Aspects of nonlinear wave equations

Sheet 3

Problem 1 For \( f \in C^2(\mathbb{R}) \), \( f(0) = 0, f'(0) = 1 \) consider the fourth order wave equation

\[
U_{tt} + U_{xxxx} + f(U) = 0.
\] (\( \ast \))

Traveling waves \( U(x, t) = v(x - \omega t) \) satisfy

\[
v^{iv} + \omega^2 v'' + f(v) = 0.
\]

If we set \( v(x) = \frac{4}{\omega^4} u \left( \frac{1}{\sqrt{2}} x \right) \), then \( u \) satisfies

\[
u^{iv} + 2u'' + g(u) = 0 \quad \text{with} \quad g(y) = f \left( \frac{4}{\omega^4} y \right).
\] (\( \ast \ast \))

(a) Find a first integral for (\( \ast \ast \)).

(b) Write (\( \ast \ast \)) as a Hamiltonian system with

\[
\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -u' - u''' \\ u + u'' \end{pmatrix}, \quad \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} u \\ u' \end{pmatrix}.
\]

(c) Show that (\( \ast \)) has periodic traveling waves for \( \omega \in (\sqrt{2}, \infty) \) by means of the Lyapunov center theorem.

Remark: The relevant matrix is

\[
\begin{pmatrix}
\frac{\partial^2 H}{\partial p_1 \partial q_1} (p_0, q_0) & \frac{\partial^2 H}{\partial q_1 \partial q_2} (p_0, q_0) \\
-\frac{\partial^2 H}{\partial p_2 \partial q_1} (p_0, q_0) & -\frac{\partial^2 H}{\partial q_2 \partial q_2} (p_0, q_0)
\end{pmatrix}.
\]
Note the minus sign in the lower right entry.

Problem 2 Consider \( u_{tt} - u_{xx} = u(1 - u^2) \). Find the explicit form of traveling wave \( u(x, t) = v(x - \omega t), \omega \in (-1, 1) \setminus \{0\} \) with a heteroclinic profile \( v(0) = 0, \lim_{s \to \pm \infty} v(s) = \pm 1 \).