Exercise 16: Let $a, b, c \in \mathbb{R}^3$ and let $\Delta$ denote a triangle in $\mathbb{R}^3$ with corners $a, b, c$ and center of mass $s := \frac{1}{3}(a + b + c)$. Consider a linear transformation $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the point $a$ to $a' = (2, -2, -1)^\top$, the point $b$ to $b' = (1, -4, 5)^\top$ and the point $s$ to $s' = (1, -3, 2)^\top$.

(a) Find the image $c'$ of the point $c$ under the transformation $\varphi$.

(b) Find two different linear transformations and two sets of points $a, b$ and $c$ which satisfy the conditions.

Exercise 17: Consider the linear transformation $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ with $\alpha(x) = Ax$, given by

$$A = \frac{1}{6} \begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}.$$ 

(a) Show that the equation $\alpha(\alpha(x)) = \alpha(x)$ holds for all $x \in \mathbb{R}^3$.

(b) Show that the equation $x \cdot u = \alpha(x) \cdot u$ is satisfied for each $u \in \text{range}(\alpha) = \{v \in \mathbb{R}^3 : v = \alpha(z) \text{ for an } z \in \mathbb{R}^3 \}$ and each $x \in \mathbb{R}^3$. Hint: The equation $(By) \cdot z = y(B^\top z)$ holds for all $y, z \in \mathbb{R}^3$ and every matrix $B \in \mathbb{R}^{3 \times 3}$.

(c) Calculate the distance from $x = (1, 5, -1)^\top$ to the subspace range($\alpha$).

Exercise 18: Consider the two planes $E : 2x_1 - x_2 - 2x_3 = 0$ and $F : x(\lambda, \mu) = (0, 1, 0)^\top + \lambda(4, 1, 3)^\top + \mu(4, -7, 3)^\top$, $\lambda, \mu \in \mathbb{R}$. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear transformation given by reflection in $E$ and let $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear transformation given by the orthogonal projection onto $F$.

Determine the transformation matrices $A$ of $\Phi$ and $B$ of $\Psi$ with respect to the standard basis. Check if $A$ or $B$ are orthogonal matrices and compute the matrix products $A^2$ as well as $B^2$.

Exercise 19: Let $\alpha, \beta \in \mathbb{R}$ and consider the following system of linear equations

$$
\begin{align*}
2x_1 + 3x_2 &+ 1x_4 = \beta \\
1x_1 &+ 1x_2 - 1x_4 = 1 \\
2x_2 &+ 1x_3 + 3x_4 = 0 \\
3x_1 &- 1x_2 + 2x_3 + 3x_4 = 2
\end{align*}
$$

(a) For which values of $\alpha, \beta \in \mathbb{R}$ does the system of linear equations have a solution?

(b) Determine the set of solutions $\mathcal{L}$ of the system of linear equations, depending on $\alpha$ and $\beta$. Moreover find the set of solutions $\mathcal{L}_0$ of the corresponding homogeneous system.

Exercise 20: Let $A, B, C$ denote the matrices given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 3 & 4 \end{pmatrix}. $$

Calculate the inverses of $A$ and $B$ and find a matrix $D \in \mathbb{R}^{3 \times 3}$, such that $ADB = C$.

Hint: The matrix multiplication is not commutative, i.e. $AB \neq BA$ in general.

Due date: Your written solutions are due on Tuesday, 20th May 2014. Please put them into the box in the student office until 2:00 PM.
Exercise T10: Consider the linear transformation \( \Phi : \mathbb{R}^2 \to \mathbb{R}^3 \) with \( x \mapsto \hat{x} = \Phi(x) := A x \) which satisfies

\[
\begin{align*}
A(\begin{pmatrix} 1 \\ 2 \end{pmatrix}) &= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, & A(\begin{pmatrix} 2 \\ 1 \end{pmatrix}) &= \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}.
\end{align*}
\]

(a) Calculate the images of the basis vectors \((1, 0)^\top\) and \((0, 1)^\top\) and the transformation matrix \(A\).

(b) Determine the kernel of \(\Phi\) and a representation of the range of \(\Phi\) in parametric form.

Exercise T11: Let \(\alpha, \beta \in \mathbb{R}\) and consider the system of linear equations \(Ax = b\) given by

\[
A = \begin{pmatrix} \alpha & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ \beta \\ 1 \end{pmatrix}.
\]

(a) For which values of \(\alpha, \beta\) is the system not solvable, uniquely solvable or has infinitely many solutions? Calculate the set of solutions depending on \(\alpha\) and \(\beta\).

(b) Determine the set of solutions \(L_0 = \{x \in \mathbb{R}^3 : Ax = 0\}\) depending on \(\alpha\), i.e., the set of solutions of the corresponding homogeneous system.

Exercise T12: Let \(\alpha \in \mathbb{R}\) and consider the following matrix

\[
A = \begin{pmatrix}
 1 & 0 & -2 & -2 \\
 2 & -2 & 1 & 2 \\
 0 & \alpha & 3 & -1 \\
-1 & 1 & 0 & 1
\end{pmatrix} \in \mathbb{R}^{4 \times 4}.
\]

For which values of \(\alpha\) is \(A\) regular? Determine the inverse matrix of \(A\) depending on \(\alpha\).

For detailed information regarding this course visit the following web page:

www.math.kit.edu/iag6/lehre/am22014s/en

Tutorial: Friday, 16th May 2014, 9:45 AM