Exercise 1

(a) (1) Let \( a_n := \frac{5+i}{4-2i} \). Since it holds \(|5+i| = \sqrt{26} > \sqrt{20} = |4-2i| \) we have \(|a_n| > 1\) for all \( n \in \mathbb{N} \). Therefore, \( a_n \not\to 0 \) for \( n \to \infty \) and thus the series is divergent.

(2) Let \( b_n := (\frac{1}{\sqrt{n}})^n (1 + \frac{i}{n})^n \). Then

\[
|b_n| = \left| \frac{1}{2i} \right|^n \left| 1 + \frac{i}{n} \right|^n = \frac{1}{2^n} \left( 1 + \frac{1}{n^2} \right)^{\frac{n^2}{2}}
\]

Since \((1 + \frac{1}{n^2})^{\frac{n^2}{2}} \to e\) \((n \to \infty)\) there is a large enough \( n \in \mathbb{N} \) for which \((1 + \frac{1}{n^2})^{\frac{n^2}{2}} \leq 2e\) and thus

\[
|b_n| \leq \sqrt{2e} \frac{1}{2^n}.
\]

From the convergence of the geometric series \( \sum_{n=0}^{\infty} \frac{1}{2^n} \), by the Majorant criterion, follows the absolute convergence of the given series.

(3) Let \( c_n := \frac{\sqrt{}}{n} \). This series is not absolutely convergent because \( \sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} \frac{1}{n} \) and this diverges.

However, the series is convergent: Due to

\[
i^{4n-3} = i, \quad i^{4n-2} = -1, \quad i^{4n-1} = -i, \quad i^{4n} = 1 \quad \text{for} \quad n \in \mathbb{N},
\]

for all \( N \in \mathbb{N} \) it follows

\[
\sum_{k=1}^{4N} \frac{j^k}{k} = \sum_{n=1}^{N} \left( \frac{i}{4n-3} + \frac{-1}{4n-2} + \frac{-i}{4n-1} + \frac{1}{4n} \right)
\]

\[
= \sum_{n=1}^{N} \left( \frac{-2}{4n(4n-2)} + i \frac{2}{(4n-1)(4n-3)} \right),
\]

which is convergent for \( N \to \infty \). Moreover, since for \( j \in \{1, 2, 3\} \) we have

\[
\left| \sum_{k=1}^{4N+j} \frac{j^k}{k} - \sum_{k=1}^{4N} \frac{j^k}{k} \right| \leq \sum_{k=4N+1}^{4N+j} \frac{1}{k} \leq \frac{3}{4N+1} \to 0 \quad \text{as} \quad N \to \infty
\]

we can conclude that the given series converges.

(b) For \( z = x + iy \) we have

\[
\sin(z) = \frac{e^{ix} - e^{-ix}}{2i} = \frac{(\cos(x) + i \sin(x))e^y + (\cos(-x) + i \sin(-x))e^{-y}}{2i}
\]

\[
= \frac{\sin(x)(e^y + e^{-y})}{2} + i \cos(x)(e^y - e^{-y}) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)
\]

and therefore

\[
|\sin(z)|^2 = \sin^2(x) \cosh^2(y) + \cos^2(x) \sinh^2(y)
\]

\[
= \sin^2(x) \cosh^2(y) - \sin^2(x) \sinh^2(y) + \sin^2(x) \sinh^2(y) + \cos^2(x) \sinh^2(y) = \sin^2(x) + \sinh^2(y)
\]

The function \( z \mapsto \sin(z) \) is unbounded in \( \mathbb{C} \) since for \( y \in \mathbb{R} \) we have

\[
|\sin(iy)| = |\sinh(y)| \to \infty \quad \text{as} \quad |y| \to \infty.
\]
Exercise 2

(a) Let \( a_n := (1 + ni)^n \). Then it holds
\[
\sqrt[n]{|a_n|} = |1 + ni| = \sqrt{1 + n^2} \to \infty \quad \text{as} \ n \to \infty,
\]
so the radius of convergence is
\[
R = \limsup_{n \to \infty} \sqrt[n]{|a_n|} = 0.
\]
So the power series converges only in \( z = 0 \).

(b) Let \( b_n = in^2 + 2^n \). Then, for all \( n \in \mathbb{N} \), it holds
\[
\sqrt[n]{|b_n|} = \sqrt[n]{|in^2 + 2^n|} \leq \sqrt[n]{2^n + 2^n} = 2 \quad \text{as} \ n \to \infty.
\]
In addition, there exists \( n_0 \in \mathbb{N} \) with \( n^2 \leq 2^n \) for all \( n \geq n_0 \) and thus
\[
\sqrt[n]{|b_n|} = \sqrt[n]{|in^2 + 2^n|} \leq \sqrt[n]{2^n + 2^n} \to 2 \quad \text{as} \ n \to \infty.
\]
Therefore, it follows that \( \sqrt[n]{|b_n|} \to 2 \) as \( n \to \infty \) and so the power series \( \sum_{n=1}^{\infty} b_n w^n \) has the convergence radius \( R = \frac{\sqrt{2}}{2} \).

(c) Let \( c_n := (1+i^n)(n+1)/2 \). Since \( |1+i^n| \leq |1| + |i^n| = 2 \) for all \( n \in \mathbb{N}_0 \) and \( |1+i^n| = 2 \) for all \( n = 4k \) \( (k \in \mathbb{N}_0) \) it follows
\[
\limsup_{n \to \infty} \sqrt[n]{|c_n|} = \lim_{n \to \infty} \sqrt[n]{2(n+1)/2} = \lim_{n \to \infty} 2^{1+1/n}/2 = 2^{1/2}.
\]
Therefore, the convergence radius of the given power series is \( R = \frac{\sqrt{2}}{2} \).

(d) Let \( d_n := \sum_{k=1}^{n} \frac{1}{k+i} \). For all \( k \in \mathbb{N} \) we have \( |k+i| \geq \text{Im}(k+i) = 1 \) and so
\[
|d_n| \leq \sum_{k=1}^{n} \frac{1}{|k+i|} \leq \sum_{k=1}^{n} 1 = n.
\]
On the other hand, it holds
\[
|d_n| \geq \text{Red}_n = \sum_{k=1}^{n} \frac{k}{k^2 + 1} \geq \frac{1}{2}
\]
Since both \( \sqrt[n]{n} \to 1 \) and \( \sqrt[n]{\frac{1}{2}} \to 1 \) as \( n \to \infty \) hold, we conclude that the radius of convergence for the given series is \( R = 1 \).

Exercise 3 For \( z \) with \( |z| < 1 \) define
\[
g(z) := \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} \right)
\]
Using that \( \sum_{k=1}^{\infty} (-1)^k z^k = \frac{1}{1+z} \), we can conclude that
\[
g'(z) = 0
\]
and thus
\[
\exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} \right) = C(1 + z), \quad C = \text{const}.
\]
For \( z = 0 \) we have
\[
\exp(0) = C \Rightarrow C = 1.
\]
Exercise 4

(a) We are looking for \(a, b, c, d \in \mathbb{C}\) with \(ad - bc \neq 0\) such that \(S(z) := \frac{az + b}{cz + d}\) satisfies the given conditions.

\[
\begin{align*}
S(0) &= -2 \implies b = -2d, \\
S(2) &= 0 \implies a = -\frac{1}{2}b = d, \\
S(i) &= \infty \implies c = di, \\
S(\infty) &= -i \implies c = di.
\end{align*}
\]

Since \(ad - bc = d^2 + 2id^2 \neq 0\), it follows that

\[
S(z) = \frac{dz - 2d}{dz + di} = \frac{z - 2}{iz - 1}.
\]

(b) We are looking for \(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{C}\) with \(\tilde{a}\tilde{d} - \tilde{b}\tilde{c} \neq 0\) such that \(T(z) := \frac{\tilde{a}z + \tilde{b}}{\tilde{c}z + \tilde{d}}\) satisfies the given conditions.

\[
\begin{align*}
T(3i) &= 0 \implies \tilde{b} = -3\tilde{a}i, \\
T(0) &= \infty \implies \tilde{d} = 0, \\
T(i) &= 1 \implies \tilde{c} = -2\tilde{a}
\end{align*}
\]

but \(T(1) = \frac{\tilde{a} - 3\tilde{a}i}{2\tilde{a}} \neq i\). Thus there exists no Möbius tranformation which satisfies the given conditions.

(c) To determine \(S^{-1}\), we have to find \(p, q, r, s \in \mathbb{C}\) with \(ps - qr \neq 0\) such that \(S^{-1}(z) := \frac{pz + q}{rz + s}\) satisfies

\[
\begin{align*}
S^{-1}(-2) &= 0, \\
S^{-1}(0) &= 2, \\
S^{-1}(\infty) &= 0, \\
S^{-1}(-i) &= \infty.
\end{align*}
\]

Analogously to (a), we find that \(p = 1, q = 2, r = -i, s = 1\), i.e.

\[
S^{-1}(z) = \frac{z + 2}{-iz + 1}.
\]

In order to find the fixed points of \(S\) we have to solve

\[
S(z) = z \iff z^2 = 2i \iff z_{1,2} = \pm(1 + i)
\]