Boundary and Eigenvalue Problems:
13th problem sheet

On this exercise sheet you can use that every eigenfunction \( v \in W^{1,2}_0(\Omega) \) of \(-\Delta\) on a bounded Lipschitz domain \( \Omega \) automatically lies in \( C^\infty(\Omega) \cap C(\overline{\Omega}) \).

**Exercise 1**

Let
\[
E := \{(x, y) \in \mathbb{R}^2 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1\}, \quad a > b,
\]
be an ellipse. Find optimal enclosures \( \lambda_1(E) \in [\lambda_1(\Omega_1), \lambda_1(\Omega_2)] \) for the first Dirichlet eigenvalue \( \lambda_1(E) \) of \(-\Delta\) on \( E \) for approximating sets \( \Omega_2 \subset E \subset \Omega_1 \) such that

a) \( \Omega_1, \Omega_2 \) are balls.

b) \( \Omega_1, \Omega_2 \) are rectangles.

**Exercise 2**

Let \( \lambda_1 \leq \ldots \leq \lambda_n < \lambda_{n+1} \leq \ldots \) denote the Dirichlet eigenvalues of \(-\Delta\) on a bounded Lipschitz domain \( \Omega \) with a corresponding \( L^2\)-ONB of eigenfunctions \( v_1, v_2, \ldots \in W^{1,2}_0(\Omega) \). We want to show that the open set \( D := \{ x \in \Omega : v_n(x) \neq 0 \} \) has at most \( n \) open disjoint components.

a) Assume for contradiction \( D = \bigcup_{j=1}^{n+1} D_j \) for pairwise disjoint open subsets \( D_j \) of \( D \).

Set \( w_j := v_{n+1, D_j} \) for \( j = 1, \ldots, n+1 \). Show that every element of \( \text{span}\{w_1, \ldots, w_{n+1}\} \) has the same Rayleigh quotient as \( v_n \).

b) Show that \( \lambda_{n+1}(W) \leq \lambda_n \) for every subspace \( W \) of \( W^{1,2}_0(\Omega) \) with \( \text{dim}(W) = n \) and establish a contradiction.

*Hint:* In a) you may assume \( w_j \in W^{1,2}_0(D_j) \).
Exercise 3

Let $\Omega$ be a bounded Lipschitz domain $\Omega$ and let $\lambda_1$ be the first Dirichlet eigenvalue of $-\Delta$ with eigenfunction $\varphi_1 \in W_0^{1,2}(\Omega)$. Our aim is to prove that $\lambda_1$ is a simple eigenvalue, i.e. its eigenspace has dimension one and that its only eigenfunction $\varphi_1$ has no zeros in $\Omega$. In the following let

$$R(u) = \frac{||\nabla u||_2^2}{||u||_2^2}, \quad u \in W_0^{1,2}(\Omega) \setminus \{0\},$$

denote the Rayleigh quotient of $u$.

a) Show that every minimizer $\tilde{u} \in W_0^{1,2}(\Omega)$ of the Rayleigh quotient, i.e. every $\tilde{u} \in W_0^{1,2}(\Omega)$ satisfying $R(\tilde{u}) = \lambda_1$, is an eigenfunction for $-\Delta$ with eigenvalue $\lambda_1$.

b) Conclude that if $\tilde{u}$ is an eigenfunction of $-\Delta$ for $\lambda_1$ then so is $|\tilde{u}|$.

c) Let $\tilde{u}$ be an eigenfunction of $-\Delta$ for $\lambda_1$. Apply the classical strong maximum principle to $|\tilde{u}|$ in order to show that $\tilde{u}$ has no zeros in $\Omega$.

d) Show that there cannot exist two linearly independent eigenfunctions of $-\Delta$ for $\lambda_1$.

Hint: In a) use that the function $t \mapsto R(\tilde{u} + t\phi)$ has a local minimum at $t = 0$ for every $\phi \in W_0^{1,2}(\Omega)$. 