Exercise 1: Let $p = (2, -2, 3)^\top$, $a = (1, -5, 6)^\top$ and let $G = \{p + ta \in \mathbb{R}^3 : t \in \mathbb{R}\}$ denote the line through the point $p$ in direction of $a$.

(a) Find a point $w \neq p$ which is contained in $G$.
(b) Are the points $(1, 1, 1)^\top$ and $(\frac{3}{2}, -\frac{7}{2}, 6)^\top$ contained in $G$?
(c) Determine a parameter form of the line which contains the points $q = (1, 2, 5)^\top$ and $r = (-4, -2, 0)^\top$.

Exercise 2: Consider the following vectors in $\mathbb{R}^3$
\[ u = (-2, 1, 3)^\top, \quad v = (-9, 2, 4)^\top, \quad w = (3, 1, 5)^\top. \]

(a) Calculate the following linear combinations of these vectors: $u + w$, $v - 3u$, $2u - v + w$.
(b) Show that every subset of two vectors in the set $\{u, v, w\}$ is linearly independent.
(c) Is the set of all three vectors linearly independent, as well?

Exercise 3: Let the set $V \subseteq \mathbb{R}^3$ be given by
\[ V = \{x \in \mathbb{R}^3 : x_1 + x_2 + 2x_3 = 0\}. \]

(a) Prove that $V$ is a linear subspace of $\mathbb{R}^3$. To this end prove that the set $V$ is algebraically closed with respect to addition and multiplication by scalars.
(b) Consider the set $E \subseteq \mathbb{R}^3$ given by
\[ E = \{x \in \mathbb{R}^3 : x_1 + x_2 + 2x_3 = 1\}. \]
Show that $E$ is an affine subspace of $\mathbb{R}^3$. To this end find a point $p \in \mathbb{R}^3$ such that $E = p + V$.
(c) Prove that the dimension of the space $V$ equals 2. To this end find two linearly independent vectors $u, v \in V$ such that every element in $V$ can be written as a linear combination of $u$ and $v$.

Note: This shows that $V$ is a plane in $\mathbb{R}^3$.

Exercise 4:

(a) Consider the following four vectors in $\mathbb{R}^4$
\[ a_1 = (1, 0, 0, 0)^\top, \quad a_2 = (0, 1, 0, 0)^\top, \quad a_3 = (1, 2, 3, 4)^\top, \quad a_4 = (4, 3, 2, 1)^\top. \]
Prove that these vectors form a basis of $\mathbb{R}^4$.

(b) Another basis is given by the four vectors
\[ b_1 = (1, 0, 0, 0)^\top, \quad b_2 = (1, 1, 0, 0)^\top, \quad b_3 = (1, 1, 1, 0)^\top, \quad b_4 = (1, 1, 1, 1)^\top. \]
Determine the coordinates of the vector $p = (1, 3, 0, 2)^\top \in \mathbb{R}^4$ (given with respect to the standard basis) with respect to the basis in part a) and the basis in part b).

Exercise 5: Let $P_n$ denote the vector space of polynomials with real coefficients and degree at most $n \in \mathbb{N}$.

(a) Does the set $\{1, x, x^2\}$ form a basis of $P_2$?
(b) Does the set $\{1, x, x^2, 1 + x^2\}$ form a basis of $P_2$?
(c) Let $q_1(x) = 1 - x + x^2$ and $q_2(x) = 1 + 2x - x^2$ denote two polynomials in $P_2$. Calculate the dimension of the subspace $U = \text{span}\{q_1, q_2\}$ and write $q_3(x) = 3x - 2x^2$ as a linear combination of $q_1$ and $q_2$.

Due date: Your written solutions are due on Tuesday, 29th April 2014. Please put them into the box in the student office until 2:00 PM.
Exercise T1: Sketch the following subsets of \( \mathbb{R}^2 \) and decide whether they are linear subspaces.

(a) \( U_1 := \left\{ \lambda_1 \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 0 \\ 4 \end{pmatrix} \mid \lambda_1, \lambda_2 \in \mathbb{R} \right\} \),

(b) \( U_2 := \left\{ \lambda \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} \mid \lambda \in [0, 1], k \in \mathbb{Z} \right\} \),

(c) \( U_3 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid 2x - y = 0, x, y \in \mathbb{R}, x \geq 0 \right\} \).

Exercise T2: Let \( \lambda \in \mathbb{R} \) and consider the following vectors in \( \mathbb{R}^3 \)

\[
U = \begin{pmatrix} \lambda \\ 1 \\ 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 \\ \lambda \\ 1 \end{pmatrix}, \quad W = \begin{pmatrix} 1 \\ 1 \\ \lambda \end{pmatrix}.
\]

(a) For which values of \( \lambda \) do these vectors form a basis of \( \mathbb{R}^3 \)?

(b) Let \( \lambda = 0 \). Determine the coordinates of the vectors \( (1, 1, 1)^\top \) and \( (3, 3, 2)^\top \) (given with respect to the standard basis) with respect to the basis in part a).

For detailed information regarding this course visit the following web page:

www.math.kit.edu/iag6/lehre/am22014s/en

Tutorial:   Friday, 25th April 2014, 9:45 AM