Exercises in Convex Geometry
Exercise Sheet No. 4 – 11/11/2008

Exercise 13
Let \( A \subset \mathbb{R}^n \) be closed and convex. A subset \( M \subset A \) is called extreme (in \( A \)), if \( M \) is convex and \( x, y \in A, (x, y) \cap M \neq \emptyset \) implies \([x, y] \subset M\). Show that:

a) Extreme sets \( M \) are closed.

b) Each support set of \( A \) is extreme.

c) If \( M, N \subset A \) are extreme, then \( M \cap N \) is extreme.

d) If \( M \) is extreme in \( A \) and \( N \subset M \) is extreme in \( M \), then \( N \) is extreme in \( A \).

Exercise 14

a) Give a detailed proof of Theorem 2.1.2.

b) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuous function satisfying

\[
f \left( \frac{x_1 + x_2}{2} \right) \leq \frac{1}{2} \left( f(x_1) + f(x_2) \right) \quad \text{for all } x_1, x_2 \in \mathbb{R}^n.
\]

Show that \( f \) is convex.

Exercise 15
Let \( f : \mathbb{R}^n \to \mathbb{R} \) be convex and \( \emptyset \neq A \subset \mathbb{R}^n \) compact and convex. Show that there is a point \( x_0 \in \text{ext } A \) such that

\[
f(x_0) = \max \{ f(x) : x \in A \}.
\]

Exercise 16
Let \( f : \mathbb{R}^n \to (-\infty, \infty] \) be convex. Show that the following assertions are equivalent.

(i) \( f \) is closed.

(ii) \( f \) is semi-continuous from above, i.e.

\[
f(x) \leq \lim \inf_{y \to x} f(y).
\]

(iii) All the sublevel sets \( \{ f \leq \alpha \}, \alpha \in \mathbb{R} \) are closed.

Turn in your solutions on Tuesday, 11/18/2008, after the lecture. Do not forget to indicate your name and student identity number (Matrikelnummer) on your solutions.