Exercises in Convex Geometry
Exercise Sheet No. 6 – 11/25/2008

Exercise 20
Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be positively homogeneous and twice continuously differentiable on \( \mathbb{R}^n \setminus \{0\} \). Show that there are convex bodies \( K, L \in \mathcal{K} \) with
\[
f = h_K - h_L.
\]

Hint: Use exercise 19.

Exercise 21
Let \( f : \mathbb{R}^n \rightarrow (-\infty, \infty] \) be convex and \( x \in \text{int dom } f \). We define the subgradient of \( f \) at \( x \) as
\[
\partial f(x) := \{ v \in \mathbb{R}^n : f(y) \geq f(x) + \langle v, y-x \rangle \text{ for all } y \in \mathbb{R}^n \}.
\]
Show:

a) \( \partial f(x) \) ist nonempty, compact and convex.

b) \( \partial f(x) = \{ v \in \mathbb{R}^n : \langle v, u \rangle \leq f'(x; u) \text{ for all } u \in \mathbb{R}^n \} \).

c) If \( f \) is differentiable at \( x \), then \( \partial f(x) = \{ \text{grad } f(x) \} \).

Exercise 22
Let \( f : \mathbb{R}^n \rightarrow [-\infty, \infty] \) be closed and convex and \( x \in \text{dom } f \). Show that
\[
s \in \partial f(x) \iff f^*(s) + f(x) = \langle s, x \rangle.
\]
Conclude that
\[
s \in \partial f(x) \iff x \in \partial f^*(s).
\]

Exercise 23
Let \( b \in \mathbb{R}^n \) and let \( Q \) be a symmetric, positive definite \( n \times n \) matrix. Define the function
\[
f(x) := \frac{1}{2} \langle x, Qx \rangle + \langle b, x \rangle, \quad x \in \mathbb{R}^n.
\]
Determine \( f^* \).

Hint: You can use the first statement from exercise 22.

Turn in your solutions on Tuesday, 12/02/2008, after the lecture. Do not forget to indicate your name and student identity number (Matrikelnummer) on your solutions.

Discussion of solutions: Wednesday, 12/03/2008.