Proof: The proof consists of two steps.

**Step 1.** We will define $S = L^{-1}$ appropriately and explain why:
(i) $S$ is compact,
(ii) $S$ is symmetric,
(iii) $S$ has only positive eigenvalues.

**Step 2.** Using Step 1 and Thm 8.1 we will obtain Thm 8.2.

---

**Step 1.** The bilinear form associated to $L$ is $B[u,v] = \sum_{i,j} \int_{\Omega} a_{ij} u_i v_j \, dx$.

So there exists $C > 0$ such that
$$|B[u,v]| \leq C \|u\|_{H_0^2(\Omega)} \|v\|_{H_0^2(\Omega)} \quad \forall u, v \in H_0^2(\Omega).$$

Also $B[u,u] = \sum_{i,j} \int_{\Omega} a_{ij} u_i u_j \, dx \geq D \|u\|_{L^2(\Omega)}^2$ due to the ellipticity condition.

Using Poincaré's inequality (Thm 4.7(i)) we obtain that there exists $D > 0$ such that
$$B[u,u] \geq D \|u\|_{H_0^2(\Omega)}^2 \quad \forall u \in H_0^2(\Omega).$$

So due to the Lax-Milgram Theorem (Thm 5.4) there is a unique solution $u$ of
$$\begin{cases}
Lu = f \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega
\end{cases}$$
for any $f \in L^2$. We define $Sf = u$. 

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We showed in proof of Thm 6.7 that $S = L^{-1} : L^2(V) \to L^2(V)$ is compact. We will show that $S$ is symmetric. Let $f, g \in L^2(V)$ and $u, v \in H_0^1(V)$ the unique functions satisfying, in the weak sense,

$$\begin{align*}
Lu &= f \text{ in } V \\
u &= 0 \text{ on } \partial V
\end{align*} \tag{1}$$

and

$$L v = g \text{ in } V \quad \{ v = 0 \text{ on } \partial V \}. \tag{2}$$

Then

$$\langle S f, g \rangle \overset{(1)}{=} \langle u, g \rangle \overset{(2)}{=} B[v, u].$$

$L^2$ inner product by definition of the weak solution of $\pm$.

Similarly

$$\langle f, S g \rangle \overset{(2)}{=} \langle f, v \rangle \overset{(1)}{=} B[u, v].$$

and since $B[u, v] = B[v, u]$, we obtain that

$$\langle S f, g \rangle = \langle f, S g \rangle$$

implying that $S$ is symmetric.

(iii) If $\lambda$ is EV of $S$ with eigenfunction

$$\| f \|_2^2 = 1 \quad \text{then} \quad \lambda = \langle S f, f \rangle = \langle u, f \rangle = B[u, u] > 0$$

because of $\text{(*)}$ and the fact that $u \neq 0$.

**Step 2.** Since $S f = u \iff Lu = f$, we have

$$S f = \eta f \iff L f = \frac{1}{\eta} f.$$ So $f$ is eigenfunction of $L$ if $f$ is eigenfunction of $S$ (due to Step 1(iii) we always have $\eta \neq 0$) and $\eta$ is eigenvalue of $S$ if $\frac{1}{\eta}$ is eigenvalue of $L$.
the theorem follows.

Definition 7.3 We call $\lambda_1$ the principal eigenvalue of $L$.

Theorem 8.4 (Variational principle for the principal eigenvalue) (Recall $Lu = -\sum_{i,j=1}^n (-a_{ij}u_{x_i}x_j)$, $a_{ij} \in C^0(U)$.

Assume $U$ open, bounded, connected
(i) We have $\lambda_1 = \min \{ B[u,u] \mid u \in H^1(U), \|u\|_2 = 1 \}$
(ii) The above minimum is attained for a function $w_1$, positive within $U$, which solves
\[
\begin{cases}
Lw_1 = \lambda_1 w_1 \text{ in } U \\
w_1 = 0 \text{ on } \partial U
\end{cases}
\]
(iii) If $u \in H^1_0(U)$ is any weak solution of
\[
\begin{cases}
Lu = \lambda_1 u \text{ in } U \\
u = 0 \text{ on } \partial U.
\end{cases}
\]
then $u$ is a multiple of $w_1$.

Proof (i): Let $(w_k)_{k \in \mathbb{N}}$ be an orthonormal basis of $L^2(U)$ consisting of eigenfunctions of $L$.

Then $B[w_k, w_k] = (\lambda_k w_k, w_k) = \lambda_k$ (1)

$B[w_k, w_l] = (\lambda_k w_k, w_l) = \delta_k^l (w_k, w_l) = 0$ (2)

we write $u = \sum_{k=1}^\infty \lambda_k d_k w_k$ and we define

$u_n = \sum_{k=1}^n \lambda_k d_k w_k$
By (1), we obtain $B[u_n, u_n] = \sum_{k=1}^{M} \lambda_k d_k^2$. (14)

Assuming that $B[u_n, u_n] \to B[u, u]$ (3),
we obtain that $B[u, u] = \sum_{k=1}^{\infty} \lambda_k d_k^2 \geq \lambda_1$ (because $\lambda_k^2 = 1$, $\forall k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} d_k^2 = 1$).

Since equality holds for $u = w_1$, assertion (i) follows given that (3) holds. We will now prove (3). Indeed, in the proof of Theorem 3.2 we showed that $\exists C, D > 0$, such that $\forall u, v \in H_0^1(U)$

$|B[u, v]| \leq C \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$

$B[u, u] \geq D \|u\|_{H_0^1(U)}^2$

implies that $B[\cdot, \cdot]$ defines an inner product in $H_0^1(U)$ which induces the natural topology in $H_0^1(U)$. Therefore, if we show that $u_n \to u$ in $H_0^1(U)$ (4) then (3) follows.

So it suffices to show (4). We first observe that from (1), it follows that $w_k$ is orthonormal in $H_0^1(U)$ w.r.t. the inner product $B$. Since $B[w_k, u] = 0 \forall k \in \mathbb{N}$

$\Rightarrow (w_k, u) = 0 \forall k \in \mathbb{N} \Rightarrow u = 0$ (since $w_k$ is an orthonormal basis of $L^2$) it follows that
\[ w_k \] is an orthonormal basis of \( H^1_0(U) \) with \( \lambda_k^{\frac{1}{2}} \).

So, \( \sum_{k=1}^{\infty} \frac{d_k w_k}{\lambda_k^{\frac{1}{2}}} \).

Since \( u = \sum_{k=1}^{\infty} d_k w_k \) it follows that \( \sum_{k=1}^{\infty} d_k^2 w_k \) is convergent in \( H^1_0(U) \). This implies \( 14 \) and completes the proof of \( 11 \).

The above proof that \( \lambda_k \leq \min_{\Omega} \langle BL, v \rangle_{H^1_0(U)} \) can be summarised to:

\[ \| u \|_{L^2(\Omega)}^2 = 1 \] (the other inequality is trivial) can be summarised to:

**Step 1.** We showed \( BL, v \rangle_{H^1_0(U)} \) defines an inner product inducing a norm which is equivalent to the \( H^1_0(U) \) norm.

**Step 2.** We showed: If \( \{ w_k \}_{k \in \mathbb{N}} \) is an orthonormal basis of \( L^2(U) \) consisting of eigenvalues of \( L \), then \( \{ \frac{w_k}{\lambda_k^{\frac{1}{2}}} \}_{k \in \mathbb{N}} \) is an orthonormal basis of \( H^1_0(U) \) with respect to the inner product \( BL, v \rangle_{H^1_0(U)} \).

**Step 3.** Let \( u \in H^1_0(U) \) and write \( u = \sum_{k=1}^{\infty} d_k w_k \).

We showed: \( u_n := \sum_{k=1}^{n} d_k w_k \rightarrow u \) in \( H^1_0(U) \).

**Step 4.** We showed that \( \langle BL, u \rangle_{H^1_0(U)} = \sum_{n=1}^{\infty} \lambda_n d_n^2 \).

Which implies that \( \lambda_k \geq 2 \lambda_1^{\frac{1}{2}} \) and \( \sum_{k=1}^{\infty} d_k^2 = 1 \).

Since \( \lambda_k \geq \min_{\Omega} \langle BL, v \rangle_{H^1_0(U)} \).