**Exercise 1** For $z = x + iy$, we have

$$F(x, y) := f(x + iy) = e^{-rac{1}{(x+iy)^2}} \quad \text{for } x \neq 0, y \neq 0,$$

and thus

$$D_1 F(x, y) = 4e^{-rac{1}{(x+iy)^2}} \frac{1}{(x+iy)^3},$$

$$D_2 F(x, y) = 4ie^{-rac{1}{(x+iy)^2}} \frac{1}{(x+iy)^3},$$

i.e., $D_2 F(x, y) = iD_1 F(x, y)$ for all $x \neq 0, y \neq 0$. Clearly, for $x = 0, y = 0$, $f$ fulfills the Cauchy-Riemann conditions.

However, near the origin $f$ is unbounded (consider it $z = \varepsilon(1+i)$), so it cannot be analytic in the whole complex plane.

**Exercise 2**

(a) 

$$u_x(x, y) = 3x^2 - 3y^2$$

From $u_x = v_y$ it follows

$$v(x, y) = \int (3x^2 - 3y^2) dy = 3x^2 y - y^3 + c(x).$$

From $u_y = -v_x$ it follows that

$$c'(x) = 0, \quad \text{i.e. } c(x) = \text{const} := 0.$$ 

Therefore, the function we are looking for is

$$f(x + iy) = x^3 - 3x^2 y + 1 + i(3x^2 y - y^3) = (x + iy)^3 + 1, \quad f(z) = z^3 + 1 \quad \text{for } z = x + iy.$$ 

(b) 

$$u_y(x, y) = \frac{-2xy}{(x^2 + y^2)^2} \text{ for all } x \neq 0, y \neq 0$$

From $u_y = -v_x$ it follows

$$v(x, y) = \int \frac{2xy}{(x^2 + y^2)^2} dx = \frac{-y}{x^2 + y^2} + c(y).$$

From $u_x = v_y$ it follows

$$c'(y) = 0 \quad \text{i.e. } c(y) = \text{const} := 0.$$ 

Therefore, the function we are looking for is

$$f(x + iy) = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}, \quad f(z) = \frac{1}{z} \quad \text{for } z = x + iy.$$
(c) \[ v_x(x, y) = e^x(x \sin(y) + y \cos(y) + \sin(y)) \]

From \( v_x = -u_y \) it follows

\[ u(x, y) = -\int e^x(x \sin(y) + y \cos(y) + \sin(y))dx = e^x(x \cos(y) - y \sin(y)) + c(x). \]

From \( v_y = u_x \) it follows that

\[ c'(x) = 0 \quad \text{i.e.} \quad c(x) = \text{const} := 0. \]

Therefore, the function we are looking for is

\[ f(x + iy) = e^x(x \cos(y) - y \sin(y)) + i(x \sin(y) + y \cos(y)) = e^{x+iy}(x + iy), \]

\[ f(z) = ze^z \quad \text{for} \ z = x + iy. \]

Exercise 3

(a) E.g. \( c = i \). Then we have

\[ a_0^{(i)} = 0, \quad a_1^{(i)} = i, \quad a_2^{(i)} = -1 + i, \quad a_3^{(i)} = -i, \quad a_4^{(i)} = a_2^{(i)}, \quad a_5^{(i)} = a_3^{(i)}, \ldots, \]

so the sequence \( \{a_n^{(i)}\} \) is divergent.

(b) First we note that if there is \( \mu \in \mathbb{N} \) such that \( |a_\mu^{(c)}| \geq \max\{|c|, 2\} \) then for all \( n \in \mathbb{N} \) with \( n \geq \mu \) we have \( |a_n^{(c)}| \geq \max\{|c|, 2\} \):

\[
|a_{\mu+1}^{(c)}| = |(a_\mu^{(c)})^2 + c| \geq |a_\mu^{(c)})^2 - |c| \geq (|a_\mu^{(c)}| - 1)|c| \geq |c|, \]

\[
|a_{\mu+1}^{(c)}| = |(a_\mu^{(c)})^2 + c| \geq |a_\mu^{(c)})^2 - |c| \geq (|a_\mu^{(c)}| - 1)|a_\mu^{(c)}| \geq |a_\mu^{(c)}|^2 \geq 2. \]

Therefore, for all \( n \in \mathbb{N} \) with \( n \geq \mu \) it holds

\[
|a_n^{(c)}| \geq (|a_\mu^{(c)}| - 1)^n |a_\mu^{(c)}| \tag{2}
\]

Let \( N \in \mathbb{N} \) be such that \( |a_N^{(c)}| > 2 \). We consider the following two cases:

Case 1: \( |c| > 2 \). By \( |a_1^{(c)}| = |c| \geq \max\{|c|, 2\} \), we get that \( \mu \) from above is 1 and by (2) it follows

\[ |a_n^{(c)}| \geq (|a_1^{(c)}| - 1)^n |a_1^{(c)}| \to \infty \quad \text{as} \ n \to \infty, \]

Case 2: \( |c| \leq 2 \). Since \( |a_N^{(c)}| > 2 \) it follows that \( |a_N^{(c)}| \geq \max\{|c|, 2\} \) and by (2) we get

\[ |a_n^{(c)}| \geq (|a_N^{(c)}| - 1)^n |a_N^{(c)}| \to \infty \quad \text{as} \ n \to \infty, \]

Therefore, \( \{a_n^{(c)}\} \) is unbounded.

(c) We will show that \( M \) is bounded and closed:

- **bounded**: Assume that there is \( c \in M \) with \( |c| > 2 \). Then \( |a_1^{(c)}| = |c| > 2 \) and by part (b) we can conclude that \( \{a_n^{(c)}\} \) is unbounded, which is a contradiction. Thus \( |c| \leq 2 \) for all \( c \in M \).

- **closed**: For every \( c \in \mathbb{C} \) we have either \( |a_n^{(c)}| \leq 2 \) for all \( n \in \mathbb{N} \) (and so \( c \in M \)) or there exists \( n \in \mathbb{N} \) with \( |a_n^{(c)}| > 2 \) (so by part (b) it follows that \( \{a_n^{(c)}\} \) is unbounded, i.e. \( c \notin M \)). Therefore

\[
\mathbb{C} \setminus M = \bigcup_{n \in \mathbb{N}} \{c \in \mathbb{C} : |a_n^{(c)}| > 2 \}. 
\]

By induction, it can be shown that \( c \mapsto a_n^{(c)} \) is a polynomial over \( \mathbb{C} \) and then, by the hint, we can conclude that \( \mathbb{C} \setminus M = \bigcup_{n \in \mathbb{N}} \{c \in \mathbb{C} : |a_n^{(c)}| > 2 \} \) is open. Thus \( M \) is closed.

(d) By part (c): \( |c| \leq 2 \) for all \( c \in M \). Consider \( \{a_n^{(c)}\} \) for \( c = -2 \). Then \( \{a_n^{(c)}\} = \{0, -2, 2, 2, 2, \ldots, \} \), and so \( -2 \in M \). Thus \( \max\{|c| : c \in M\} = 2 \).
Exercise 4 From the convergence of the series $\sum_{k=1}^{\infty} z_k$ follows the convergence of $\sum_{k=1}^{\infty} \text{Re} z_k$, which means that $\text{Re} z_k \to 0$ as $k \to \infty$. Since we have $\text{Re} z_k > 0$ it follows that $\text{Re} z_k \in [0, 1]$ for all large enough $k \in \mathbb{N}$. Therefore we have

$$0 \leq (\text{Re} z_k)^2 \leq \text{Re} z_k.$$ 

By the Majorant criterion, we can conclude that $\sum_{k=1}^{\infty} (\text{Re} z_k)^2$ converges.

From the convergence of $\sum_{k=1}^{\infty} z_k^2$ follows the convergence of

$$\sum_{k=1}^{\infty} \text{Re}(z_k)^2 = \sum_{k=1}^{\infty} \text{Re}(x_k^2 + 2ix_ky_k + i^2 y_k)$$

$$= \sum_{k=1}^{\infty} (x_k^2 - y_k^2)$$

$$= \sum_{k=1}^{\infty} ((\text{Re} z_k)^2 - (\text{Im} z_k)^2)$$

and therefore we conclude the convergence of

$$2 \sum_{k=1}^{\infty} (\text{Re} z_k)^2 - \sum_{k=1}^{\infty} ((\text{Re} z_k)^2 - (\text{Im} z_k)^2) = \sum_{k=1}^{\infty} ((\text{Re} z_k)^2 + (\text{Im} z_k)^2) = \sum_{k=1}^{\infty} |z_k|^2.$$ 

Counterexample: Consider $\{z_k = i(-1)^k\}_{k \in \mathbb{N}}$. Then it holds $\text{Re} z_k = 0 \geq 0$. Furthermore $\sum_{k=1}^{\infty} z_k = i \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ and $\sum_{k=1}^{\infty} z_k^2 = - \sum_{k=1}^{\infty} \frac{1}{k^2}$ are convergent. But $\sum_{k=1}^{\infty} |z_k| = \sum_{k=1}^{\infty} \frac{1}{k}$ is divergent.