Exercise 1
Let $\Omega$ be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant $C$, depending only on $\Omega$, such that
\[ \max_{\Omega} |u| \leq C (\max_{\partial \Omega} |g| + \max_{\Omega} |f|) \]
whenever $u$ is a smooth solution of
\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega.
\end{cases}
\]
(Hint: $-\Delta \left( u + \frac{|x|^2}{2n} \lambda \right) \leq 0$, for $\lambda := \max_{\Omega} |f|$.)

Exercise 2
Let $\Omega$ be a bounded, open subset of $\mathbb{R}^n$. Let $u_n \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be harmonic in $\Omega$ with $u_n = g_n$ on $\partial \Omega$, where $g_n \in C^0(\partial \Omega)$ with
\[ \sup_{\partial \Omega} |g_n - g_m| \to 0 \quad \text{as } n, m \to \infty. \]
Show that $(u_n)_{n \in \mathbb{N}}$ converges uniformly to a function $u : \overline{\Omega} \to \mathbb{R}$, which is harmonic in $\Omega$ and continuous on $\overline{\Omega}$.

Exercise 3
Let $\Omega \subset \mathbb{R}^n$ be bounded with $C^1$-boundary. For $v \in C^1(\overline{\Omega})$ the Dirichlet energy is defined as
\[ E(v) := \frac{1}{2} \int_{\Omega} |Dv|^2 \, d\mu. \]
Show that for $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ the following are equivalent:
\begin{enumerate}
  \item $\Delta u = 0$
  \item $E(u) = \min \{ E(v) : v \in C^1(\overline{\Omega}), v = u \text{ on } \partial \Omega \}$
\end{enumerate}