Boundary and Eigenvalue Problems:
9th problem sheet

In the following let \( l^2 = \{ (x_i)_{i \in \mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^2 < \infty \} \) be the normed space of square summable real sequences equipped with the inner product \( \langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i \). You may use that \((l^2, \langle \cdot, \cdot \rangle)\) is a real Hilbert space.

Exercise 1

Let \((a_{ij})_{i,j \in \mathbb{N}}\) be a double sequence such that \( \sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty \) and such that \( A_N = (a_{ij})_{1 \leq i,j \leq N} \) is positive semidefinite for all \( N \in \mathbb{N} \).

a) Prove that \( T : l^2 \to l^2, v \mapsto (\sum_{i,j=1}^{\infty} a_{ij}v_j)_{i \in \mathbb{N}} \) is well-defined, linear and compact.

b) Use Fredholm’s Alternative to prove that the equation

\[
    u_i + \sum_{j=1}^{\infty} a_{ij}u_j = f_i \quad (i \in \mathbb{N})
\]

is uniquely solvable for all \( f \in l^2 \).

Hint for a): Given a bounded sequence \((v^n) \in l^2 \) use Cantor’s diagonal argument to select a subsequence \((v^{n_j})\) such that every component sequence \((v^{n_j}_i)_{j \in \mathbb{N}}\) converges.

Exercise 3

Let \( I := [\alpha, \beta] \subset \mathbb{R} \) be a bounded interval and \( a \in C^1(I), c \in C(I) \) such that \( a(x) \geq \lambda > 0 \). Our aim is to derive Fredholm’s alternative by elementary means for the ODE boundary value problem

\[
    -(a(x)u')' + c(x)u = f(x) \quad (x \in I),
\]

\[
    u'(\alpha) = u'(\beta) = 0.
\]

\((*)\)
a) Write down a solution formula of the differential equation in terms of a fundamental system \( \{\xi, \eta\} \) of the homogeneous differential equation. Use variation of constants.

b) Prove that the boundary value problem \((*)\) is uniquely solvable for all \( f \in C(I) \) if and only if \( \int_I f(x)w(x) \, dx = 0 \) for all \( w \) satisfying

\[
-(a(x)w')' + c(x)w = 0 \quad (x \in I),
\]

\[
w'(\alpha) = w'(\beta) = 0.
\]

Determine \( \gamma \in \mathbb{R} \) such that the problem

\[
\begin{align*}
u'' + u &= e^x - \gamma x \quad (x \in (0, \pi)), \\
u'(0) &= u'(\pi) = 0
\end{align*}
\]

is solvable and calculate all solutions.

*Hint:* Use (and maybe prove) that the Wronskian \( W(x) = \det \begin{pmatrix} \xi(x) & \eta(x) \\ \xi'(x) & \eta'(x) \end{pmatrix} \) of the fundamental system \( \{\xi, \eta\} \) is constant.