2e^S - T^2 = 0,$$

hence $U = \delta_S(T^2) = T^2S - ST^2 \in N(f(\delta_S))$. Proposition 2.5(c) gives now

$$0 = SU + US = ST^2 - S^2T^2 + T^2S^2 - ST^2 = T^2S^2 - S^2T^2.$$

■

Proof of Theorem 1.3.

(a) We know from Theorem 1.1 that $e^ST = Te^S$, thus

$$f(\delta_S)(\delta_S(T)) = e^{-S}Te^S - T = 0,$$

therefore $TS - ST \in N(f(\delta_S))$. Use again Proposition 2.5(c) to see that

$$0 = S(TS - ST) + (TS - ST)S = TS^2 - S^2T$$

(b) Proposition 2.5(d) gives $N(f(\delta_S)) = \{0\}$. Hence $TS = ST$. ■

Proof of Theorem 1.4. Proposition 2.2(g) shows that

$$\begin{aligned} f(\delta_{T+S})(\delta_{T+S}(e^T)) &= e^{-(T+S)}e^Te^{T+S} - e^T \\ &= e^{-S}e^{-T}e^Te^{T+S} - e^T \\ &= e^{-S}e^Se^T - e^T = 0, \end{aligned}$$

therefore $U = e^T(T+S) - (T+S)e^T = e^TS - Se^T \in N(f(\delta_{T+S}))$. Since $N(f(\delta_{T+S})) = \{0\}$ (Proposition 2.5(d)), it follows that $U = 0$, hence $e^TS = Se^T$, therefore

$$\begin{aligned} f(\delta_{T+S})(\delta_{T+S}(S)) &= e^{-(T+S)}Se^{T+S} - S \\ &= e^{-S}e^{-T}Se^Te^S - S \\ &= 0. \end{aligned}$$

Hence we see that $S(T + S) - (T + S)S = ST - TS \in N(f(\delta_{T+S})) = \{0\}$. ■

Proof of Theorem 1.5. Since

$$f(\delta_S)(\delta_S(T)) = e^{-S}Te^S - T = e^{-T}Te^T - T = 0,$$

we have $TS - ST \in N(f(\delta_S))$, thus, by Proposition 2.5(c)

$$0 = S(TS - ST) + (TS - ST)S = TS^2 - S^2T,$$

hence $TS^2 = S^2T$.

If $\sigma_p(S) \cap \{k\pi i : \ k = 1, 2, \ldots\} = \emptyset$, we see from Proposition 2.5(d) that $N(f(\delta_S)) = \{0\}$, thus $TS = ST$. ■

4 Corollaries

Corollary 4.1 *If $A \in \mathcal{L}(\mathcal{H})$ then*

$$A \quad \textit{is normal} \quad \Leftrightarrow \quad e^A e^{A^*} = e^{A+A^*} = e^{A^*} e^A.$$

Proof. The implication „$\Rightarrow$“ is clear.
„$\Leftarrow$“: Since $A + A^*$ is selfadjoint, $\sigma(A + A^*) \subseteq \mathbb{R}$. Thus $A + A^* \in \hat{\Sigma}(\mathcal{H})$ and $\sigma_p(A + A^*) \cap \{k\pi i : k = 1, 2, \ldots\} = \emptyset$. Theorem 1.4 shows now that $AA^* = A^*A$. ■

Corollary 4.2 *If $A, B \in \mathcal{L}(\mathcal{H})$ are selfadjoint then*

$$A = B \quad \Leftrightarrow \quad e^A = e^B.$$

Proof. The implication „$\Rightarrow$“ is clear.
„$\Leftarrow$“: Since $A \in \hat{\Sigma}(\mathcal{H})$ and $\sigma_p(A) \cap \{k\pi i : k = 1, 2, \ldots\}$ we see from Theorem 1.5 that $AB = BA$. Thus $A - B$ is selfadjoint and $e^{A-B} = I$. Take $\lambda \in \sigma(A - B)$. Thus $\lambda \in \mathbb{R}$ and $e^\lambda = 1$, hence $\lambda = 0$. This gives $\sigma(A - B) = \{0\}$. From $\|A - B\| = r(A - B) = 0$ we get $A = B$. ■

Corollary 4.3 *Suppose that A and B are normal operators in $\mathcal{L}(\mathcal{H})$ and that $e^A = e^B$. Then*

$$A + A^* = B + B^*.$$

Proof. Use Corollary 4.1 to see that $e^{A+A^*} = e^{B+B^*}$. By Corollary 4.2, $A + A^* = B + B^*$. ■

Corollary 4.4 *If $A \in \mathcal{L}(\mathcal{H})$ is normal then*

$$A = -A^* \quad \Leftrightarrow \quad e^A \quad \textit{is unitary.}$$

Proof. The implication „$\Rightarrow$“ is clear.
„$\Leftarrow$“: Since A is normal,

$$e^{A+A^*} = e^A e^{A^*} = e^A (e^A)^* = I = e^0.$$

Now use Corollary 4.2 to derive $A + A^* = 0$. ■

For our next result we need the following lemma (see also [8, Theorem 12.37]).

Lemma 4.1 *If $T \in \mathcal{L}(\mathcal{H})$ is invertible then there are selfadjoint operators A and B in $\mathcal{L}(\mathcal{H})$ such that*

$$T = e^{iA}e^{B}, \quad \sigma(A) \subseteq [-\pi, \pi] \quad \textit{and} \quad \pi \notin \sigma_p(A).$$

Proof. If T is invertible, so are T^* and T^*T. Theorem 12.33 in [8] shows that the positive square root $(T^*T)^{1/2}$ is also invertible. By [8, Theorem 12.35] there is a unitary $U \in \mathcal{L}(\mathcal{H})$ with $T = U(T^*T)^{1/2}$. Since $\sigma((T^*T)^{1/2}) \subseteq (0, \infty)$, log is a continuous real function on $\sigma((T^*T)^{1/2})$. Thus the symbolic calculus for selfadjoint operators shows that there is a selfadjoint $B \in \mathcal{L}(\mathcal{H})$ such that $(T^*T)^{1/2} = e^B$. A. Wintner has shown in [16] that there is a selfadjoint $A \in \mathcal{L}(\mathcal{H})$ such that $U = e^{iA}$, $\sigma(A) \subseteq [-\pi, \pi]$ and $\pi \notin \sigma_p(A)$. ∎

Remarks.
(1) It is shown in [13] that if $U \in \mathcal{L}(\mathcal{H})$ is unitary then there is a *unique* selfadjoint operator $A \in \mathcal{L}(\mathcal{H})$ such that

$$U = e^{iA}, \quad \sigma(A) \subseteq [-\pi, \pi] \quad \text{and} \quad \pi \notin \sigma_p(A).$$

For related results see [9].

(2) Lemma 4.1 shows that an invertible operator in $\mathcal{L}(\mathcal{H})$ is the product of two exponentials. It is natural to ask whether every invertible operator is an exponential, rather than merely the product of two exponentials. The answer is affirmative if $\dim \mathcal{H} < \infty$, as a consequence of [8, Theorem 10.30]. But in general the answer is negative, as one can see from [8, Theorem 12.38]. For normal and invertible operators we have the following results.

Corollary 4.5 *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is invertible. The following assertions are equivalent:*

(a) *T is normal.*

(b) *There is some normal $S \in \mathcal{L}(\mathcal{H})$ such that $T = e^S$.*

Proof. (b) $\Rightarrow$ (a): Clear.
(a) $\Rightarrow$ (b): By Lemma 4.1 there are selfadjoint operators $A, B \in \mathcal{L}(\mathcal{H})$ such that

$$T = e^{iA}e^{B}$$

and

$$\sigma(A) \subseteq [-\pi, \pi] \text{ and } \pi \notin \sigma_p(A). \tag{1}$$

From $T^* = e^B e^{-iA}$ and the normality of T we see that

$$e^{2B} = T^*T = TT^* = e^{iA}e^{2B}e^{-iA},$$

thus

(2) $$e^{2B}e^{iA} = e^{iA}e^{2B}.$$

Use (1) to get

(3) $$iA \in \hat{\Sigma}(\mathcal{H}) \text{ and } \sigma_p(iA) \cap \{k\pi i : k = 1, 2, \ldots\} = \emptyset.$$

Since $2B$ is selfadjoint, we have

(4) $$2B \in \hat{\Sigma}(\mathcal{H}) \text{ and } \sigma_p(2B) \cap \{k\pi i : k = 1, 2, \ldots\} = \emptyset.$$

Therefore it follows from (2), (3), (4) and Theorem 1.3(b) that $AB = BA$. Thus $T = e^{iA+B}$. Put $S = iA + B$. Then $T = e^S$ and S is normal. ■

Corollary 4.6 *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is invertible and normal. Then there is a unique normal operator $S \in \mathcal{L}(\mathcal{H})$ such that*

$$T = e^S, \quad r(S - S^*) \leq 2\pi \quad \textit{and} \quad 2\pi i \notin \sigma_p(S - S^*).$$

Proof. The proof of Corollary 4.5 shows that there is a normal $S \in \mathcal{L}(\mathcal{H})$ with $T = e^S$, $S = iA + B$, where A and B are selfadjoint, $AB = BA$, $\sigma(A) \subseteq [-\pi, \pi]$ and $\pi \notin \sigma_p(A)$. Since $S - S^* = 2iA$, we get $r(S - S^*) \leq 2\pi$ and $2\pi i \notin \sigma_p(S - S^*)$. Now suppose that $R \in \mathcal{L}(\mathcal{H})$ is normal, $T = e^R$, $r(R - R^*) \leq 2\pi$ and $2\pi i \notin \sigma_p(R - R^*)$. Then there are selfadjoint operators $C, D \in \mathcal{L}(\mathcal{H})$ with

$$R = iC + D \quad \text{and} \quad CD = DC.$$

From $R - R^* = 2iC$ we see that

$$\sigma(C) \subseteq [-\pi, \pi] \quad \text{and} \quad \pi \notin \sigma_p(C).$$

It follows from $e^S = e^R$ that $T^* = e^B e^{-iA} = e^D e^{-iC}$, thus $e^{2B} = T^*T = e^{2D}$. Now use Corollary 4.2 to derive $B = D$. From $e^{iA}e^B = e^{iC}e^D$ we see that

$$e^{iA} = e^{iC}.$$

It is shown in [13] that then $A = C$ (see Remark (1)). Hence $S = T$. ■

Our final result reads as follows:

Corollary 4.7 *For $P \in \mathcal{L}(\mathcal{H})$ the following assertions are equivalent:*

(a) $e^{T+P} = e^T$ *for all $T \in \mathcal{L}(\mathcal{H})$.*

(b) *There is some $k \in \mathbb{Z}$ such that $P = 2k\pi i I$.*

Proof. (b) $\Rightarrow$ (a): Clear.
(a) $\Rightarrow$ (b): Take $T \in \mathcal{L}(\mathcal{H})$ with $r(T) < \pi$. Proposition 2.2(e) shows that $r(\delta_T) < 2\pi$. Thus, by Proposition 2.2(c), $M_T = \emptyset$, hence $N(f(\delta_T)) = \{0\}$ (Proposition 2.2(a)). From

$$\begin{aligned} f(\delta_T)(\delta_T(T+P)) &= e^{-T}(T+P)e^{T} - (T+P) \\ &= e^{-(T+P)}(T+P)e^{T+P} - (T+P) \\ &= 0 \end{aligned}$$

we see that $(T+P)T = T(T+P)$, hence $TP = PT$. Therefore we have shown that

$$TP = PT \text{ for each } T \in \mathcal{L}(\mathcal{H}) \text{ with } r(T) < \pi. \tag{5}$$

Now take $T \in \mathcal{L}(\mathcal{H})$ with $r(T) \geq \pi$ and put $T_0 = \frac{\pi}{2r(T)}T$. Then $r(T_0) = \frac{\pi}{2}$. (5) shows that $T_0P = PT_0$. Therefore we have that $TP = PT$ for all $T \in \mathcal{L}(\mathcal{H})$. Thus $P = \alpha I$ for some $\alpha \in \mathbb{C}$. Since $e^P = I$, $I = e^{\alpha}I$, hence $e^{\alpha} = 1$. ∎

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Christoph Schmoeger
Mathematisches Institut I
Universität Karlsruhe
D-76128 Karlsruhe
Germany
E-mail: christoph.schmoeger@math.uni-karlsruhe.de