Exercise Sheet No. 5
Advanced Mathematics II

Exercise 21: Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear transformation given by $\alpha(x) = Ax$ with

$$A = \begin{pmatrix} 2 & -1 & 5 \\ 3 & 3 & 0 \\ 1 & 4 & 1 \end{pmatrix}$$

and let $B = \begin{pmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ denote a basis of $\mathbb{R}^3$.

(a) Determine the transformation matrix $A_{B,B}$ of $\alpha$ with respect to the basis $B$ of $\mathbb{R}^3$.

(b) Let $v = (1, 2, 0)^\top$. Calculate the coordinate vector $[\alpha(v)]_B$ of $\alpha(v)$ with respect to $B$.

Exercise 22:

(a) Let $a, b, c \in \mathbb{C}^3$. Prove the following identities for the cross product

$$(a1) \quad a \times (b + c) = (a \times b) + (a \times c), \quad (a2) \quad a \times (b \times c) = (a \cdot c)b - (a \cdot b)c.$$

(b) Calculate the following cross products

$$(b1) \quad \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad (b2) \quad \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad (b3) \quad \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}, \quad (b4) \quad \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

Exercise 23: Let $a \in \mathbb{R}$ and let $U = (u_1, u_2, u_3, u_4)$ denote an (ordered) set of vectors in $\mathbb{R}^4$ given by

$u_1 = (a + 1, a + 2, -a)^\top, \quad u_2 = (a, 2, 1, -1)^\top, \quad u_3 = (3, 8, 3, a - 5)^\top, \quad u_4 = (2 - a, a, 1, 1 - a)^\top.$

(a) Use an appropriate determinant to check for which values of $a$ the set $U$ is a basis of $\mathbb{R}^4$.

(b) Let $a = -1$. Find the transformation matrix $A_{U,U}$ of the linear transformation $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ with $\phi((x_1, x_2, x_3, x_4)^\top) = \sum_{k=1}^{4} kx_k(1, 1, 1)^\top$ with respect to the basis $U$.

Exercise 24: Calculate the following determinants

$$(a) \quad \det \begin{pmatrix} -1 & -2 \\ -3 & -8 \end{pmatrix}, \quad (b) \quad \det \begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & -1 \\ -2 & 7 & 1 \end{pmatrix}, \quad (c) \quad \det \begin{pmatrix} -1 & 1 & 0 \\ 5/3 & -1/5 & 1 \\ 2/3 & 2/5 & -3/2 \end{pmatrix}, \quad (d) \quad \det \begin{pmatrix} 1 & 2 & \pi & 1 \\ 2 & 4 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 4 & -1 & 0 \end{pmatrix}.$$

Exercise 25: Let the matrices $A, B \in \mathbb{C}^{4 \times 4}$ be given by

$$A = \begin{pmatrix} 5 & i - 1 & 7 & -4 \\ -1 & -1 & -4 & 1 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -i - 1 & -2 & -2 \\ 2 & -i - 1 & -4 & -4 \\ 3 & -5i - 5 & -4 & -4 \\ 4 & -7i - 7 & -6 & -7 \end{pmatrix}.$$

Calculate $\det(A)$ and $\det(B)$, as well as $\det(AB^*)$ and $\det(A^{-1}B)$.

Due date: Your written solutions are due on Tuesday, 27th May 2014. Please put them into the box in the student office until 2:00 PM.
Exercise T13: Consider the linear transformation \( \psi : \mathbb{R}^3 \to \mathbb{R}^3 \) represented with respect to the standard basis by the following matrix

\[
A = \begin{pmatrix}
-1 & -1 & 2 \\
-2 & -1 & 2 \\
-2 & -2 & 3
\end{pmatrix}.
\]

Let \( b_1 = (1,1,1)^\top \) and \( b_2 = (0,1,1)^\top \). Determine a vector \( b_3 \neq 0 \) which satisfies the equation \( \psi(b_3) = b_3 \) and show that \( B = (b_1, b_2, b_3) \) is a basis. Find the transformation matrix \( A_{B,B} \) of \( \psi \) with respect to the basis \( B \).

Exercise T14: Calculate the determinant

\[
D := \begin{vmatrix}
-3 & 0 & 0 & 7 \\
6 & 4 & -1 & -3 \\
0 & -5 & 2 & 2 \\
3 & -7 & 1 & 0
\end{vmatrix}
\]

(a) by expanding along the first row,

(b) by expanding along the last column,

(c) using Gaussian elimination.

Exercise T15: Calculate the determinant of the matrix \( C := (AB)^{-1} \) with

\[
A := \begin{pmatrix}
1 & 2 & 3 & -2 & 2 \\
3 & -1 & 4 & 2 & 0 \\
8 & -3 & -2 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
B := \begin{pmatrix}
-1 & 3 & -2 & 2 & -1 \\
0 & 2 & -3 & -3 & 1 \\
0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 2 & 5 \\
0 & 2 & 0 & 0 & 6
\end{pmatrix}.
\]

For detailed information regarding this course visit the following web page:

www.math.kit.edu/iag6/lehre/am22014s/en

Tutorial: Friday, 23rd May 2014, 9:45 AM