Exercise 1

(a) It holds:

\[ |z - a| < |1 - \bar{a}z| \iff |z - a|^2 < |1 - \bar{a}z|^2 \]
\[ \iff (z - a)(\bar{z} - \bar{a}) < (1 - \bar{a}z)(1 - a\bar{z}) \]
\[ \iff \bar{z} - a\bar{z} - z\bar{a} + a\bar{a} < 1 - a\bar{z} - z\bar{a} + a\bar{a}\bar{z} \]
\[ \iff |z|^2 + |a|^2 < 1 + |a|^2|z|^2 \]
\[ \iff |z|^2(1 - |a|^2) < 1 - |a|^2 \]
\[ \iff |a|^2 < 1 \]
\[ \iff |z|^2 < 1 \]
\[ \iff |z| < 1 \]

(b) Note: due to $|a| < 1$, $|z| \leq 1$, it follows that $1 - \bar{a}z \neq 0$. It holds:

\[ \left( \frac{|z - |a||}{1 - |az|} \right)^2 = \frac{(|z| - |a|)^2}{(1 - |az|)^2} \]
\[ = \frac{|z|^2 - 2|a||z| + |a|^2}{(1 - |az|)^2} \]
\[ = 1 - \frac{1 + |a|^2|z|^2 - |z|^2 - |a|^2}{(1 - |az|)^2}. \]

Since $|\bar{a}| = |a| < 1$ and $|z| \leq 1$, it follows that $1 + |a|^2|z|^2 - |z|^2 - |a|^2 = (1 - |z|^2)(1 - |a|^2) \geq 0$ and $0 < 1 - |az| = 1 - |az| = |1 - |az|| \leq |1 - \bar{a}z|$. It follows

\[ 1 - \frac{1 + |a|^2|z|^2 - |z|^2 - |a|^2}{(1 - |az|)^2} \leq 1 - \frac{1 + |a|^2|z|^2 - |z|^2 - |a|^2}{|1 - \bar{a}z|^2} \]
\[ = \frac{|1 - \bar{a}z|^2 - 1 - |a|^2|z|^2 + |z|^2 + |a|^2}{|1 - \bar{a}z|^2} \]
\[ = 1 - \bar{a}z - a\bar{z} + |a|^2|z|^2 - 1 - |a|^2|z|^2 + |z|^2 + |a|^2 \]
\[ = -\bar{a}z - a\bar{z} + |z|^2 + |a|^2 \]
\[ = \frac{(z - a)^2}{|1 - \bar{a}z|^2} \]
\[ = \frac{|z - a|^2}{|1 - \bar{a}z|^2}. \]

Thus, it follows

\[ \frac{|z - |a||}{1 - |az|} < \frac{|z - a|}{1 - \bar{a}z}. \]
Exercise 2

(a) $f_1$ is not continuous in 0:
For $n \in \mathbb{N}$ let $z_n = \frac{1}{n}$. Then $z_n \xrightarrow{n \to \infty} 0$, but $f_1(z_n) \to 0 = f_1(0)$ for $n \to \infty$, because

$$f_1(z_n) = \frac{1}{z_n} = \frac{1}{\frac{1}{n}} = -i \quad \text{for all } n \in \mathbb{N}.$$ 

$f_2$ is continuous in 0:
For every $z \in \mathbb{C} \setminus \{0\}$ it holds

$$|f_2(z)| = |z^{-1}||\text{Im}z^2| \leq |z^{-1}||z^2| = |z| \xrightarrow{z \to 0} 0.$$ 

Therefore $f_2(z) \xrightarrow{z \to 0} 0 = f_2(0)$.

$f_3$ is continuous in 0:
For every $z \in \mathbb{C} \setminus \{0\}$ it holds

$$|f_3(z)| = |z^{-1}||\text{Im}z^2|^2 \leq |z^{-1}||z^2|^2 = |z|^3 \xrightarrow{z \to 0} 0.$$ 

Therefore $f_3(z) \xrightarrow{z \to 0} 0 = f_3(0)$.

(The following shows that the function $\tilde{f}_2 = z^{-2}\text{Im}(z^2)$ is not continuous in 0 and the function $\tilde{f}_3 = z^{-3}(\text{Im}(z^2))^2$ is continuous in 0:

$f_2$ is not continuous in 0:
For $n \in \mathbb{N}$ let $z_n = \frac{1+i}{n}$. Then $z_n \xrightarrow{n \to \infty} 0$. Furthermore, $z_n^2 = (1+i)^2 = \frac{2}{n}$ and therefore

$$\tilde{f}_2(z_n) = \frac{2}{n^2} = -i \quad \text{for all } n \in \mathbb{N}.$$ 

Thus $\tilde{f}_2(z_n) \to 0 = \tilde{f}_2(0)$ for $n \to \infty$.

$f_3$ is continuous in 0:
For every $z \in \mathbb{C} \setminus \{0\}$ it holds

$$|\tilde{f}_3(z)| = |z^{-2}||\text{Im}z^2|^2 \leq |z^{-2}||z^2|^2 = |z|^2 \xrightarrow{z \to 0} 0.$$ 

Therefore $\tilde{f}_3(z) \xrightarrow{z \to 0} 0 = \tilde{f}_3(0)$. )

(b) If both of the properties were true, then it would hold

$$1 = (f(1))^2 = f(1)f(1) = f(1.1) = f(1),$$

$$-1 = (f(-1))^2 = f(-1)f(-1) = f((-1)(-1)) = f(1),$$

which is a contradiction.

(c) Let $a \in \mathbb{C} \setminus \{0\}$. Define the function

$$g(z) := \frac{f(a)f(z)}{f(az)}, \quad z \in \mathbb{C} \setminus \{0\}.$$ 

It takes only the values $\pm 1$. Since $g$ is continuous, it follows that it is constant and the value of this constant is either 1 or $-1$. Assume that $g(z) \equiv 1$. Then we have

$$f(a)f(z) = f(az)$$

and we can use part (b).

Analogously, if we assume that $g(z) \equiv -1$, it would follow that

$$f(a)f(z) = -f(az)$$

and similarly to part (b), we would have

$$1 = (f(1))^2 = f(1)f(1) = -f(1.1) = -f(1),$$

$$-1 = (f(-1))^2 = f(-1)f(-1) = -f((-1)(-1)) = -f(1),$$

which is a contradiction again.
Exercise 3 We know that
\[ \varepsilon |z|^2 + \alpha z + \alpha \bar{z} + \beta = 0, \quad z, \alpha \in \mathbb{C}, \beta \in \mathbb{R} \] (1)
defines a circle (for \( \varepsilon = 1 \) and \( \beta < |\alpha|^2 \)) or a line (for \( \varepsilon = 0 \)) in \( \mathbb{C} \).
Furthermore, we know that a circle on \( \Sigma \) is defined by
\[ Ax_1 + Bx_2 + Cx_3 + D = 0 \]
for \( A, B, C, D \in \mathbb{R} \) and \( x_1^2 + x_2^2 + x_3^2 = 1 \).

(a) By the definition of \( p = \pi^{-1} \) it follows that for \( C \ni z = x + iy \) we have \( p(z) = \frac{1}{x^2+y^2+1}(2x,2y,x^2+y^2-1) \).
Plugging this in (\(*\)), we get
\[ \frac{2Ax}{x^2+y^2+1} + \frac{2By}{x^2+y^2+1} + \frac{C(x^2+y^2-1)}{x^2+y^2+1} + D = 0, \]
which is equivalent to
\[ (C+D)(x^2+y^2) + 2Ax + 2By + (D-C) = 0 \]
and this is similar to [1].

(b) Let \( \alpha = a + ib \). Furthermore, let \( Z = (x_1,x_2,x_3) \in \Sigma \) (i.e., \( x_1^2 + x_2^2 + x_3^2 = 1 \)) and \( z \in \mathbb{C} \) be the projection of \( Z \) on \( \mathbb{C} \), i.e., \( z = \pi(x_1,x_2,x_3) = \frac{x_1}{1-x_3^2} + i\frac{x_2}{1-x_3^2} \).
Then \( |z|^2 = \frac{1+x_3^2}{1-x_3^2} \).
Therefore, by [1], we have
\[ \varepsilon(1+x_3) + 2ax_1 + 2bx_2 + \beta(1-x_3) = 0 \iff 2ax_1 + 2bx_2 + (\varepsilon-\beta)x_3 = -\beta - \varepsilon, \]
which defines a level on \( \Sigma \), i.e. a circle.

Exercise 4

(a) \( d(z,w) \geq 0 \) for every \( z, w \in \hat{\mathbb{C}} \) and \( d(z,w) = 0 \iff |z-w| = 0 \), i.e., \( z = w \).

(b) \( d(z,w) = d(w,z) \) for every \( z, w \in \hat{\mathbb{C}} \).

(c) We will show that
\[ d(z,w) := \frac{|z-w|}{\sqrt{|z|^2+1}} = ||p(z) - p(w)|| =: \chi(z,w) \quad z, w \in \hat{\mathbb{C}} \]

We have
\[
||p(z) - p(w)||^2 = \sum_{k=1}^{n} (x_k - \eta_k)^2
= \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} \eta_k^2 - 2 \sum_{k=1}^{n} x_k \eta_k
= 2 - 2\frac{z+w(w+w) + i(z-w)(w+w) + (|z|^2-1)(|w|^2-1)}{(|z|^2+1)(|w|^2+1)}
= 4\left(\frac{|w|^2 + |z|^2 - z\bar{w} - \bar{z}w}{(|z|^2+1)(|w|^2+1)}\right)
= 4\left(\frac{|z-w|^2}{(|z|^2+1)(|w|^2+1)}\right),
\]
i.e.,
\[ d(z,w) = ||p(z) - p(w)||, \]
Note that
\[ d(z,\infty) = ||p(z) - p(\infty)|| = \sqrt{x_1^2 + x_2^2 + (x_3-1)^2} = \frac{2}{\sqrt{|z|^2+1}} \]
Thus, we have
\[ d(z,w) = ||p(z) - p(w)|| \leq ||p(z) - p(w)|| + ||p(v) - p(w)|| = d(z,v) + d(v,w) \quad \text{for all } z, v, w \in \hat{\mathbb{C}}. \]