Abstracts

Multiscale preconditioning

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1. Introduction

In this contribution, we consider the numerical solution of discrete equations arising from the discretisation of a multiscale problem. We show that the multiscale nature can be used to construct an efficient preconditioner. In contrast to [8], we do not require conditions of cell symmetry here, but require a sufficient resolution of the fine scale problem which allows to transfer the continuous approximation result of homogenisation theory to the discrete situation.

2. Microscopic problem

For $k \geq 1$, let $A \in C^{0,1}_{\mathrm{per}}(\mathbb{R}^d, \mathbb{R}^{d\times d})$ be a Lipschitz-continuous, 1-periodic, matrix-valued function which satisfies $A_{ij}(x) = A_{ji}(x)$ for all $x \in \mathbb{R}^d$ together with the ellipticity and continuity condition that there exist constants $0 < \lambda_1 \leq \lambda_2 < \infty$ such that

$$\lambda_1 |\xi|^2 \leq (\xi, A(x)\xi) \leq \lambda_2 |\xi|^2 , \quad x \in \mathbb{R}^d . \tag{1}$$

Now, let $\varepsilon = 1/N$ for some $N \in \mathbb{N}$ and $\Omega = (0,1)^d$. Given $f \in L^2(\Omega)$, we search for $u^\varepsilon \in H^1_0(\Omega)$ satisfying

$$a^\varepsilon(u^\varepsilon, v) := \int_\Omega \nabla v(x) A(\frac{x}{\varepsilon}) \nabla u^\varepsilon(x) dx = \int_\Omega f(x) v(x) dx \tag{2}$$

for all test functions $v \in H^1_0(\Omega)$. The Lemma of Lax-Milgram then guarantees unique solvability of this problem.

3. Homogenised problem

Homogenisation theory (c.f. [7]) proves that, for small ε, a good approximation of the solution u^ε of problem (2) can be constructed by solving the following *homogenised* problem: find $u^0 \in H^1_0(\Omega)$ satisfying

$$a^0(u^0, v) := \int_\Omega \nabla v : A^0 \, \nabla u^0 dx = \int_\Omega v \, f dx, \quad v \in H^1_0(\Omega), \tag{3}$$

where the constant coefficient matrix A^0 is defined as an average

$$A^0_{ij} = \int_Y \left(A_{ij}(y) + \sum_{l=1}^{d} A_{il}(y) \frac{\partial N_j}{\partial y_l}(y) \right) dy, \tag{4}$$

over $Y = (0,1)^d$ with functions $N_k \in \dot{H}^1(Y), k = 1, \ldots, n$ satisfying

$$a(N_k, V) := \int_Y \nabla V(y) : A(y) \nabla N_k(y) dy = - \int_Y \nabla \varphi(y) : A(y) e_k dy \tag{5}$$

for all $V \in \dot{H}^1(Y)$, where $e_k \in \mathbb{R}^d$ is the vector with components $(\delta_{jk})_{j=1,\dots,n}$.

Theorem 1. *Let u^ε be the solution of (2), N_k be the solutions of (5), and u^0 be the solution of (3). Consider the first-order corrector*

$$u^{1,\varepsilon}(x) := u^0(x) + \varepsilon \sum_{k=1}^{d} N_k(\frac{x}{\varepsilon}) \frac{\partial u^0}{\partial x_k}(x) \tag{6}$$

and the boundary correction $\theta^\varepsilon = \theta^\varepsilon(\nabla u^0)$ defined by $\theta^\varepsilon = u^{1,\varepsilon}$ on $\partial\Omega$ and

$$\int_{\Omega^\varepsilon} \nabla v^\varepsilon(x) \cdot A^\varepsilon(x) \nabla \theta^\varepsilon(x) dx = 0, \qquad v^\varepsilon \in V_0^\varepsilon \,. \tag{7}$$

Then we have the error estimate

$$\|\nabla(u^\varepsilon - u^{1,\varepsilon} + \theta^\varepsilon)\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon \|f\|_{L^2(\Omega)} \,. \tag{8}$$

Proof. See, e.g., [7], [8], [9]. □

4. Discretisation

Let $T_{\hat{h}}$ denote a quasi-uniform mesh of the unit cell Y which fits across the identified boundary ∂Y. Thus, $T_{\hat{h}}$ can be scaled by ε and repeated periodically, such that we obtain a ε-periodic mesh T_h on Ω of meshsize $h = \varepsilon\hat{h}$. Both T_h and $T_{\hat{h}}$ may contain simplices or tensor products of simplices (i.e. triangles and/or quadrangles for $n = 2$). With T_H we denote the cube mesh of meshsize $H = \varepsilon$.

On T_h, we discretise problem (2) with standard conforming finite elements of order 1 and denote the ansatz space with Dirichlet boundary conditions by $S^1_{h,0}$. In the same way, we discretise problem (5) on $T_{\hat{h}}$, and denote the ansatz space consisting of periodic functions with zero mean value by $S^1_{\hat{h}}$. Solving the arising finite-dimensional problem gives us approximations $\mathbf{N}_{\hat{h}} \in S_{\hat{h}}$ of N, and using $\mathbf{N}_{\hat{h}}$ instead of $\mathbf{N}_{\hat{h}}$ in (4) gives us an approximation $\mathbf{A}^0_{\hat{h}}$ to $\mathbf{A}^0$. Finally, we can use $\mathbf{A}^0_{\hat{h}}$ for discretizing (3) using an ansatz space $S^1_{H,0}$ consisting of conforming multilinear finite elements on T_H.

Remark 2. *Because of Theorem 1, the most interesting case is $h \leq \varepsilon^2$, because otherwise the solution to the homogenised problem (3) together with a boundary layer approximation according to (7) would yield an equally good or even better approximation at lower computational cost.*

5. Multiscale preconditioning

If h is small (e.g. of size ε^2 as indicated in Remark 2), the efficient solution of problem (2) on $S^1_{h,0}$ becomes large, and efficient solution techniques are called for. Here, we propose to use a domain decomposition with respect to the subdomains

$$\Omega^\varepsilon_{\vec{k}} = \Omega \cap \varepsilon(\vec{k} + (-1,1)^d) \,, \quad 1 \leq \max_{i=1,\dots,d} |k_i| \leq \frac{1}{\varepsilon} - 1 \tag{9}$$

together with a coarse space defined by solving the discrete homogenised problem. More precisely, let $V_h = S^1_{h,0}$, and for $1 \le i \le m = \frac{1}{(1/\varepsilon-1)^d}$, let $i \mapsto \vec{k}^{(i)}$ be a numbering of the set $\{\vec{k} \in \mathbb{Z}^d : 0 \le k_j \le \frac{1}{\varepsilon} - 1\}$.

Theorem 3. *Let*

$$V_i = \{v \in V_h : \mathrm{Supp}(v) \subset \Omega_{\vec{k}^{(i)}}\}\,, \quad i = 1, \dots, n\,, \tag{10}$$

and set $V_0 = \mathrm{Range}(p^h_H)$, *where* $p^h_H : V_H \to V_h$ *is defined as*

$$u_H \mapsto I_h \circ (u_H + \varepsilon \sum_{k=1}^{d} N_k \frac{\partial u_H}{\partial x_k}) \tag{11}$$

with $I_h : C^0(\Omega) \to S_h$ *denoting the projection operator defined by nodal evaluation. For* $i = 1, \dots, n$ *let* $A_i : V_i \to V_i'$ *be defined by restricting the bilinear form* a^ε *from (2) to* V_i. *For* $i = 0$, *choose* A_0 *to be a discretisation of the homogenised problem.*

If $\hat{h} \lesssim \varepsilon$, *and* $H \sim \varepsilon$, *then every* $v \in V_h$ *has a decomposition* $v = \sum_{i=0}^n v_i$ *with* $v_i \in V_i$ *such that for some* $K_1 > 0$ *independent of* ε *and* h, *we have*

$$\Big(\sum_{i=0}^{n} \|v_i\|^2_{A_i}\Big)^{\frac{1}{2}} \le K_1 \|v\|_A\,. \tag{12}$$

and there exists $K_2 > 0$ *such that for arbitrary* $u_i \in V_i$,

$$\sum_{k=0}^{n}\sum_{i=0}^{n} a(v_k, u_i) \le K_2 \|v\|_A \Big(\sum_{i=1}^{n} \|u_i\|^2_{A_i}\Big)^{\frac{1}{2}}\,. \tag{13}$$

Proof. For $v_h \in V_h$, let v_H denote the solution of

$$a_H(v_H, w_H) = a^\varepsilon(v_h, p^h_H w_H)\,, \quad w_H \in V_H\,,$$

and set $v_0 = p^h_H v_H$. Then it is a straightforward combination of standard finite element error estimates and Theorem 1 to show that the remainder $w = v_h - v_0$ satisfies $\|w\|_{L^2(\Omega)} \lesssim \varepsilon$. This is sufficient to show that (12) is satisfied for the decomposition $w = \sum_{i=1}^n v_i$, if v_i is defined as $I_h \circ (w\psi_i)$ for $\{\psi_i\}_{i=1,\dots,m}$ denoting the set of standard nodal basis functions in S_H with $\psi_i(\vec{k}^{(j)}) = \delta_{ij}$. (13) follows, because no more than $2^d + 1$ of the subspaces have nonempty intersection. □

Corollary 4. *A parallel subspace correction using the subspaces* $V_0, \dots, V_n$ *defined above yields an optimal preconditioner for problem (2) discretised on* $V_h = S^1_{h,0}$.

6. Numerical results

We look at a special case of problem (2), with a scalar diffusion coefficient

$$\mathbf{A}_{ij}(y) = \frac{\delta_{ij}}{(2 + 1.8\sin(2\pi y_1))(2 + 1.8\sin(2\pi y_2))}\,.$$

We discretize and solve this problem according to Sections 4 and 5. As solver for the local subproblems we used a simple CG iteration with Jacobi preconditioning. The implementation was done using the program Femlisp, see [5]. We vary $\hat{h} = \varepsilon$,

ε	1/2	1/4	1/8	1/16
ρ	0.0004	0.36	0.45	0.5

TABLE 1. Convergence rates for solving the diffusion problem.

ε	1/2	1/4	1/8	1/16
ρ	0.65	0.69	0.69	0.7

TABLE 2. Convergence rates for solving the Stokes problem.

$H = \varepsilon$, $h = \varepsilon^2$ simultaneously with ε and obtain the convergence rates shown in Table 1. Of course, a more practical version of this algorithm would not use an exact solver on each subspace, but an approximate hierarchical solver. The numerical results do not change significantly in this case and also Theorem 3 can be adapted easily.

Finally, we want to remark that the technique described here can also be applied to higher order discretisations, systems of equations like linear elasticity, and even to situations where the micro- and macroproblem are of rather different character. For example, consider the problem of solving Stokes equation in a domain Ω^ε with periodically distributed holes of size ε, with no-slip boundary conditions posed on $\partial\Omega^\varepsilon$. Here the homogenised equation is a diffusion equation (Darcy's law) with a certain permeability which can be computed by solving flow problems on a representative cell. Preliminary results for $\hat{h} = \text{const}$ (which corresponds to the setting of [8]) and a very cheap choice of overlapping subspaces centered at vertices of T_h show a robust convergence independent of ε, see Table 2. However, the theoretical analysis of this method as well as the analogous method with $\hat{h} \sim \varepsilon$ is still open.

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