Note: We freely use the notation and definitions of my lecture notes Functional Analysis, Spectral Theory and (most importantly) Evolution Equations.

These lecture notes are based on my course from the summer semester 2011 though there are some minor changes in the contents and the numbering. Since this course had only two hours per week, some of the following proofs were not given in the lectures.

For these and previous lecture notes, I have received detailed comments and corrections from the students of my classes. I am glad to thank them for their fruitful response.

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Contents

Chapter 1. Stability of positive semigroups ........................................... 1

Chapter 2. Interpolation theory and regularity ........................................ 9
  1. Real interpolation spaces for semigroups ..................................... 9
  2. Regularity of analytic semigroups ............................................. 19

Chapter 3. Semilinear parabolic problems ........................................... 23
  1. Wellposedness ........................................................................... 23
  2. Convergence to equilibria ....................................................... 33

Chapter 4. The nonlinear Schrödinger equation ...................................... 43
  1. Basic properties and the linear problem ..................................... 43
  2. Local wellposedness ................................................................ 54
  3. Asymptotic behavior .............................................................. 60

Bibliography ..................................................................................... 65
CHAPTER 1

Stability of positive semigroups

Evolution equations often describe the behavior of positive quantities, such as the concentration of a species or the distribution of mass or temperature. It is then a crucial property of the system that positive initial values lead to positive solutions. This property of positivity has to be verified in the applications, of course, and we will see below that it implies many additional useful features of the semigroup solving the equation. To deal with positivity, we consider as state spaces only the following classes of Banach spaces $E$ consisting of scalar valued functions.

**Standing hypothesis.** In this chapter, $E$ denotes a function space of the type $L^p(\mu), C_0(U)$ or $C(K)$, where $p \in [1, \infty)$, $(S, \mu)$ is a $\sigma$–finite measure space (e.g., a Borel set $S \subseteq \mathbb{R}^d$ with the Lebesgue measure), $U$ is a locally compact metric space (e.g., an open subset of $\mathbb{R}^d$), or $K$ is a compact metric space, respectively.

We stress that we still take $C$ as the scalar field in order to use spectral theory. Actually, we could work in the more general class of (complex) Banach lattices $E$, but for simplicity we restrict ourselves to the above indicated setting. It suffices for the typical applications; however for certain deeper investigations one actually needs the more abstract framework. We refer to the monograph [Na-Ed] for a thorough discussion of positive $C_0$–semigroups in Banach lattices.

In the spaces $E$ given by the standing hypothesis, we have the usual concept of positive functions $f \geq 0$, of positive and negative parts $f_\pm$ and domination $f \leq g$ of real valued functions, and of the absolute value $|f|$. We write $E_+ = \{f \in E \mid f \geq 0\}$ for the cone of positive functions, which is closed in $E$. For all $f, g \in E$, it holds $\|f\| = \|g\|$, and $0 \leq f \leq g$ implies that $\|f\| \leq \|g\|$.

An operator $T \in \mathcal{B}(E)$ is called positive if $Tf \geq 0$ for every $f \in E_+$. One then writes $T \geq 0$. A $C_0$–semigroup $T(\cdot)$ is positive if each operator $T(t), t \geq 0$, is positive. We discuss a few basic properties of positive operators $T,S \in \mathcal{B}(E)$ which are used below without further notice. First, products of positive operators are positive. Next, for all $f,g \in E$ with $f \geq g$ we have $T(f-g) \geq 0 \iff Tg$. For real valued $f$, also the image $Tf = Tf_+ - Tf_- \leq T|f|$ has real values. Moreover, $|Tf| \leq Tf_+ + Tf_- = T|f|$. For complex valued $f$, we fix a point $x$ in $\Omega \in \{S,U,K\}$. Take a number $\alpha$ such that $|\alpha| = 1$ and $|Tf(x)| = \alpha Tf(x)$, where we fix a representative of $Tf$ if $E = L^p$. It follows that

$$|Tf(x)| = \alpha Tf(x) = T(\text{Re}(\alpha f))(x) + iT(\text{Im}(\alpha f))(x) = T(\text{Re}(\alpha f))(x) \leq T(\text{Re}(\alpha f))(x) \leq T(|\alpha f|)(x) = T(|f|)(x).$$
Consequently,
\[ |Tf| \leq T|f| \quad \text{holds for all } f \in E. \]
We further write \( 0 \leq T \leq S \) if \( 0 \leq Tf \leq Sf \) for all \( f \in E_+. \) Let \( 0 \leq T \leq S. \)
Then \( |Tf| \leq T|f| \leq S|f| \) is true for all \( f \in E, \) and hence
\[ \|T\| = \sup_{\|f\| \leq 1} \|Tf\| = \sup_{\|f\| \leq 1} \|Tf\| \leq \sup_{\|f\| \leq 1} \|S|f|\| \leq \|S\|. \]

As a first result we recall Corollary 3.25 of [EE] characterizing the positivity of a semigroup by the resolvent of its generator.

**Proposition 1.1.** Let \( A \) generate the \( C_0 \)-semigroup \( T(\cdot) \) on \( E. \) Then \( T(\cdot) \)
is positive if and only if there exists a number \( \omega \geq \omega_0(A) \) such that \( R(\lambda, A) \geq 0 \)
for all \( \lambda > \omega. \)

This proposition easily follows from the formulas
\[ R(\lambda, A)f = \int_0^\infty e^{-\lambda t}T(t)f \, dt \quad \text{and} \quad T(t)f = \lim_{n \to \infty} \left[ \frac{n}{T} R\left( \frac{n}{T}, A \right) \right]^n f \]
which are valid for all \( f \in E, \lambda > \omega_0(A) \) and \( t \geq 0, \) see Proposition 1.14 and Corollary 3.23 of [EE]. Of course one would like to determine the positivity of the semigroups by its generator. For contraction semigroups this can be done by a version of the Lumer–Phillips theorem due to Phillips, see Theorem C-II.1.2 in [Na-Ed]. Here we restrict ourselves to the Dirichlet and Neumann Laplacians in \( L^p. \) The case \( E = C_0 \) was treated in Example 3.26 of [EE] (for Dirichlet boundary conditions).

To discuss the Neumann Laplacian we need a few tools. Let \( U \subseteq \mathbb{R}^d \) be open and bounded with \( \partial U \in C^2 \) and outer unit normal \( \nu. \) For \( g \in W^{1}_p(U)^d, \)
\( v \in W^{1}_p(U) \) and \( p \in [1, \infty], \) we have the divergence theorem in Sobolev spaces:
\[ \int_U \text{div}(g)v \, dx = -\int_U g \cdot \nabla v \, dx + \int_{\partial U} (\nu \cdot \text{tr} g) \, \text{tr} \, v \, d\sigma, \quad (1.1) \]
where \( \text{tr} \in B(W^{1}_p(U), L^p(U)) \) is the trace operator extending the map \( v \mapsto v|\partial U \)
defined for \( v \in C(\overline{U}) \cap W^{1}_p(U), \) see Theorem 3.32 in [ST]. Equation (1.1) can be shown by approximation as Green’s formula in Theorem 3.34 in [ST], starting from the divergence theorem in \( C^1(\overline{U}). \) We now define the Neumann trace by
\[ \partial_{\nu} v = \nu \cdot \text{tr} \, \nabla v = \sum_{j=1}^{d} \nu_j \text{tr} \, \partial_j v \quad \text{for } u \in W^{2}_p(U). \]

We next state an important property of the Laplacian for later use.

**Lemma 1.2** (Hopf). Let \( B = B(y, \rho) \subseteq \mathbb{R}^d \) be an open ball and \( w \) belong to \( W^{2}_p(B) \) for all \( p \in (1, \infty) \) and satisfy \( 0 \leq \Delta w \in C(\overline{B}). \) Assume that there is
an \( x_0 \in \partial B \) such that \( w(x_0) > w(x) \) for all \( x \in B. \) Then \( \partial_{\nu} w(x_0) > 0 \) for the outer normal \( r = |x - y| \) of \( \partial B. \)

For \( w \in C^2(B), \) this lemma is a special case of Lemma 2.1.2 of [Jo]. Looking at the arguments given in [Jo], one sees that this additional regularity of \( w \) is only needed in the proof of the earlier Lemma 2.1.1. But, under our conditions one can prove this lemma as in [Jo] using Proposition 3.1.10 of [Lu1].
Example 1.3. Let \( U \subseteq \mathbb{R}^d \) be open and bounded with boundary of class \( C^2 \), or let \( U = \mathbb{R}^d \). Set \( E = L^p(U) \) for \( p \in (1, \infty) \).

(a) We first study the Dirichlet Laplacian \( Au = \Delta u \) with domain \( D(A) = D(A_p) = W^2_p(U) \cap W^1_p(U) \) in \( E \). Recall that \( W^1_p(U) \) is the kernel of the trace operator by Theorem 4.8 in \([EE]\). It is known that \( A \) generates an analytic contraction semigroup on \( E \) (see e.g. Example 2.19 of \([EE]\)). Let \( \lambda > 0 \) and \( 0 \neq f \in E_+ \cap L^2(U) \). Then \( 0 \neq u := R(\lambda, A)f \) is the unique solution in \( D(A_p) \) of the equation

\[
\lambda u - \Delta u = f \quad \text{on} \quad U.
\]

Since \( v := \text{Im} u \in D(A_p) \) satisfies \( \lambda v - \Delta v = 0 \), we have \( v = 0 \); i.e., \( u \) is real. We will prove that \( u \geq 0 \) in the next paragraphs. By approximation, the positivity of \( u \) then holds for all \( f \in E_+ \) implying that \( T(\cdot) \geq 0 \) due to Proposition 1.1.

If \( p > 2 \), we have \( u \in D(A_2) \). If \( p < 2 \), we observe that we also have a solution \( w \in D(A_2) \) of (\( * \)). Since then \( w \in D(A_p) \), the uniqueness in \( L^p \) yields \( u = w \in D(A_2) \) by uniqueness in \( L^p \). In any case, \( u \) belongs to \( u \in D(A_2) \).

Corollary 3.10 of \([ST]\) now shows that \( u_- \in W^1_2(U) \) and \( \nabla u_- = -\int_{\{u<0\}} \nabla u \).

To verify that \( u_- \in W^1_2(U) \), we take \( u_n \in C^\infty(U) \) converging to \( u \) in \( W^1_2(U) \).

As in the proof of Corollary 3.9, we find \( C^1 \)-functions \( h_\varepsilon, \varepsilon > 0 \), such that \( |h'_\varepsilon| \leq 1, h_\varepsilon = 0 \) on \( \mathbb{R}_+ \) and \( |h_\varepsilon(t)| \leq |t| \) on \( \mathbb{R}_- \), as well as \( h_\varepsilon(t) \to -t \) and \( h'_\varepsilon(t) \to -1 \) for \( t < 0 \) as \( \varepsilon \to 0 \). The proof of Proposition 3.9 in \([ST]\) (with \( K \) replaced by \( U \)) yields that \( h_\varepsilon(u_n) \to h_\varepsilon(u) \) in \( W^1_2(U) \) as \( n \to \infty \), for each fixed \( \varepsilon > 0 \). Since \( h_\varepsilon(u_n) \) has compact support, we deduce that \( h_\varepsilon(u) \in W^1_2(U) \). Lebesgue’s theorem further implies that \( h_\varepsilon(u) \to u_- \) in \( W^1_2(U) \) as \( \varepsilon \to 0 \), see the proof of Corollary 3.9 in \([EE]\). As a result, \( u_- \in W^2_1(U) \).

We next multiply (\( * \)) by \( u_- \) and integrate over \( U \). The divergence theorem (1.1) yields

\[
\int_U f u_- \, dx = \lambda \int_U u u_- \, dx - \int_U u_- \Delta u \, dx = -\lambda \int_{\{u<0\}} |u|^2 \, dx - \int_{\{u<0\}} |\nabla u|^2 \, dx
\]

Because the left hand side is positive, the set \( \{u < 0\} \) must have measure 0, and hence \( u \geq 0 \).

(b) The Neumann Laplacian on \( E \) is given by \( \Delta_N u = \Delta u \) on \( D(\Delta_N) = \{ u \in W^2_p(U) \mid \partial_u u = 0 \} \). Based on (1.1) one sees as in Example 2.18 and 2.19 of \([EE]\) that the operator \( e^{\theta} \Delta_N \) is dissipative on \( L^p(U) \), if \( 0 \leq |\theta| \leq \arccot(\frac{p-2}{2\sqrt{p-1}}) \in (0, \pi/2] \). Theorem 9.3.5 in \([Kr]\) further implies that that \( I - \Delta_N \) is surjective. Consequently, \( \Delta_N \) generates a contractive analytic \( C_0 \)-semigroup on \( E \) by Corollary 2.14 in \([EE]\). The positivity is shown as in (a) using again (1.1).

The next result collects the basic features of the spectral theory of positive semigroups.

**Theorem 1.4.** Let \( A \) generate the positive \( C_0 \)-semigroup \( T(\cdot) \) on \( E \). Then the following assertions hold.

(a) \( \text{Let } \text{Re } \lambda > s(A) \text{ and } f \in E. \text{ Then the improper Riemann integral} \)

\[
\int_0^\infty e^{-\lambda t} T(t)f \, dt = R(\lambda, A)f
\]

(1.2)
exists. Moreover, \( \|R(\lambda, A)\| \leq \|R(Re \lambda, A)\| \).

(b) \( s(A) = s_0(A) := \inf\{r > s(A) \mid \sup_{Re \mu \geq r} \|R(\mu, A)\| < \infty\} \).

(c) If \( \sigma(A) \neq \emptyset \), then \( s(A) \in \sigma(A) \).

(d) For \( \lambda \in \rho(A) \), the resolvent \( R(\lambda, A) \) is positive if and only if \( \lambda > s(A) \).

(e) \( s(A) = \omega_1(A) := \inf\{\omega \in \mathbb{R} \mid \exists M_\omega \geq 1 : \|T(t)x\| \leq M_\omega e^{\omega t} \|x\|_A (\forall t \geq 0, x \in D(A))\} \). In particular, if \( s(A) < 0 \), then there are \( N, \delta > 0 \) such that \( \|T(t)x\| \leq Ne^{-\delta t} \|x\|_A \) for all \( x \in D(A) \) and \( t \geq 0 \).

**Proof.** (a) For \( \lambda > \omega_0(A) \), Proposition 1.1 yields that \( R(\lambda, A) \geq 0 \). If \( \mu \in (s(A), \lambda) \) with \( 0 < \lambda - \mu < \|R(\lambda, A)\|^{-1} \), the Neumann series gives

\[
R(\mu, A) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda, A)^{n+1} \geq 0.
\]

Since \( \|R(r, A)\| \) is bounded for \( r \geq s(A) + \varepsilon \) and any fixed \( \varepsilon > 0 \), we deduce the positivity of \( R(\mu, A) \) for all \( \mu > s(A) \) (establishing one implication in assertion (d)). Let \( \mu > s(A), Re \alpha > 0, f \in E \) and \( t \geq 0 \). We set

\[
V(t)f = \int_{0}^{t} e^{-\mu s}T(s)f ds.
\]

From Lemma 1.12 in [EE] we deduce that

\[
0 \leq V(t)f = R(\mu, A)f - R(\mu, A)e^{-\mu t}T(t)f \leq R(\mu, A)f
\]

for all \( f \in E_+ \). Hence, \( \|V(t)\| \leq \|R(\mu, A)\| \) for all \( t \geq 0 \), and thus the function \( \mathbb{R}_+ \ni t \mapsto e^{-\alpha t}V(t)f \) is integrable. Integrating by parts, we deduce

\[
\int_{0}^{t} \alpha e^{-\alpha s}V(s)f ds + e^{-\alpha t}V(t)f = \int_{0}^{t} e^{-\alpha s}e^{-\mu s}T(s)f ds
\]

for all \( f \in E \). We can now let \( t \to \infty \), obtaining the integral in (1.2) with \( \lambda = \mu + \alpha \) on the right hand side. Proposition 1.14 in [EE] then yields \( \lambda \in \rho(A) \) and (1.2). Since we can vary \( \mu > s(A) \), these results also hold for \( Re \alpha \geq 0 \). It further follows that

\[
|R(\mu + \alpha, A)f| \leq \int_{0}^{\infty} e^{-(\mu + Re \alpha)t} |T(t)f| dt \leq \int_{0}^{\infty} e^{-\mu t}T(t)|f| dt = R(\mu, A)|f|.
\]

This inequality implies that \( \|R(\mu + \alpha, A)\| \leq \|R(\mu, A)\| \), and thus the second assertion in (a) is true.

(b) It is clear that \( s(A) \leq s_0(A) \). The converse inequality follows from (a) and the fact that \( \|R(r, A)\| \) is bounded for \( r \geq s(A) + \varepsilon \) and any fixed \( \varepsilon > 0 \).

(c) Assume that \( \sigma(A) \neq \emptyset \). We can find \( \lambda_n \in \rho(A) \) tending to \( \sigma(A) \) with \( Re \lambda_n > s(A) \to -\infty \). Assertion (a) and Theorem 1.13 in [ST] imply that

\[
\|R(Re \lambda_n, A)\| \geq \|R(\lambda_n, A)\| \geq d(\lambda_n, \sigma(A))^{-1} \to \infty
\]
as \( n \to \infty \). If \( s(A) \in \rho(A) \), then \( R(Re \lambda_n, A) \) would converge to \( R(s(A), A) \) leading to a contradiction. The spectral bound thus belongs to \( \sigma(A) \).

(d) Let \( R(\lambda, A) \) be positive for some \( \lambda \in \rho(A) \). Take \( 0 \neq f \in E_+ \). The function \( 0 \neq u := R(\lambda, A)f \) is also positive and \( Au = \lim_{t \to 0} \frac{1}{t} (T(t)f - f) \) is
real valued. Hence, \( \lambda u = f + Au \) is real, so that \( \lambda \in \mathbb{R} \). Let \( \mu > \max\{\lambda, s(A)\} \).

Part (a) of the proof shows that \( R(\mu, A) \geq 0 \), and thus

\[
R(\lambda, A) = R(\mu, A) + (\mu - \lambda)R(\mu, A)R(\lambda, A) \geq R(\mu, A) \geq 0.
\]

Using \( s(A) \in \sigma(A) \) and Theorem 1.13 in [ST], we deduce that

\[
\frac{1}{\mu - s(A)} \leq \frac{1}{d(\mu, \sigma(A))} \leq \|R(\mu, A)\| \leq \|R(\lambda, A)\|.
\]

If \( \lambda \leq s(A) \), the limit \( \mu \to s(A) \) would give a contradiction. Hence, (d) holds.

(e) Let \( \lambda > s(A) \) and \( f \in D(A) \). Assertion (a) then implies that

\[
e^{-\lambda t}T(t)f = f + \int_0^t e^{-\lambda s}T(s)(A - \lambda I)f\,ds \to f + R(\lambda, A)(A - \lambda I)f = 0
\]

as \( t \to \infty \). Hence, \( e^{-\lambda t}T(t) \) is bounded in \( B([D(A)], X) \) uniformly for \( t \geq 0 \) by the principle of uniform boundedness. This fact implies that \( \omega_1(A) \leq s(A) \).

Conversely, let \( \Re \lambda > \omega_1(A) \) and \( f \in D(A) \). Then the integral

\[
\int_0^t e^{-\lambda t}T(t)f\,dt =: R_\lambda f
\]

converges in \( E \). As in the proof of Proposition 1.14 of [EE], it follows that \( R_\lambda f \in D(A) \) and \( (A - \lambda I)R_\lambda f = f \). Moreover, \( R_\lambda(A - \lambda I)f = f \) if \( f \in D(A^2) \). We denote by \( A_1 \) the restriction of \( A \) to \( X_1 := [D(A)] \) with domain \( D(A_1) = D(A^2) \). We have shown that \( \lambda \in \rho(A_1) \). Since \( A \) and \( A_1 \) are similar via the isomorphism \( R(\lambda, A) : D(A) \to D(A^2) \), we arrive at \( \lambda \in \rho(A) \); i.e., \( s(A) = \omega_1(A) \).

The next corollary immediately follows from part (b) of the above theorem and Gearhart’s stability theorem, see Theorem 4.4 in [EE].

**Corollary 1.5.** Every generator \( A \) of a positive semigroup on \( E = L^2(\mu) \) satisfies \( s(A) = \omega_0(A) \).

**Remark 1.6.** The above corollary is due to Greiner and Nagel, [GN1]. It actually holds for all our spaces \( E \), see Satz 3.3 in [De] for \( E = L^1(\mu) \) and \( E = C(K) \), Theorem B-IV.1.4 of [Na-Ed] for \( E = C_0(U) \), and [We] for \( E = L^p(\mu) \), but it fails already on \( L^p \cap L^q \) with \( 1 < p < q < \infty \), see e.g. Example IV.3.3 in [EN]. For any generator \( A \), it holds \( s(A) \leq \omega_1(A) \leq s_0(A) \leq \omega_0(A) \). (These inequalities follow from the proof of Theorem 1.4(e), Theorem 3.2 in [WW] and Proposition 1.14 in [EE].) Hence, in Theorem 1.4 assertion (e) follows directly from (b) due to the (more difficult) general result in [WW]. There are positive semigroups with \( s_0(A) < \omega_0(A) \), see e.g. Example 4.3 in [EE]. Moreover, there are (non positive) semigroups on Banach spaces \( X \) such that \( s(A) < \omega_1(A) \), see e.g. Example 1.2.4 in [vN], or \( \omega_1(A) < s_0(A) \), see Example 4.2 in [Wr].

**Example 1.7.** **Cell division.** Let \( \int_a^b u(t, s)\,ds \) be the number of cells of a certain species at time \( t \geq 0 \) of size \( s \in [a, b] \). We make the following assumptions on this species.

- Each cell grows linearly with time at (normalized) velocity 1.
- Cells of size \( s \geq \alpha > 0 \) divide with per capita rate \( b(s) \geq 0 \) in two daughter cells of equal size, where \( b = 0 \) on \([1, \infty)\) and on \( [\alpha/2, \alpha] \).
• Cells of size $s$ die with per capita rate $\mu(s) \geq 0$.
• The functions $b \neq 0$ and $\mu$ are continuous, and $\alpha > 1/2$.
• There are no cells at size $\alpha/2$.

It is just a normalization that the cells divide up size $s = 1$. The assumptions of linear growth and that $\alpha > 1/2$ are made for simplicity, see [GN2] for the general case. The assumptions on $b$ indicate that the interesting cell sizes belong to $J = [\alpha/2, 1]$ (for others one only has growth and death), so that we choose as state space $E = L^1(J)$. Hence, the norm $\|u(t)\|_1$ equals the number of (relevant) cells at time $t$, if $u \geq 0$. It can be shown that under the above assumptions smooth cell size distributions $u$ satisfy the equations

$$\partial_t u(t, s) = -\partial_s u(t, s) - \mu(s)u(t, s) - b(s)u(t, s) + 4b(2s)u(t, 2s), \quad t \geq 0, \ s \in J,$$

$$u(0, s) = u_0(s), \quad s \in J.$$

Note that $b(2s) = 0$ for $s \geq 1/2$. For such $s$ we put $v(2s) := 0$ for any function $v$ on $J$. We take $0 \leq u_0 \in D(A) := \{v \in W^1_1(J) \mid v(\alpha/2) = 0\}$ and define

$$Av = -v' - \mu v - bv + Bv, \quad Bv(s) = 4b(2s)v(2s),$$

for $v \in D(A)$, resp. $v \in E$ and $s \in J$. Observe that $B$ is a bounded (and positive) operator on $E$ because

$$\|Bv\|_1 \leq 4 \|b\|_\infty \int_{\alpha/2}^{1/2} |v(2s)| \, ds \leq 2 \|b\|_\infty \|v\|_1.$$

Since $-d/ds$ with domain $D(A)$ generates a positive $C_0$–semigroup on $E$ (the nilpotent translations), the bounded perturbation theorem shows that also $A$ generates a positive $C_0$–semigroup $T(\cdot)$ on $E$, see Theorem 3.4 and Example 3.5 in [EE]. It is clear that the function $u(t, s) = (T(t)u_0)(s)$ with $t \geq 0$ and $s \in J$ belongs to $C^1(\mathbb{R}^+, E) \cap C(\mathbb{R}^+, W^1_1(J))$ and satisfies the system (1.3), where the first line holds for a.e. $s \in J$. Moreover, $u$ is positive. On the other hand, each solution $u \in C^1(\mathbb{R}^+, E) \cap C(\mathbb{R}^+, W^1_1(J))$ of (1.3) is given by $T(\cdot)$.

The embedding $D(A) \hookrightarrow E$ is compact due to Theorem 3.28 in [ST]. Therefore the resolvent of $A$ is compact and $\sigma(A)$ consists of eigenvalues only, see Remark 2.14 and Theorem 2.16 of [ST]. We can even determine the eigenvalues by the zeros of a holomorphic function $\xi$. (The assumption $\alpha > 1/2$ is only needed to obtain below the very simple formula of $\xi$.)

**Lemma 1.8.** Let $A$ be given by (1.4). Then a number $\lambda \in \mathbb{C}$ belongs to $\sigma(A)$ if and only if

$$0 = \xi(\lambda) := -1 + \int_{\alpha/2}^{1/2} 4b(2\sigma) \exp \left( - \int_{\sigma}^{2\sigma} (\lambda + \mu(\tau) + b(\tau)) \, d\tau \right) d\sigma.$$

**Proof.** As noted above, we have $\sigma(A) = \sigma_p(A)$. Hence, $\lambda \in \mathbb{C}$ belongs to $\sigma(A)$ if and only if there is a $0 \neq v \in D(A)$ with $\lambda v = v'$. Equivalently, $0 \neq v \in W^1_1(J)$ satisfies

$$v'(s) = - (\lambda + b(s) + \mu(s)) v(s), \quad 1/2 \leq s \leq 1,$$

$$v'(s) = - (\lambda + b(s) + \mu(s)) v(s) + 4b(2s)v(2s), \quad \alpha/2 \leq s < 1/2,$$
\[ v(\alpha/2) = 0. \]

These equations are only satisfied by the function
\[ v(s) = c \exp \left( \int_{s}^{1} (\lambda + b(\sigma) + \mu(\sigma)) \, d\sigma \right), \quad \frac{1}{2} \leq s \leq 1, \]
\[ v(s) = c \exp \left( \int_{s}^{1} (\lambda + b(\sigma) + \mu(\sigma)) \, d\sigma \right) \cdot \left[ 1 - \int_{s}^{1/2} 4b(2\sigma) \exp \left( - \int_{\sigma}^{2\sigma} (\lambda + \mu(\tau) + b(\tau)) \, d\tau \right) \, d\sigma \right], \quad \alpha / 2 \leq s < \frac{1}{2}, \]
for any constant \( c \neq 0 \). Clearly, this \( v \) belongs to \( W^{1}_{1}(J) \), and it satisfies \( v(\alpha/2) = 0 \) if and only if \( \xi(\lambda) = 0 \). □

Theorem 1.4 shows that \( \omega_{1}(A) = s(A) \), and Remark 1.6 even yields \( \omega_{0}(A) = s(A) \). In Proposition VI.1.4 of [EN] it is further shown that \( t \mapsto T(t) \) is continuous in operator norm for \( t > 1 - \frac{\alpha}{2} \). (Here one uses the nilpotency of the semigroup generated by \( A_{0} := A - B \) and the Dyson–Phillips series for \( A = A_{0} + B \), see formula (3.5) in [EE].) Therefore even the spectral mapping theorem \( \sigma(T(t)) = e^{\sigma(A)} \setminus \{0\} \) is true implying again \( \omega_{0}(A) = s(A) \), see Theorem 4.13 and Corollary 4.14 in [EE]. Positivity even yields a very simple criterion for \( \omega_{0}(A) = s(A) < 0 \).

**Theorem 1.9.** The semigroup generated by \( A \) from (1.4) is exponentially stable on \( E \) if and only if
\[ \xi(0) = -1 + \int_{\alpha/2}^{1/2} 4b(2\sigma) \exp \left( - \int_{\sigma}^{2\sigma} (\mu(\tau) + b(\tau)) \, d\tau \right) \, d\sigma < 0. \]

In particular, there are constants \( N, \delta > 0 \) such that \( \|u(t)\|_{1} \leq Ne^{-\delta t} \|u_{0}\|_{1} \) for all \( t \geq 0 \) and all solutions \( u \in C^{1}(\mathbb{R}_{+}, E) \cap C(\mathbb{R}_{+}, W^{1}_{1}(J)) \) of (1.3).

**Proof.** In view of Lemma 1.8 and the discussion above the statement of the theorem, we have to show that all zeros of \( \xi \) have strictly negative real parts. To characterize this property, we use the positivity of the semigroup in a crucial way. Theorem 1.4 says that \( s(A) \in \sigma(A) \). Thus \( \omega_{0}(A) < 0 \) if and only if all real zeros of \( \xi \) are strictly negative. On \( \mathbb{R} \), the function \( \xi \) is continuous and strictly decreasing from \( \infty \) to \(-1 \). Consequently, \( \xi \) has exactly one real zero, which is strictly negative if and only if \( \xi(0) < 0 \). □
CHAPTER 2

Interpolation theory and regularity

Interpolation theory is an independent branch of functional analysis which has important applications in many fields of mathematics. Among others, the monographs [BB], [BL], [Lu2] and [Tr] give an introduction to this theory. The applications to evolution equations are stressed in [BB], [Lu1] and [Lu2]. We focus here on these applications and do not develop interpolation theory explicitly, though it is hidden in some of the proofs. In this sense the next section is similar to Section II.5 of [EN] (which treats the spaces $D_A(\alpha, \infty)$ and $D_A(\alpha)$ in our notation), but we are closer to interpolation theory omitting certain other aspects investigated in [EN].

1. Real interpolation spaces for semigroups

In this section we always work in the following setting, sometimes adding more restrictions and assumptions.

**Standing hypothesis.** In this section, $A$ generates the $C_0$–semigroup $T(\cdot)$ on a Banach space $X$, $\alpha \in (0, 1)$, $p \in [1, \infty]$, and we put $M_0 := \sup_{t \in [0,1]} \|T(t)\|$. Recall that $T(\cdot)x$ is continuous for $x \in X$ and continuously differentiable for $x \in D(A)$. One can define the so called real interpolation spaces between $X$ and $D(A)$ by looking at $x \in X$ such that $T(\cdot)x$ is Hölder continuous (or satisfies an $L^p$ variant of this property). To that purpose, we define

$$\varphi_{\alpha,x}(t) = t^{-\alpha} \|T(t)x - x\| \quad \text{for } x \in X, \ t > 0.$$

If $T(\cdot)$ is a $C_0$–group, we set $\varphi_{\alpha,x}(t) = |t|^{-\alpha} \|T(t)x - x\|$ for all $t \neq 0$. We further introduce the space $L^p_{\alpha}(J) = L^p(J, dt/|t|)$ for any set $J \subseteq \mathbb{R} \setminus \{0\}$. Observe that $L^p_{\alpha}(J)$ is endowed with the norm given by

$$\|f\|_{L^p_{\alpha}(J)} = \int_J |f(t)|^p \frac{dt}{|t|}$$

if $p \in [1, \infty)$ and that $L^\infty_{\alpha}(J) = L^\infty(J)$. For $0 < a < b$ and $x \in X$, an elementary estimate yields

$$\|x\| + \|\varphi_{\alpha,x}\|_{L^p_{\alpha}((0,a])} \leq \|x\| + \|\varphi_{\alpha,x}\|_{L^p_{\alpha}((0,b])} \leq \|x\| + \|\varphi_{\alpha,x}\|_{L^p_{\alpha}((0,a])} + c_0 \|x\|,$$  \hspace{1cm} (2.1)

where the constant $c_0$ only depends on $a$, $b$ and $M_0$. We now define

$$L^p_{\alpha} = L^p_{\alpha}(0,1), \quad [x]_{\alpha,p} = \|\varphi_{\alpha,x}\|_{L^p_{\alpha}} \in [0, \infty], \quad \|x\|_{\alpha,p} = \|x\| + [x]_{\alpha,p}$$

and the real interpolation space between $X$ and $D(A)$ of order $\alpha \in (0, 1)$ and exponent $p \in [1, \infty]$ by

$$D_A(\alpha, p) = \{ x \in X \mid [x]_{\alpha,p} < \infty \}.$$
Observe that this space is defined just by an estimate (and not by a limit such as the space of continuous functions). It is straightforward to check that $D_A(\alpha, p)$ is a vector space with norm $\| \cdot \|_{\alpha,p}$. If one replaces here $J = (0, 1]$ by any other interval $(0, b]$, one obtains an equivalent norm, due to (2.1). Sometimes it is convenient to work with $J = (0, \infty) =: \mathbb{R}^+$, assuming that $M := \sup_{t \geq 0} \| T(t) \| < \infty$.

We then have $|\varphi_{\alpha,x}(s)| \leq s^{-\alpha} (1 + M) \| x \|$, and thus

$$\| \varphi_{\alpha,x} \|_{L_\infty^p((1,\infty))} \leq c(\alpha, p) (1 + M) \| x \|,$$

where $c(\alpha, \infty) = 1$ and $c(\alpha, p) = (\alpha p)^{-1/p} \leq \gamma \alpha^{-1}$ for some number $\gamma > 0$ and all $p \in [1, \infty)$. As a result,

$$\| x \|_{\alpha,x} \leq \| x \| + \| \varphi_{\alpha,x} \|_{L_\infty^p(\mathbb{R}^*_+)} \leq \| x \|_{\alpha,x} + c_1 \| x \|, \quad (2.2)$$

where $c_1 = 1 + M$ for $p = \infty$ and $c_1 = \gamma \alpha^{-1} (1 + M)$ for $p < \infty$. Let $T(\cdot)$ be a $C_0$-group bounded by $M$ on $\mathbb{R}$. Using that $\varphi_{\alpha,x}(s) = (-s)^{-\alpha} \| T(s) (x - T(-s)x) \| \leq M \varphi_{\alpha,x}(-s)$ for $s < 0$, we obtain

$$\| x \|_{\alpha,x} \leq \| x \| + \| \varphi_{\alpha,x} \|_{L_\infty^p(\mathbb{R} \setminus \{0\})} \leq (1 + M) \| x \|_{\alpha,x} + c_1 (1 + M) \| x \|. \quad (2.3)$$

We further want to check that also a rescaling of the semigroup just gives an equivalent interpolation norm. In fact, let $\omega \in \mathbb{R}$, $t \in (0, 1]$ and $x \in X$. It holds

$$\varphi_{\alpha,x}(t) = e^{\omega t} \varphi_{\alpha,x}(t) = \left( \begin{array}{c} \| e^{\omega t} - e^{-\omega t} \|_p \| x(t) - x \| + t^{-\alpha} \| e^{-\omega t} - 1 \| \| x \| \\ \| e^{-\omega t} T(t) x(t) - x \| + t^{-\alpha} \| e^{-\omega t} - 1 \| \| x \| \end{array} \right) \leq c(\| x \| + t^{-\alpha} \| e^{\omega t} T(t) x(t) - x \|), \quad (2.4)$$

where the constants only depend on $\omega$.

We next check that $D_A(\alpha, p)$ is a Banach space for $\| \cdot \|_{\alpha,p}$. In fact, let $(x_n)$ be a Cauchy sequence in $D_A(\alpha, p)$. Since $X$ is a Banach space, the vectors $x_n$ converge to some $x$ in $X$. Hence, $\varphi_{\alpha,x_n}$ tends pointwise to $\varphi_{\alpha,x}$ as $n \to \infty$. Note that $\| \varphi_{\alpha,x_n} \|_{L_\infty^p} \leq \| x_n \|_{\alpha,p}$ is bounded in $n \in \mathbb{N}$. We thus obtain $x \in D_A(\alpha, p)$, using Fatou’s lemma for $p < \infty$. Let $\varepsilon > 0$. There is an $N_\varepsilon \in \mathbb{N}$ such that $\| x - x_n \|_{\alpha,p} \leq \varepsilon$ for all $n, m \geq N_\varepsilon$. Since also $\varphi_{\alpha,x_m - x_n}$ tends pointwise to $\varphi_{\alpha,x - x_n}$, we further obtain

$$\| \varphi_{\alpha,x - x_n} \|_{L_\infty^p} \leq \liminf_{m \to \infty} \| \varphi_{\alpha,x_m - x_n} \|_{L_\infty^p} \leq \varepsilon$$

for all $n \geq N_\varepsilon$. As a consequence, $x_n$ converges to $x$ in $D_A(\alpha, p)$.

We finally define the continuous interpolation space

$$D_A(\alpha) = \overline{D(\alpha)^{\| \cdot \|_{\alpha,\infty}}}$$

and equip it with the norm $\| \cdot \|_{\alpha,\infty}$ It is thus a closed subspace of $D_A(\alpha, \infty)$.

In Example 2.2 below we see that $D_A(\alpha) \neq D_A(\alpha, \infty)$, in general.

We collect several basic properties of the real interpolation spaces in the next proposition. Here and below, we write $a \lesssim_M b$ if $a \leq cb$ for a constant depending on $M$ and $a \asymp_M b$ if $a \lesssim_M b$ and $b \lesssim_M a$.

**Proposition 2.1.** Let $A$ generate the $C_0$-semigroup $T(\cdot)$ on $X$, $0 < \alpha \leq \beta < 1$, and $1 \leq p \leq q < \infty$. Then the following assertions hold.

(a) $[D(A)] \hookrightarrow D_A(\beta, p) \hookrightarrow D_A(\alpha, p) \hookrightarrow X$.

(b) If $p \in [1, \infty)$, then $D(A)$ is dense in $D_A(\alpha, p)$. 


(c) $D_A(\alpha, 1) \hookrightarrow D_A(\alpha, p) \hookrightarrow D_A(\alpha, q) \hookrightarrow D_A(\alpha) \subseteq D_A(\alpha, \infty)$.
(d) $x \in D_A(\alpha, \infty)$ if and only if $T(t)x \in C^\alpha([0, b], X)$ for all $b > 0$.
(e) $x \in D_A(\alpha)$ if and only if $t^{-\alpha}(T(t)x - x) \to 0$ in $X$ as $t \to 0$.

**Proof.** (a) The last embedding is clear. Since $s^{-\alpha} \leq s^{-\beta}$ for $s \in (0, 1]$, we have $\varphi_{\alpha,x} \leq \varphi_{\beta,x}$ on $[0, 1]$ and thus $D_A(\beta, p) \hookrightarrow D_A(\alpha, p)$. For $x \in D(A)$, it holds $\varphi_{\beta,x}(s) = s^{-\beta}\|T(s)x - x\| \leq s^{1-\beta}M_0\|Ax\|$, so that $[D(A)] \hookrightarrow D_A(\beta, p)$.
(b) Let $x \in D_A(\alpha, p)$ and $n \in \mathbb{N}$ with $n > \omega_0(A)$. We set $x_n = nR(n, A)x \in D(A)$. Since $nR(n, A)$ is uniformly bounded for $n > \omega_0(A)$, we have

\[ \varphi_{\alpha,x-x_n}(s) = s^{-\alpha} \|(T(s) - I)(x-x_n)\| \to 0, \quad \text{as } n \to \infty, \]

\[ 0 \leq \varphi_{\alpha,x-x_n}(s) \leq \varphi_{\alpha,x}(s) + s^{-\alpha}nR(n, A)(T(s) - I)x \leq c\varphi_{\alpha,x}(s), \]

for $s \in (0, 1]$. Using $p < \infty$ and dominated convergence, we obtain that $\varphi_{\alpha,x-x_n} \to 0$ in $L^p_x$ as $n \to \infty$ which yields assertion (b).
(c) Let $x \in D_A(\alpha, r)$ for $r \in [1, \infty)$ and $t \in (0, 1]$. We compute

\[ t^{-\alpha}\|T(t)x - x\| = \left( \frac{1}{\alpha r} \int_t^2 s^{-\alpha r-1} ds - 2^{-\alpha r} \right)^{\frac{1}{2}} \|T(t)x - x\| \]

\[ \lesssim_{\alpha, M_0} \|x\| + \left( \int_t^2 s^{-\alpha r} \|T(t)x \pm T(s)x - x\| \frac{ds}{s} \right)^{\frac{1}{2}} \]

\[ \leq \|x\| + \left( \int_t^2 (s-t)^{-\alpha r} \|T(t)(x - T(s-t)x)\| \frac{ds}{s-t} \right)^{\frac{1}{2}} \]

\[ + \left( \int_0^2 s^{-\alpha r} \|T(s)x - x\| \frac{ds}{s} \right)^{\frac{1}{2}} \]

\[ \lesssim_{\alpha, M_0} \|x\| + \left( \int_0^2 \tau^{-\alpha r} \|T(\tau)x - x\| \frac{d\tau}{\tau} \right)^{\frac{1}{2}} \]

\[ \lesssim_{\alpha, M_0} \|x\|_{\alpha, r}, \]

where we substituted $\tau = s - t$ and used (2.1). It follows that $D_A(\alpha, r) \hookrightarrow D_A(\alpha, \infty)$. For $x \in D_A(\alpha, q)$, assertion (b) yields $x_n \in D(A)$ converging to $x$ in $D_A(\alpha, q)$, and hence in $D_A(\alpha, \infty)$. As a consequence, $D_A(\alpha, q)$ is even embedded into $D_A(\alpha)$. We further deduce

\[ \|\varphi_{\alpha,x}\| \leq \|\varphi_{\alpha,x}\|_p^p \|\varphi_{\alpha,x}\|_\alpha^{1-p} \lesssim_{\alpha, M_0} \|\varphi_{\alpha,x}\|_p, \]

establishing assertion (b).
(d) The implication ‘$\Leftarrow$’ is clear. For the other implication, let $x \in D_A(\alpha, \infty)$, $0 \leq s < t \leq b$, and $M := \sup_{t \in [0, b]} \|T(t)\|$, where we may assume that $b \geq 1$. If $t - s \geq 1$, we trivially have $(t-s)^{-\alpha}\|T(t)x - T(s)x\| \leq 2M\|x\|$. If $t - s \leq 1$, then $(t-s)^{-\alpha}\|T(t)x - T(s)x\| \leq M(t-s)^{-\alpha}\|T(t-s)x - x\| \leq M\|x\|_{\alpha, \infty}$.
(e) For $x \in D_A(\alpha)$ and $\varepsilon > 0$, there is a vector $y \in D(A)$ such that $\|x - y\|_{\alpha, \infty} \leq \varepsilon$. We can thus estimate

\[ \limsup_{t \to 0} t^{-\alpha}\|T(t)x - x\| \leq \|x - y\|_{\alpha, \infty} + \limsup_{t \to 0} t^{-\alpha}\|T(t)y - y\| \]

\[ \leq \varepsilon + \limsup_{t \to 0} t^{1-\alpha}\|Ay\| = \varepsilon \]

11
proving the ‘only if’ part. Conversely, assume that \( \varphi_{a,x}(s) = s^{-\alpha}(T(s)x - x) \) tends to 0 as \( s \to 0 \). Let \( \varepsilon > 0 \) and \( s, t \in (0, 1] \). First, there is a \( \delta \in (0, 1] \) with
\[
s^{-\alpha} \|(T(s) - I)(T(t)x - x)\| \leq (1 + M_0)s^{-\alpha} \|T(s)x - x\| \leq \varepsilon
\]
for all \( s \in (0, \delta] \) and \( t \in (0, 1] \). Second, we find an \( \eta \in (0, 1] \) such that
\[
s^{-\alpha} \|(T(s) - I)(T(t)x - x)\| \leq (1 + M_0)\delta^{-\alpha} \|T(t)x - x\| \leq \varepsilon
\]
for all \( s \in (\delta, 1] \) and \( t \in (0, \eta] \). This means that \( T(t)x \) converges to \( x \) in \( D_A(\alpha, \infty) \) as \( t \to 0 \). Hence, the vectors \( n \int_0^{1/n} T(t)x \, dt \in D(A) \) tend to \( x \) in \( D_A(\alpha, \infty) \) as \( n \to \infty \), i.e., \( x \in D_A(\alpha) \).

We note that the above proofs give explicit estimates for the norm of the embeddings in the statement of the proposition. We next describe the interpolation spaces for the translation group.

**Example 2.2.** On \( X = L^q(\mathbb{R}) \) for some \( q \in [1, \infty) \) or \( X = C_0(\mathbb{R}) \) for \( q = \infty \), we consider the (isometric) translation group given \( T(t)f = f(\cdot + t) \) for \( f \in X \) and \( t \in \mathbb{R} \). It has the generator \( A = d/ds \) with domain \( D(A) = W^p_q(\mathbb{R}) \) or \( D(A) = C_0^1(\mathbb{R}) \). Due to (2.3), the interpolation norms are given by
\[
\|f\|_{\alpha,p} \approx \|f\|_q + \left( \int_{\mathbb{R}} |t|^{-\alpha p-1} \|f(\cdot + t) - f\|_{L^q(\mathbb{R})} \, dt \right)^{\frac{1}{p}}
\]
\[
= \|f\|_q + \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{|f(s + t) - f(s)|^q}{|t|^{\alpha q+\frac{1}{p}}} \, ds \right)^{\frac{q}{q}} \, dt \right)^{\frac{1}{p}},
\]
\[
\|f\|_{\alpha,\infty} = \|f\|_\infty + \sup_{t \in \mathbb{R}} \left( \int_{\mathbb{R}} \frac{|f(s + t) - f(s)|^q}{|t|^{\alpha q}} \, ds \right)^{\frac{1}{q}}
\]
for \( p, q \in [1, \infty) \), and for \( p = q = \infty \) by the Hölder norm
\[
\|f\|_{\alpha,\infty} = \|f\|_\infty + \sup_{t, s \in \mathbb{R}} \frac{|f(s + t) - f(s)|}{|t|^{\alpha}}.
\]
For \( q < \infty \), the space \( D_A(\alpha, p) \) coincides with the Besov space \( B^\alpha_{qp}(\mathbb{R}) \), see [Tr] and [Tr2].1 In the special case \( p = q \in [1, \infty) \), the space \( B^\alpha_{qp}(\mathbb{R}) = W^\alpha_p(\mathbb{R}) \) is called Slobodetskiĭ space (or fractional Sobolev space) and has the simpler norm
\[
\|f\|_{\alpha,p} \approx \|f\|_p + \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(\tau) - f(\tau)|^p}{|\tau - s|^{\alpha p+1}} \, d\tau \, ds \right)^{\frac{1}{p}}.
\]
(Here one uses Fubini’s theorem and the substitution \( \tau = s + t \).)

There are \( f \in C_0(\mathbb{R}) \) with a finite Hölder norm \( \|f\|_{\alpha,\infty} \) such that \( f(s) = s^\alpha \) for \( s \in [0, 1] \), and thus \( t^{-\alpha} \|T(t)f - f\|_\infty \geq t^{-\alpha} |f(t) - f(0)| = 1 \) for all \( t \in (0, 1] \). Proposition 2.1 then yields \( f \in D_A(\alpha, \infty) \setminus D_A(\alpha) \).

We next see that the semigroup \( T(\cdot) \) behaves nicely on its interpolation spaces. In general, it is not strongly continuous on \( D_A(\alpha, \infty) \). Consider for instance the translation group on \( C_0(\mathbb{R}) \) in Example 2.2, and take a function \( f \in C_0(\mathbb{R}) \) such that \( f(t) = (t - n)^\alpha \) on \([n, n + 1/n]\). It then holds \( \|T(1/n)f - f\|_{C_0} \geq n^\alpha |f(n + 1/n) - f(n)| = 1 \) for all \( n \in \mathbb{N} \). Nevertheless one

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1To obtain the Besov spaces \( B^\alpha_{\infty,p}(\mathbb{R}) \) one has to start with \( C_0(\mathbb{R}) \) instead with \( C_0(\mathbb{R}) \), which is not possible in our setting, but see Example 1.9 of [Lu2] or [Tr], [Tr2].
could also work on $D_A(\alpha, \infty)$, see Section II.5 in [EN] and Section 2.2 in [Lu1] for analytic semigroups.

**Proposition 2.3.** Let $A$ generate the $C_0$-semigroup $T(\cdot)$ on $X$, $\alpha \in (0, 1)$, and $p \in [1, \infty]$. We have $T(t)D_A(\alpha, p) \subseteq D_A(\alpha, p)$ and $T(t)D_A(\alpha) \subseteq D_A(\alpha)$ for all $t \geq 0$. The norms of the restrictions $T_{\alpha,p}(t) = T(t)|D_A(\alpha, p)$ and $T_{\alpha}(t) = T(t)|D_A(\alpha)$ are less or equal $\|T(t)\|_{B(X)}$ for each $t \geq 0$. The operator families $T_{\alpha,p}(\cdot)$ and $T_{\alpha}(\cdot)$ are $C_0$-semigroups if $p < \infty$. The generators are the restrictions $A_{\alpha,p}$ (with $p < \infty$) and $A_{\alpha}$ of $A$ in the respective spaces having the domains

$$D(A_{\alpha,p}) = \{ x \in D(A) \mid Ax \in D_A(\alpha, p) \} =: D_A(1 + \alpha, p),$$

$$D(A_{\alpha}) = \{ x \in D(A) \mid Ax \in D_A(\alpha) \} =: D_A(1 + \alpha).$$

Let $\lambda \in \rho(A)$. We then obtain $\lambda \in \rho(A_{\alpha,p})$ and $\lambda \in \rho(A_{\alpha})$, with $R(\lambda, A_{\alpha,p}) = R(\lambda, A)|D_A(\alpha, p)$ and $R(\lambda, A_{\alpha})|D_A(\alpha)$. These resolvents have norm less or equal $\|R(\lambda, A)|B(X)\|$. Of course, the restrictions are still semigroups. Let $x \in D(A)$. Proposition 2.1 yields that

$$\|T(t)x - x\|_{A} \leq c \|T(t)x - x\|_{A} \rightarrow 0$$

as $t \rightarrow 0$. Since the restrictions are locally bounded, $T(\cdot)$ is strongly continuous on $D_A(\alpha)$ and, due to the density proved in Proposition 2.1, also on $D_A(\alpha, p)$ if $p < \infty$.

Let $B$ be the generator of $T_{\alpha,p}(\cdot)$ and let $A_{\alpha,p}$ be defined as in the statement. Let $x \in D(B) \subseteq D_A(\alpha, p)$. Then $\frac{1}{t}(T(t)x - x)$ converges to $Bx$ in $D_A(\alpha, p)$, as $t \rightarrow 0$. Since $D_A(\alpha, p) \hookrightarrow X$, it also converges to $Bx$ in $X$. This means that $x \in D(A)$ and $Ax = Bx \in D_A(\alpha, p)$, and hence $B \subseteq A_{\alpha,p}$. Let $\lambda \in \rho(A)$. We show $\lambda \in \rho(A_{\alpha,p})$, implying that $\rho(B)$ and $\rho(A_{\alpha,p})$ both contain a left half plane and, hence, $B = A_{\alpha,p}$. Let $x \in D_A(\alpha, p)$. Then $AR(\lambda, A)x = \lambda R(\lambda, A)x - x$ also belongs to $D_A(\alpha, p)$, so that $R(\lambda, A)x \in D_A(\alpha + 1, p)$ and

$$\lambda I - A_{\alpha,p})R(\lambda, A)x = (\lambda I - A)R(\lambda, A)x = x$$

due to the definition of $A_{\alpha,p}$. For $x \in D_A(\alpha + 1, p)$, we also have

$$R(\lambda, A)(\lambda I - A_{\alpha,p})x = R(\lambda, A)(\lambda I - A)x = x.$$

This means that $\lambda \in \rho(A_{\alpha,p})$ and $R(\lambda, A_{\alpha,p}) = R(\lambda, A)|D_A(\alpha, p)$. The estimate for $R(\lambda, A_{\alpha,p})$ is then shown as for $T_{\alpha,p}(t)$. The remaining results for $D_A(\alpha)$ are proved in the same way.

We note that it even holds $\sigma(A_{\alpha,p}) = \sigma(A_{\alpha}) = \sigma(A)$ due to Proposition IV.2.17 in [EN].

**Example 2.4.** We continue with Example 2.2 taking only $p = q \in [1, \infty)$ for simplicity. Let $X = L^p(\mathbb{R})$, $A = d/ds$ and $D(A) = W^1_p(\mathbb{R})$. One defines

$$W^{1+\alpha}_p(\mathbb{R}) := \{ u \in W^\alpha_p(\mathbb{R}) \mid u' \in W^\alpha_p(\mathbb{R}) \} = D_A(1 + \alpha, p).$$
Investing more theory, one can show that $W^{1+\alpha}_p(\mathbb{R}) = D\mathcal{A}^\alpha(\frac{1}{2} + \frac{\alpha}{p}, p)$, see Example 5.14 in [Lu2]. Observe that $A^2$ is the one-dimensional Laplacian. One defines the Slobodetskii space on $\mathbb{R}^d$ as

$$W^\alpha_p(\mathbb{R}^d) = \left\{ u \in L^p(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(y) - u(x)|^p}{|y-x|^{\alpha p + d}} \, dx \, dy < \infty \right\},$$

$$W^{1+\alpha}_p(\mathbb{R}^d) = \{ u \in W^\alpha_p(\mathbb{R}^d) \mid \nabla u \in W^\alpha_p(\mathbb{R}^d) \}.$$

Let $p \in (1, \infty)$ and $D(\Delta) = W^2_p(\mathbb{R}^d)$. With some more effort it can be shown

$$W^\alpha_p(\mathbb{R}^d) = D_\Delta(\frac{\alpha}{p}, p) \quad \text{and} \quad W^{1+\alpha}_p(\mathbb{R}^d) = D_\Delta(\frac{1}{2} + \frac{\alpha}{p}, p).$$

See Examples 5.16 and 5.15 in [Lu2], where also the cases $p = 1, \infty$ are treated. When comparing with the examples in [Lu2], one has to use Proposition 5.7 in this book.

One can characterize the interpolation spaces in terms of the resolvent of $A$. In the proof we need the following facts. The multiplicative group $(0, \infty)$ possesses the invariant measure $dt/t$, i.e.,

$$\int_0^\infty f(\lambda t) \frac{dt}{t} = \int_0^\infty f(s) \frac{ds}{s}$$

holds for every positive measurable function $f$ and each $r > 0$, due to the substitution $s = \lambda t$. Similarly, one obtains

$$\int_0^\infty f(1/t) \frac{dt}{t} = \int_0^\infty f(s) \frac{ds}{s}.$$

Exactly as for the additive group $\mathbb{R}$ and the Lebesgue measure one can now prove Young’s convolution estimates for the convolution integral

$$h(\lambda) = \int_0^\infty f(\lambda t) g(t) \frac{dt}{t}$$

in the spaces $L^p(\mathbb{R}_+)$. Recall that $\|R(t, A)\| \leq M(\lambda - \omega)^{-1}$ for all $\lambda > \omega$ if $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and some constants $M \geq 1$ and $\omega \in \mathbb{R}$.

**Proposition 2.5.** Let $A$ generate the $C_0$-semigroup $T(t)$ on $X$, $\alpha \in (0, 1)$ and $p \in [1, \infty]$. Fix some $\omega \geq 0$ with $\omega > \omega_0(A)$. We set $\chi_{\alpha,x}(\lambda) = \lambda^\alpha \|AR(\lambda, A)x\|$ for $\lambda \geq \omega$ and $x \in X$. Then the norms $\|x\|_{\alpha,p}$ and $\|x\| + \|\chi_{\alpha,x}\|_{L^p((\omega, \infty))}$ are equivalent. Moreover, $x$ belongs to $D_A(\alpha)$ if and only if $\chi_{\alpha,x}(\lambda) \to 0$ as $\lambda \to \infty$.

**Proof.** If we change $\omega \in (\omega_0(A), \infty) \cap \mathbb{R}_+$, then the norm defined by $\chi_{\alpha,x}$ varies to an equivalent one, since $|\chi_{\alpha,x}(\lambda)| \leq c(a, b) \|x\|$ for $\lambda \in [a, b] \subseteq (\omega_0(A), \infty)$. Next, we check that also a rescaling of $A$ changes this norm to an equivalent one. Let $r \in \mathbb{R}$, $\lambda \geq \max\{\omega, r, 1\}$ and $x \in X$. We then have

$$\|((\lambda - r)^\alpha(A - rI)R(\lambda - r, A - rI)x\|

\leq (\lambda - r)^\alpha \lambda^{-\alpha} \chi_{\alpha,x}(\lambda) + (\lambda - r)^\alpha \|rR(\lambda, A)x\|

\leq c (\|x\| + \chi_{\alpha,x}(\lambda)),

\chi_{\alpha,x}(\lambda) \leq \lambda^\alpha \|\lambda^{-\alpha} (((\lambda - r)^\alpha(A - rI)R(\lambda - r, A - rI)x\| + \lambda^\alpha \|rR(\lambda, A)x\|

\leq c (\|((\lambda - r)^\alpha(A - rI)R(\lambda - r, A - rI)x\| + \|x\|),$$

14
Moreover, we can thus assume that \( \omega \in \omega \). We can now use Young’s convolution inequality to deduce that

\[
AR(\lambda, A)x = \lambda R(\lambda, A)x - x = \int_0^\infty \lambda e^{-\lambda t} t^{-\alpha} (T(t)x - x) dt,
\]

and Proposition 2.1(e) implies that

\[
\|\varphi_{\alpha,x}\|_{L^p_{\omega}(\mathbb{R}^+_\lambda)} \leq \|\varphi_{\alpha,x}\|_{L^p_{\omega}(\mathbb{R}^+_\lambda)} \int_0^\infty \lambda s^{\alpha-1} e^{-s} \varphi_{\alpha,x}(s/\lambda) ds.
\]  

(2.5)

In view of the discussion above, we can now use Young’s convolution inequality to deduce that

\[
\|\chi_{\alpha,x}\|_{L^p_{\omega}(\mathbb{R}^+_\lambda)} \leq \|\varphi_{\alpha,x}\|_{L^p_{\omega}(\mathbb{R}^+_\lambda)} \int_0^\infty s^{\alpha-1} e^{-s} \frac{ds}{s} = \Gamma(1 + \alpha) \|\varphi_{\alpha,x}\|_{L^p_{\omega}(\mathbb{R}^+_\lambda)}
\]

with the classical Gamma function. If \( x \in D_A(\alpha) \), then \( \varphi_{\alpha,x}(s/\lambda) \leq \|x\|_{\omega, \alpha, \infty} \) and Proposition 2.1(e) implies that \( \varphi_{\alpha,x}(s/\lambda) \to 0 \) as \( \lambda \to \infty \), for all \( s > 0 \). By means of dominated convergence we then deduce from (2.5) that \( \chi_{\alpha,x}(\lambda) \to 0 \).

For the converses, let \( \chi_{\alpha,x} \in L^p_{\omega}(\mathbb{R}^+_\lambda) \) and \( t > 0 \). We write \( x = t^{-1} R(t^{-1}, A)x - AR(t^{-1}, A)x =: x_1 - x_2 \) and estimate

\[
\|T(t)x_1 - x_1\| \leq \int_0^t \|T(s)Ax_1\| ds \leq tM t^{-1} \chi_{\alpha,x}(1/t),
\]

\[
\|T(t)x_2 - x_2\| \leq (M + 1)t^{\alpha} \chi_{\alpha,x}(1/t).
\]

It follows that \( \varphi_{\alpha,x}(t) \leq (1 + 2M) \chi_{\alpha,x}(1/t) \) and, hence, \( \|\varphi_{\alpha,x}\|_{L^p_{\omega}(\mathbb{R}^+_\lambda)} \leq (1 + 2M) \|\chi_{\alpha,x}\|_{L^p_{\omega}(\mathbb{R}^+_\lambda)} \). Using also the estimates (2.2), the first assertion is shown. Proposition 2.1(e) then yields the second claim.

The next proposition determines the interpolation spaces of an analytic semigroup in terms of its time derivative \( \frac{d}{dt} T(t) = AT(t) \). This result will be crucial for our later applications to parabolic problems. We point out that for \( p \in (1, \infty) \) it implies the equivalence

\[
x \in D_A(1 - \frac{1}{p}, p) \iff AT(\cdot)x \in L^p((0, 1], X).
\]

Proposition 2.6. Let \( A \) generate the analytic \( C_0 \)-semigroup \( T(\cdot) \) on \( X \), \( \alpha \in (0, 1) \), and \( p \in [1, \infty] \). For \( x \in X \), we set \( \psi_{\alpha,x}(t) = t^{1-\alpha} \|AT(t)x\| \) for \( t \in (0, 1] \). Then the norms \( \|x\|_{\omega,p} \) and \( \|x\| + \|\psi_{\alpha,x}\|_{L^p_{\omega}(0,1]} \) are equivalent. Moreover, \( x \) belongs to \( D_A(\alpha) \) if and only if \( \psi_{\alpha,x}(t) \to 0 \) as \( t \to 0 \).

Proof. Let \( x \in X \) and \( t \in (0, 1] \). We first estimate

\[
\varphi_{\alpha,x}(t) = \lim_{\varepsilon \to 0} t^{-\alpha} \left\| \int_\varepsilon^t AT(s)x ds \right\| \leq \limsup_{\varepsilon \to 0} t^{-\alpha} \int_\varepsilon^t s^{\alpha-1} s^{1-\alpha} \|AT(s)x\| ds \leq t^{-\alpha} \int_0^t s^{\alpha-1} s^{1-\alpha} \|AT(s)x\| ds \leq \frac{1}{\alpha} \sup_{0<s\leq t} \psi_{\alpha,x}(s).
\]

15
This inequality yields the first half of the equivalence for \( p = \infty \) and of the last assertion (using also Proposition 2.1(e)). For \( p < \infty \), we obtain similarly
\[
[x]_{\alpha,p}^p \leq \int_0^1 t^{-\alpha p} \left( \int_0^t s \| AT(s)x \| \frac{ds}{s} \right)^p \frac{dt}{t}
\leq \alpha^{-p} \int_0^1 s^{-\alpha p}s^p \| AT(s)x \| p \frac{ds}{s} = \alpha^{-p} \| \psi_{\alpha,x} \|_{L^p}^p,
\]
where the second estimate is Hardy’s inequality, see e.g. (1.2.16) in [Lu1].

For the converse, we put \( M_1 := \sup_{0 < t \leq 1} \| tAT(t) \| \). We compute
\[
t^{1-\alpha} AT(t)x = t^{-\alpha} T(t)(T(t)x - x) - t^{-\alpha} AT(t) \int_0^t (T(s)x - x) \, ds,
\]
\[
\psi_{\alpha,x}(t) \leq M_0 \varphi_{\alpha,x}(t) + M_1 t^{-1-\alpha} \int_0^t s^{\alpha} \varphi_{\alpha,x}(s) \, ds.
\]
For the case \( p = \infty \), we further estimate
\[
\psi_{\alpha,x}(t) \leq M_0 \varphi_{\alpha,x}(t) + \frac{M_1}{1 + \alpha} \sup_{s \in (0,t]} \varphi_{\alpha,x}(s),
\]
and deduce the asserted equivalence for \( p = \infty \) and the final assertion. For \( p < \infty \), we write
\[
\psi_{\alpha,x}(t) \leq M_0 \varphi_{\alpha,x}(t) + M_1 t^{-\alpha} \int_0^t s^{\alpha} \varphi_{\alpha,x}(s) \frac{ds}{s},
\]
and use again Hardy’s inequality to finish the proof. \( \square \)

The next theorem describes the most important property of the spaces \( D_A(\alpha,p) \) and \( D_A(\alpha) \). For an operator \( T \in \mathcal{B}(X,Y) \) mapping a subspace \( X_0 \subseteq X \) into a subspace \( Y_0 \subseteq Y \), we denote the restriction of \( T \) acting from \( X_0 \) to \( Y_0 \) by the same symbol.

**Theorem 2.7 (Interpolation property).** Assume that \( A \) and \( B \) generate \( C_0 \)-semigroups \( T(\cdot) \) and \( S(\cdot) \) on Banach spaces \( X \) and \( Y \), respectively, and that the operator \( T \in \mathcal{B}(X,Y) \) satisfies \( TD(A) \subseteq D(B) \) and \( T \in \mathcal{B}([D(A)], [D(B)]) \). Let \( 0 < \alpha < 1 \) and \( 1 \leq p \leq \infty \). Then \( T \) maps \( D_A(\alpha,p) \) into \( D_B(\alpha,p) \) and \( D_A(\alpha) \) into \( D_B(\alpha) \), we have \( T \in \mathcal{B}(D_A(\alpha,p), D_B(\alpha,p)) \) and \( T \in \mathcal{B}(D_A(\alpha), D_B(\alpha)) \), and it holds
\[
\| T \|_{\mathcal{B}(D_A(\alpha,p), D_B(\alpha,p))} \leq c \| T \|_{[D(A), [D(B)]]} \| T \|_{B([D(A)], [D(B)])}^{\alpha},
\]
for a constant not depending on \( T \).

**Proof.** In view of (2.4), after rescaling if necessary, we may assume that the semigroups are exponentially stable so that \( A \) and \( B \) are invertible. Let \( x \in X, t \in (0,1] \) and \( s > 0 \). We define the so called \( k \)-function by
\[
k(s,x) = \inf \{ \| x_1 \|_X + s \| x_2 \|_A | x = x_1 + x_2, x_1 \in X, x_2 \in D(A) \}.
\] (2.6)
Since the result is trivially true for \( T = 0 \), we may assume that \( T \neq 0 \). We set \( \| T \|_0 = \| T \|_{\mathcal{B}(X,Y)}, \| T \|_1 = \| T \|_{\mathcal{B}(D_A(\alpha,p), D_B(\alpha))}, \text{ and } N_0 = \sup_{0 < t \leq 1} \| S(t) \| \). Let
$x = x_1 + x_2$ for some $x_1 \in X$ and $x_2 \in D(A)$. Employing $Tx_2 \in D(B)$, we estimate
\[
\|S(t)T x - T x\|_Y \leq \|S(t)T x_1 - T x_1\|_Y + \|S(t)T x_2 - T x_2\|_Y
\]
\[
\leq (N_0 + 1) \|T x_1\|_Y + \int_0^t \|S(s)BT x_2\|_Y \, ds
\]
\[
\leq (N_0 + 1) \|T x_1\|_Y + tN_0 \|T x_2\|_B
\]
\[
\leq (N_0 + 1) \|T\|_0 \|\|x_1\|_X + t \|T\|_1 \|T\|_0^{-1} \|x_2\|_A\).
\]
Taking the infimum as in (2.6), we deduce
\[
\varphi_{\alpha, T x}(t) \leq (N_0 + 1) \|T\|_0 t^{-\alpha} k(t \|T\|_1 \|T\|_0^{-1}, x),
\]
where we use here and below an obvious notation. Choosing $x_1 = AR(1/s, A)x$ and $x_2 = s^{-1}R(1/s, A)x$, we can dominate the $k$-functional by
\[
k(s, x) \leq \|AR(1/s, A)x\|_X + s \|s^{-1}R(1/s, A)x\|_A \leq (2 + \|A^{-1}\|) \|AR(1/s, A)x\|_X
\]
For $p < \infty$, the invariance of the measure $dt/t$ then implies
\[
[Tx]_{\alpha, p}^B \leq (N_0 + 1) \|T\|_0 \left(\int_0^\infty t^{-\alpha p} k(t \|T\|_1 \|T\|_0^{-1}, x) \frac{dt}{t}\right)^{\frac{1}{p}}
\]
\[
= (N_0 + 1) \|T\|_0 \left(\int_0^\infty \frac{ds}{s} \|T\|_0^{-\alpha p} k(s, x) \right)^{\frac{1}{p}}
\]
\[
\leq (N_0 + 1) (2 + \|A^{-1}\|) \|T\|_0^{1-\alpha} \|T\|_1^\alpha \left(\int_0^\infty \frac{d\lambda}{\lambda} \|AR(\lambda, A)x\|_X \right)^{\frac{1}{p}}
\]
\[
\leq (N_0 + 1) (2 + \|A^{-1}\|) \|T\|_0^{1-\alpha} \|T\|_1^\alpha \|\chi_{\alpha, p}\|_{L^p(\mathbb{R}_+)}.
\]
Proposition 2.5 then yields the assertion for $p < \infty$. The case $p = \infty$ can be handled in a similar, but simpler way. The remaining result then follows from $TD_A(\alpha) = TD(A) \subseteq TD(B) \subseteq D(B) = D_B(\alpha)$ with closures in the $(\alpha, \infty)$ norms, using that $T$ is continuous in these norms.

**Corollary 2.8.** Let $A$ generate the $C_0$-semigroup $T(\cdot)$ on $X$, $x \in D(A)$, $\alpha \in (0,1)$ and $p \in [1, \infty]$. We then have
\[
\|x\|_{\alpha, p} \leq c \|x\|_X^{1-\alpha} \|x\|_A^\alpha
\]
for a constant $c > 0$ independent of $x$.

**Proof.** For $x \in D(A)$, we consider the operator $T_{x} : \mathbb{C} \to D(A)$ given by $T_{x} \mu = \mu x$. On $\mathbb{C}$ we choose the semigroup $R(\cdot) = I$ generated by the 0 operator with domain $\mathbb{C}$. Observe that $T_{x}$ has the norms $\|x\|_X$ in $B(\mathbb{C}, X)$, $\|x\|_{\alpha, p}$ in $B(\mathbb{C}, D_A(\alpha, p))$, and $\|x\|_A$ in $B(\mathbb{C}, [D(A)])$. Hence, the corollary follows from the above theorem.

**Remark 2.9.** (a) We point out that the interpolation estimate in Corollary 2.8 does not imply the interpolation property expressed by Theorem 2.7.
Let $B \in \mathcal{B}(D_A(\alpha, p), X)$ for some $\alpha \in [0, 1)$ and $p \in [1, \infty]$. Using Corollary 2.8 and Young’s inequality, for all $a > 0$ we find a number $b = b(a) \geq 0$ such that

$$\|Bx\| \leq c \|x\|_{\alpha,p} \leq a \|Ax\| + b \|x\|$$

holds for all $x \in D(A)$. This estimate allows to apply the perturbation theorems for analytic or dissipative semigroups, see Section 3.1 in [EE].

(b) For any subspace $X_1 \subseteq X$ one can define the $k$–functional as in (2.6). Setting $\kappa_{\alpha,x}(t) = t^{-\alpha}k(t, x)$ for $t > 0$, one then introduces the real interpolation space

$$(X, X_1)_{\alpha,p} = \{x \in X \mid \kappa_{\alpha,x} \in L_p^p\}$$

endowed with the norm $\|x\| + \|\kappa_{\alpha,x}\|_{L_p}$. The proof of Theorem 2.7 implies that this space coincides with our real interpolation space if $X_1$ is the domain of a generator, see also Proposition 5.7 in [Lu2]. This observation tells us that the real interpolation spaces do not depend on the generator itself, but only on the Banach spaces $X$ and $[D(A)]$. One can replace the assumption $X_1 \subseteq X$ by the condition that $X_1$ and $X$ are embedded into the same normed vector space $Z$ and define the spaces $(X, Z)_{\alpha,p}$ in a similar way, see [BL], [Lu2] or [Tr].

Any spaces $E$ and $F$ satisfying the conclusion of Theorem 2.7 are called interpolation spaces (of order $\alpha$) between $X$ and $[D(A)]$ and between $Y$ and $[D(B)]$, respectively. Another important class of such spaces are the so called complex interpolation spaces $(X, X_1)_\alpha$ of order $\alpha \in (0, 1)$. It can be shown that $[L^p(\mu), L^q(\mu)]_\alpha = L^r(\mu)$ for $r = (1 - \alpha)p + \alpha q$ and $1 \leq p, q \leq \infty$, see e.g. Example 2.11 in [Lu2]. In this case the assertion of Corollary 2.8 is one of the standard consequences of Hölder’s inequality. The real interpolation spaces between $L^p(\mu)$ and $L^q(\mu)$ are the ‘Lorentz spaces’, see Example 1.27 in [Lu2].

In the next example we see a typical application of the interpolation property to the theory of function spaces.

**Example 2.10.** Let $U \subseteq \mathbb{R}^d$ be a bounded open subset with $\partial U \in C^2$ and take $\alpha \in (0, 1)$ with $\alpha \neq \frac{1}{2}$. Let $p \in (1, \infty)$. On $L^p(U)$ we consider the generator $A = \Delta$ with $D(A) = W^{2}_p(U) \cap W^{1}_p(U)$, and on $L^p(\mathbb{R}^d)$ the generator $B = \Delta$ with $D(B) = W^{2}_p(\mathbb{R}^d)$. There is an (extension) operator $E \in B(L^p(U), L^p(\mathbb{R}^d)) \cap B(W^{2}_p(U), W^{2}_p(\mathbb{R}^d))$ such that $Ef = f$ on $U$ for all $f \in L^p(U)$, see Theorem 3.23 in [ST]. Observing that $E \in B([D(A)], [D(B)])$, we deduce from Theorem 2.7 that $E \in B(D_A(\alpha, p), D_B(\alpha, p))$. Example 5.15 in [Lu2] yields $D_B(\alpha, p) = W^{2\alpha}_p(\mathbb{R}^d)$, see also Example 2.4 above. Using the restriction operator $Rg = g|U$ on $L^p(\mathbb{R}^d)$, we thus obtain the embedding

$$RE : D_A(\alpha, p) \hookrightarrow W^{2\alpha}_p(U) := \{u \in L^p(U) \mid \exists v \in W^{2\alpha}_p(\mathbb{R}^d) : v|U = u\}.$$ 

By the same reasoning, we have $D_G(\alpha, p) \hookrightarrow W^{2\alpha}_p(U)$ for any generator $G$ on $L^p(\mathbb{R}^d)$ such that $D(G) \subseteq W^{2}_p(U)$ and the graph norm of $G$ is equivalent to the norm of $W^{2\alpha}_p(U)$.

In the above definition, the norm of $u$ in $W^{2\alpha}_p(U)$ is given by the infimum of $\|v\|_{W^{2\alpha}_p(\mathbb{R}^d)}$ with $v|U = u$. However, $W^{2\alpha}_p(U)$ also possesses an equivalent norm $\|u\|_{W^{2\alpha}_p(U)} = \left(\int (\int |\nabla^\alpha u|^2)^\frac{p}{2}dx\right)^\frac{1}{p}$, see [EE] for details.

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2Actually, it suffices that $Z$ is a ‘Hausdorff topological vector space’.
‘intrinsic’ norm of the same type as those in Example 2.4, see Theorem 4.4.2.2 in [Tr]. Actually, one can show much more, namely

\[
D_A(\alpha, p) = \begin{cases} 
W_p^{2\alpha}(U), & 0 < \alpha < \frac{1}{2}, \\
\{u \in B_{pp}^1(U) \mid \text{tr} u = 0\}, & \alpha = \frac{1}{2}, \\
\{u \in W_p^{2\alpha}(U) \mid \text{tr} u = 0\}, & \frac{1}{2} < \alpha < 1,
\end{cases}
\]

see Theorem 4.3.3 in [Tr]. Here, the (integer) Besov space \(B_{pp}^1(U)\) (≠ \(W_p^1(U)\)) is described in Theorem 4.4.2.2 of [Tr].

Finally, we state a result on compact embeddings needed in the next chapter. The proof requires more facts from interpolation theory not presented here.

**Proposition 2.11.** Let \(A\) generate the \(C_0\)-semigroup \(T(\cdot)\) on \(X\). For \(\alpha, \beta \in (0, 1)\) we consider \(X_\alpha \in \{D_A(\alpha, p), D_A(\alpha) \mid p \in [1, \infty]\}\) and \(X_\beta \in \{D_A(\beta, q), D_A(\beta) \mid q \in [1, \infty]\}\). Assume that the embedding \(X_1 := [D(A)] \hookrightarrow X \Rightarrow X_0\) is compact. Then also the embeddings \(X_\beta \hookrightarrow X_\alpha\) are compact for all \(0 \leq \alpha < \beta \leq 1\).

**Proof.** For \(\alpha \in (0, 1)\) and the spaces \(D_A(\cdot, p)\) the assertion is shown in Corollary 3.8.2 of [BL]. For the remaining cases we take \(0 \leq \alpha < \alpha' < \beta' < \beta \leq 1\). Proposition 2.1 yields \(X_\beta \hookrightarrow D_A(\beta', 1) \hookrightarrow D_A(\alpha', 1) \hookrightarrow X_\alpha\). Since the embedding in the middle is compact, the embedding \(X_\beta \hookrightarrow X_\alpha\) is compact as the product of compact and continuous operators.

\(\Box\)

2. Regularity of analytic semigroups

In this section we treat basic regularity properties of parabolic evolution equations complementing the results established in Theorem 2.20 of [EE]. The term ‘parabolic’ means that we assume that \(A\) generates the analytic \(C_0\)-semigroup \(T(\cdot)\) on \(X\) and it is motivated by the applications to diffusion equations. These examples will be discussed in the next chapter.

Throughout this section, we let \(J \in \{(0, b), [0, b] \mid b > 0\}\) and study the homogeneous problem

\[
u'(t) = Au(t) + f(t), \quad t \in J, \quad u(0) = x,
\]

for given \(x \in X\) and \(f \in C([0, b], X)\). A (classical) solution of \((2.7)\) is a function \(u \in C([0, b], X) \cap C^1(J, X)\) such that \(u'(t) \in D(A)\) for all \(t \in J\) and \((2.7)\) holds.

If a solution of \((2.7)\) exists, it is uniquely given by the mild solution

\[
u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds =: T(t)x + v(t), \quad t \in [0, b],
\]

see Proposition 2.6 in [EE]. Lemma 2.8 in [EE] further says that the mild solution \(v\) is a classical one on \(J\) if and only if \(v \in C^1(J, X)\) if and only if \(v(t) \in D(A)\) for all \(t \in J\) and \(Av \in C(J, X)\). Recall that \(T(\cdot)x\) solves the problem \((2.7)\) with \(f = 0\) on \((0, \infty)\), and on \(\mathbb{R}_+\) if \(x \in D(A)\).

For convenience, we write \(X_\alpha\) for any of the spaces \(D_A(\alpha, p)\) or \(D_A(\alpha)\) with \(\alpha \in (0, 1)\) and \(p \in [1, \infty]\). We further set \(X_k = [D(A^k)]\) for \(k \in \mathbb{N}\) and \(X_0 := X\), writing \(\|x\|_{\alpha, p} := \|x\|_{X_\alpha}\) for \(\alpha \in [0, 1]\). In Proposition 2.3 we have seen that the ‘parts’ \(A_{\alpha, p}\) and \(A_\alpha\) of \(A\) in \(D_A(\alpha, p)\) and \(D_A(\alpha)\) generate \(C_0\)-semigroups in
these spaces, respectively, which are the restrictions of \( T(\cdot) \) to the respective space. For simplicity, we now use the symbols \( A \) and \( T(t) \) for all these objects.

We first describe the regularizing effect of these semigroups in the scale of interpolation spaces, if we start with an analytic semigroup on \( X \).

**Proposition 2.12.** Let \( A \) generate the analytic \( C_0 \)-semigroup \( T(\cdot) \) on \( X \), \( \alpha \in [0,1] \) and \( b > 0 \). Then the following assertions hold.

(a) The restrictions of \( T(\cdot) \) to \( X_{\alpha} \) are also analytic \( C_0 \)-semigroups, denoted by the same symbol.

(b) \( \|A^kT(t)x\|_{\alpha} \lesssim_b t^{\beta - \alpha - k} \|x\|_{\beta} \) for all \( x \in X_\beta \), \( \beta \in [0,\alpha] \), \( k = 0, 1 \), and \( t \in (0, b] \). If \( T(\cdot) \) is bounded, the estimate is independent of \( b > 0 \).

(c) \( T(\cdot)x \) belongs to \( C^{1-\alpha}([\varepsilon, b], X_{\alpha}) \) if \( x \in X_1 \) and \( T(\cdot)x \in C^{1-\alpha}(\varepsilon, b], X_{\alpha}) \) for any \( \varepsilon \in (0, b) \) if \( x \in X \).

**Proof.** (a) The analyticity of the semigroup on \( X_{\alpha} \) can be deduced from that on \( X \) using that \( AT(t) = T(t)A \) on \( X_1 \) for \( \alpha = 1 \) and the resolvent estimate in Proposition 2.3 combined with the generation theorem for analytic semigroups for \( \alpha \in (0,1) \).

(b) The proof of the estimate in (b) is done in a few steps, where \( x \in X \) and \( t \in (0, b] \). We note that the constants below do not depend on \( b > 0 \) if \( T(\cdot) \) is bounded, so that the last part of (b) also follows.

1) For \( \beta = 0 \), Corollary 2.8 and the analyticity of \( T(\cdot) \) on \( X \) yield

\[
\|A^kT(t)x\|_{\alpha} \lesssim \|T(t/2)A^kT(t/2)x\|_{0} \lesssim_k \|T(t/2)A^kT(t/2)x\|_{0}^{1-\alpha}
\lesssim_b (t/2)^{-\alpha} \|A^kT(t/2)x\|_{0} \lesssim t^{-\alpha-k} \|x\|_0.
\]

2) Let \( \beta > 0 \). If \( \alpha = \beta \), the assertion follows from the analyticity of \( T(\cdot) \) on \( X_{\alpha} \). So, let \( \alpha > \beta \). We first take \( k = 1 \) and \( t \in (0,1] \). In this case, part 1) and Propositions 2.6 and 2.1 imply that

\[
\|tAT(t)x\|_{\alpha} \lesssim \|T(t/2)\|_{\mathcal{B}(X_{\alpha})} \cdot \|T(t/2)\|^{1-\beta} \|AT(t/2)x\|_0
\lesssim t^{\beta-\alpha} \|x\|_{\beta, \infty} \lesssim t^{\beta-\alpha} \|x\|_\beta.
\]

For \( k = 0 \) and \( t \in (0,1] \), we deduce from Step 1) and Proposition 2.1 that

\[
\|T(t)x\|_{\alpha} = \left\langle T(1)x - \int_0^1 AT(s)x \, ds \right\rangle_{\alpha} \lesssim \|x\| + \int_t^1 s^{1-\alpha - 1} \|x\|_{\beta} \, ds \lesssim t^{\beta-\alpha} \|x\|_\beta.
\]

If \( b > 1 \), we estimate

\[
\|A^kT(t)x\|_{\alpha} \lesssim_b t^{-\alpha-k} \|x\|_0 \lesssim t^{-\alpha-k} \|x\|_\beta
\]

for \( t \in [1, b] \) by means of Step 1) and Proposition 2.1.

(c) Let \( 0 \leq s < t \leq b \) and \( x \in X_1 \). Assertion (b) then yields

\[
\|T(t)x - T(s)x\|_{\alpha} = \|((T(t-s) - I)T(s)x\|_{\alpha} \lesssim_b \int_0^{t-s} \|T(r+s)A\|_{\alpha} \, dr
\lesssim_b \int_0^{t-s} r^{-\alpha} \, dr \|x\|_A \lesssim_b (t-s)^{1-\alpha} \|x\|_1.
\]

The last claim then follows from \( T(t)x - T(s)x = (T(t-s) - T(s-s))T(\varepsilon)x \) and \( T(\varepsilon)x \in D(A) \). \( \square \)
By induction, one can define the scale \( X_\alpha \) to all \( \alpha \geq 0 \) and extend the above result to this setting, see e.g. Proposition 2.2.9 in [Lu1]. As observed after (2.8), the solvability of (2.7) is determined by the regularity of \( v \) given by (2.8). This topic is treated next, where we denote by \( B(M, Z) \) the space of bounded functions from a set \( M \) to a Banach space \( Z \), endowed with the supnorm \( \|f\|_{\infty, Z} \).

**Theorem 2.13.** Let \( A \) generate the analytic \( C_0 \)-semigroup \( T(\cdot) \) on \( X \), \( f \in C([0, b], X) \) and \( v \) be given by (2.8). Then the following assertions hold.

(a) \( v \in C^{1-\alpha}([0, b], X_\beta) \) if either \( \alpha \in (0, 1) \) and \( \beta \in [0, \alpha] \) or if \( \alpha = 1 \) and \( \beta \in [0, 1) \).

(b) If \( f \in C^\alpha([0, b], X) \) or if \( f \in B([0, b], X_\alpha) \) for some \( \alpha \in (0, 1) \), then \( v \) solves (2.7) on \( [0, b] \) with \( x = 0 \).

**Proof.** (a) In view of Proposition 2.1, it suffices to show that \( \|v\|_{\alpha,b} \leq \|f\|_{\infty, X} \) and 
\[
\|v(t) - v(s)\|_{\alpha,b} \leq b \lim_{\varepsilon \to 0} \int_s^{s-\varepsilon} (T(t - \tau) - T(s - \tau)) f(\tau) \, d\tau + \int_s^t T(t - \tau) f(\tau) \, d\tau
\]
\[
= \lim_{\varepsilon \to 0} \int_s^{s-\varepsilon} \int_{s-\tau}^{t-\tau} A T(\sigma) f(\tau) \, d\sigma \, d\tau + \int_s^t T(t - \tau) f(\tau) \, d\tau,
\]
\[
\|v(t) - v(s)\|_{\alpha,b} \leq b \lim_{\varepsilon \to 0} \int_s^{s-\varepsilon} \int_{s-\tau}^{t-\tau} \sigma^{-\alpha - 1} \|f\|_{\infty, X} \, d\sigma \, d\tau + \int_s^t (t - \tau)^{-\alpha} \|f\|_{\infty, X} \, d\tau
\]
\[
\leq \left( \frac{1}{\alpha} \int_0^1 ((s - \tau)^{-\alpha} - (t - \tau)^{-\alpha}) \, d\tau + \frac{1}{1 - \alpha} (t - s)^{1-\alpha} \right)\|f\|_{\infty, X}
\]
\[
\leq \frac{1}{\alpha(1 - \alpha)} (t - s)^{1-\alpha} \|f\|_{\infty, X}.
\]

(b) The first part was shown in Theorem 2.20 of [EE]. Let \( f \in B([0, b], X_\alpha) \) for \( \alpha \in (0, 1) \) and \( 0 < s \leq t \leq b \). Due to Proposition 2.12, the function \( \varphi(s) = \|AT(t-s)f(s)\|_{\infty, X_\alpha} \leq \|f\|_{\infty, X_\alpha} \) is integrable on \([0, t]\). Since \( A \) is closed, we deduce that \( v(t) \in D(A) \). As in Step (a), we then obtain
\[
\|Av(t)\| \leq \int_0^t \|T(t-s)f(s)\|_{1,b} \, ds \leq \int_0^t (t-s)^{\alpha-1} \|f\|_{\infty, X_\alpha} \, ds = \frac{t^\alpha}{\alpha} \|f\|_{\infty, X_\alpha},
\]
\[
\|Av(t) - Av(s)\| \leq \lim_{\varepsilon \to 0} \int_s^{s-\varepsilon} \int_{s-\tau}^{t-\tau} \|AT(\sigma)f(\tau)\|_1 \, d\sigma \, d\tau + \int_s^t \|T(t-\tau)f(\tau)\|_1 \, d\tau
\]
\[
\leq b \int_s^t \int_{s-\tau}^{t-\tau} f(\tau) \, d\sigma \, d\tau + \int_s^t (t-\tau)^{\alpha-1} \|f(\tau)\|_{\alpha,b} \, d\tau
\]
\[
\leq \left( \frac{1}{1 - \alpha} \int_0^1 (s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1} \, d\tau + \frac{(t-s)^\alpha}{\alpha} \right) \|f\|_{\infty, X_\alpha}.
\]
\begin{equation}
\leq \frac{1}{\alpha(1-\alpha)} (t-s)^\alpha \|f\|_{\infty,X_\alpha}.
\end{equation}
As a result, \( Av \in C([0,b],X) \) since \( v(0) = 0 \).

We note that in part (b) of the above proof we have even shown that \( Av \in C^\alpha([0,b],X) \). Actually, it can proved that the terms \( u' \) and \( Au \) have the same regularity as the given function \( f \), if we work in \( D_\alpha(\alpha,\infty) \) and assume that \( f(0) \in D_\alpha(\alpha,\infty) \) in the case \( f \in C^\alpha \). This property of 'maximal regularity of type \( C^\alpha \)' is shown in Theorem 4.3.1 and Corollary 4.3.9 of [Lu1]. In Chapter 4 of this monograph many variants and refinements of the maximal regularity results and of Theorem 2.13(a) are established. Here one uses in the time variable spaces like \( C \), \( C^\alpha \) or \( C^1 \). One can obtain analogous results in the framework of \( L^p \), \( W_\alpha^p \) and \( W^1_p \) by similar methods, see [DB].
CHAPTER 3

Semilinear parabolic problems

Semilinear parabolic problems are evolution equations involving a generator of an analytic semigroup and a nonlinearity of ‘lower order’. They can be handled in a way rather similar to ordinary differential equations. Typical examples are reaction diffusion systems. In this chapter we always work in the following setting.

Standing hypothesis. Let $A$ generate the analytic $C_0$–semigroup $T(\cdot)$ on $X$, $\alpha \in [0, 1)$, $X_\alpha \in \{D_A(\alpha, p), D_A(\alpha) \mid \alpha \in (0, 1), p \in [1, \infty)\}$. We set $X_0 = X$, $X_1 = [D(A)]$, $\parallel \cdot \parallel = \parallel \cdot \parallel_{X_\beta}$, $B_\beta(r) = B_{X_\beta}(0, r)$, and $B_\beta(x, r) = B_{X_\beta}(x, r)$ for $\beta \in [0, 1]$, $x \in X_\beta$ and $r \geq 0$. Let $F : X_\alpha \to X$ be locally Lipschitz, i.e., for each $r > 0$ there is an $L(r) \geq 0$ such that $\|F(x) - F(y)\|_0 \leq L(r) \|x - y\|_\alpha$ for all $x, y \in \overline{B}_\alpha(r)$ and some $\alpha \in [0, 1)$. Without loss of generality we assume that $r \mapsto L(r)$ is nondecreasing.

1. Wellposedness

Let $u_0 \in X_\alpha$ and $J$ be an interval with $J^0 \neq \emptyset$ and $\inf J = 0$. We set $\bar{J} = J \cup \{0\}$. We investigate the semilinear parabolic problem

$$u'(t) = Au(t) + F(u(t)), \quad t \in J, \quad u(0) = x \quad (3.1)$$

A function $u$ is called a (classical) solution of (3.1) on $J$ if $u$ belongs to $C(\bar{J}, X_\alpha) \cap C^1(J, X) \cap C(J, X_1)$ and satisfies (3.1). In this case, $f = F(u) : \bar{J} \to X$ is continuous so that Proposition 2.6 of [EE] says that $u$ is a mild solution of (3.1), i.e., a function $u \in C(\bar{J}, X_\alpha)$ fulfilling the fixed point problem

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s)) \, ds, \quad t \in \bar{J}. \quad (3.2)$$

This ‘mild equation’ is time invariant in the following sense. Let $u$ be a mild solution and $0 < t_0 \in J$. Equation (3.2) and the substitution $\sigma = s - t_0$ yield

$$u(t) = T(t-t_0)(T(t_0)u_0 + \int_0^{t_0} T(t_0-s)F(u(s)) \, ds) + \int_0^{t-t_0} T(t-s)F(u(s)) \, ds$$

$$= T(t-t_0)u(t_0) + \int_0^{t-t_0} T(t-t_0-\sigma)F(u(\sigma + t_0)) \, d\sigma$$

for $t \in J$ with $t \geq t_0$. Setting $\tau = t - t_0$, we obtain

$$u(\tau + t_0) = T(\tau)u(t_0) + \int_0^{\tau} T(\tau-s)F(u(s + t_0)) \, ds \quad (3.3)$$

for all $\tau \in \mathbb{R}_+ \cap (J - t_0)$. Therefore, the translated function $v = u(\cdot + t_0)$ solves (3.2) on the shifted time interval with the initial value $u(t_0)$.
In the same way one sees that one can 'glue' matching mild solutions of (3.1): If \( u \) is a mild solution on \([0, t_0]\) with \( u(0) = u_0 \) and \( v \) is a mild solution on \([0, t_1]\) with \( v(0) = u(t_0) \), then the function

\[
w(t) = \begin{cases} u(t), & 0 \leq t \leq t_0, \\ v(t - t_0), & t_0 \leq t \leq t_0 + t_1, \end{cases}
\]

is a mild solution on \([0, t_0 + t_1]\) with \( w(0) = u_0 \). (Compare Step 2) in the proof of Theorem 5.1 in [EE].

We next solve the fixed point problem (3.2) on a possibly small time interval by means of the contraction mapping principle. This approach is of course inspired by the usual arguments in the case of ordinary differential equations, which also guides us in many other respects (e.g., when defining and controlling maximal existence times). However, our reasoning crucially depends on the (linear) regularity theory established in the previous section.

**Proposition 3.1.** Let \( A \) generate the analytic \( C_0\)-semigroup \( T(\cdot) \) on \( X \), \( \alpha \in [0, 1) \), and \( F : X_\alpha \to X \) be locally Lipschitz. Then, for each radius \( \rho > 0 \) there is a time \( b_0(\rho) > 0 \) such that for every \( u_0 \in \overline{B}_\alpha(\rho) \) there is a unique mild solution \( u = u(\cdot ; u_0) \) of (3.1) on \([0, b_0(\rho)]\). It solves (3.1) in the classical sense on \([0, b_0(\rho)]\), and on \([0, b_0(\rho)]\) if \( u_0 \in D(A) \). The mapping \( u_0 \mapsto u(\cdot ; u_0) \) from \( \overline{B}_\alpha(\rho) \) to \( C([0, b_0(\rho)], X_\alpha) \) is Lipschitz.

**Proof.** 1) (Existence) The numbers

\[ M_0 = \sup_{t \in [0,1]} \| T(t) \|_{\mathcal{L}(X_\alpha)} \geq 1 \quad \text{and} \quad M_1 = \sup_{t \in [0,1]} \| t^\alpha T(t) \|_{\mathcal{L}(X,X_\alpha)} \]

are finite due to Proposition 2.12. For each given \( \rho > 0 \) we define \( r = M_0\rho + 1 \). Let \( b \in (0,1] \). We then introduce the space

\[ E(b) = \{ u \in C([0,b], X_\alpha) | \| u \|_\infty \leq \sup_{0 \leq t \leq b} \| u(t) \|_\alpha \leq r \}. \]

Note that \( E(b) \) is complete for the metric \( d(u,v) = \| u - v \|_\infty \). Let \( u_0 \in \overline{B}_\alpha(\rho) \) be the initial value. For \( u,v \in E(b) \) and \( t \in [0,b] \), we define

\[
[\Phi u_0](t)(t) = [\Phi(u)](t) = T(t)u_0 + \int_0^t T(t-s)F(u(s)) \, ds.
\]

Clearly, a fixed point \( \Phi(u) = u \) is a mild solution of (3.1). We will obtain such a fixed point for sufficiently small \( b > 0 \).

Since \( F \circ u \in C([0,b], X) \) and \( u_0 \in X_\alpha \), Proposition 2.12 and Theorem 2.13 imply that \( \Phi(u) \in C([0,b], X_\alpha) \) and

\[
\| \Phi(u)(t) \|_\alpha \leq M_0 \| u_0 \|_\alpha + M_1 \int_0^t (t-s)^{-\alpha} \| F(u(s)) - F(0) \|_0 \, ds
\]

\[
\leq M_0\rho + M_1 \int_0^t (t-s)^{-\alpha} (L_r) \| u(s) - 0 \|_\alpha + \| F(0) \|_0 \, ds
\]

\[
\leq M_0\rho + \frac{M_1}{1-\alpha} (rL_r + \| F(0) \|_0) b^{1-\alpha} \leq r,
\]

24
if we choose $b \in (0, 1]$ such that

$$b \leq b_1(\rho) := \left(\frac{1 - \alpha}{M_1(rL(r) + \|F(0)\|_0)}\right)^{\frac{1}{1 - \alpha}}.$$  

In the same way we estimate

$$\|\Phi(u)(t) - \Phi(v)(t)\|_\alpha \leq M_1 \int_0^t (t - s)^{-\alpha} \|F(u(s)) - F(v(s))\|_0 \, ds$$

$$\leq M_1 \int_0^t (t - s)^{-\alpha} L(r) \|u(s) - v(s)\|_\alpha \, ds$$

$$\leq M_1 L(r) \frac{1}{1 - \alpha} b^{1-\alpha} \|u - v\|_\infty \leq \frac{1}{2} \|u - v\|_\infty,$$

if we choose $b \in (0, 1]$ such that

$$b \leq b_2(\rho) := \left(\frac{1 - \alpha}{2M_1 L(r)}\right)^{\frac{1}{1 - \alpha}}.$$  

As a result, for every $0 < b \leq b_0(\rho) := \min\{1, b_1(\rho), b_2(\rho)\}$ the map $\Phi : E(b) \to E(b)$ is a strict contraction, and thus there is a unique mild solution $u = u(\cdot; u_0) = \Phi(u)$ of (3.1) on $[0, b]$ which belongs to $E(b)$.

2) (Uniqueness) Let $v$ be a mild solution of (3.1) on an interval $[0, \beta]$ with $\beta > 0$. If $u \neq v$ on $[0, \min\{\beta, b_0(\rho)\}] =: [0, a]$, then there would exist $t_0 \in [0, a)$ and $t_n \in (t_0, a]$ such that $u(t_0) = v(t_0)$, $u(t_n) \neq v(t_n)$ for all $n \in \mathbb{N}$, and $t_n \to t_0$ as $n \to \infty$. Due to (3.3), both functions $u(\cdot + t_0)$ and $v(\cdot + t_0)$ are mild solutions of (3.1) with initial value $u(t_0)$. Step 1) of the proof yields a unique mild solution of (3.1) with initial value $u(t_0)$ in the space $E(b)$ for all $b \in (0, b_0([\|u(t_0)\|_\alpha])]$, where $E(b)$ is defined with $r' := 1 + M_0 \|u(t_0)\|_\alpha$ instead of $r$. Due to $\|u(t_0)\|_\alpha < r'$ and the continuity of $u$ and $v$ in $X$, we find a time $b' \in (0, b_0([\|u(t_0)\|_\alpha]) \cap [0, a - t_0]$ such that $u(\cdot + t_0)$ and $v(\cdot + t_0)$ belong to $E(b')$. The two functions thus coincide in $[0, b']$ which contradicts the properties of $t_n$, so that $u = v$ on $[0, a]$.

3) (Lipschitz continuity) Let $u_0, v_0 \in \mathcal{B}_\alpha(\rho)$ for some $\rho > 0$ and put $u = u(\cdot ; u_0)$ and $v = u(\cdot; v_0)$. Step 1) with $b = b_0(\rho)$ then yields

$$\|u - v\|_\infty \leq \|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_\infty + \|\Phi_{v_0}(u) - \Phi_{v_0}(v)\|_\infty$$

$$\leq \frac{1}{2} \|u - v\|_\infty + M_0 \|u_0 - v_0\|_\alpha.$$  

It follows that $\|u - v\|_\infty \leq 2M_0 \|u_0 - v_0\|_\alpha$, as required.

4) (Regularity) Proposition 2.12 and Theorem 2.13 show that $u$ belongs to $C^{1-\alpha}([\varepsilon, b_0(\rho)], X_\alpha)$ for any $\varepsilon \in (0, b_0(\rho))$ and $\varepsilon = 0$ if $x \in D(A)$, where we may assume that $\alpha \in (0, 1)$. Thus, $F \circ u \in C^{1-\alpha}([\varepsilon, b_0(\rho)], X)$ for such $\varepsilon$. If $x \in D(A)$, we deduce that $u$ is a classical solution on $[0, b_0(\rho)]$ from Proposition 2.12 and Theorem 2.13. Otherwise, we note that $u$ is a mild solution of (3.1) on $[\varepsilon, b_0(\rho)]$ with initial value $u(\varepsilon)$ at initial time $\varepsilon$. Theorem 2.13 and the properties of analytic semigroups then imply that $u$ is a classical solution on $(\varepsilon, b_0(\rho))$, so that it is one on $(0, b_0(\rho))$. \qed
Observe that \( b_0(\rho) \) decreases in \( \rho \) by the above proof. In view of Proposition 3.1, we can define the \textit{maximal existence time} for each \( u_0 \in X_\alpha \) by setting
\[
b(u_0) = \sup\{b \geq b_0(\|u_0\|_\alpha) \mid \exists \text{ a mild solution of (3.1) on } [0, b]\}.
\]
It holds \( b(u_0) \in (b_0(\|u_0\|_\alpha), \infty] \) since we can restart the problem at time \( b = b_0(\|u_0\|_\alpha) \) with initial value \( u(b) \in X_\alpha \). Well known one dimensional examples such as \( u' = u^2 \) show that the case \( b(u_0) < \infty \) can easily occur.

**Theorem 3.2.** Let \( A \) generate the analytic \( C_0 \)-semigroup \( T(\cdot) \) on \( X, \alpha \in [0, 1), u_0 \in X_\alpha \) and \( F : X_\alpha \to X \) be locally Lipschitz. Then the following assertions hold.

(a) The problem (3.1) has a unique mild solution \( u = u(\cdot; u_0) \) on \([0, b(u_0))\). It solves (3.1) in the classical sense on \([0, b(u_0))\), and on \([0, b(u_0))\) if \( u_0 \in D(A) \).

(b) Let \( b \in (0, b(u_0)) \). Then there is a radius \( r = r(b) \) such that for each \( v_0 \in \overline{B}_\alpha(u_0, r) \) we have \( b(v_0) > b \) and the map \( v_0 \mapsto u(\cdot; v_0) \) from \( \overline{B}_\alpha(u_0, r) \) to \( C([0, b], X_\alpha) \) is Lipschitz.

(c) If \( b(u_0) < \infty \), then \( \limsup_{t \to b(u_0)^{-}} \|u(t)\|_\alpha = \infty \).

**Proof.** (a) The uniqueness and the regularity of the solution on \([0, b(u_0))\) are shown in Proposition 3.1.

(b) Let \( b \in (0, b(u_0)) \) and \( C := \sup_{t \in [0, b]} \|u(t)\|_\alpha < \infty \). We iterate the procedure of Step 1) of the proof of Proposition 3.1 with initial times \( b_0 = 0, b_{j+1} = b_j + b_0(\rho_j) \), radii \( \rho_j = \|u(b_j)\|_\alpha + 1 \) and initial values \( u(b_j) \). Observe that \( b_0(\rho_j) \geq b_0(C + 1) \) for all \( j \) since \( b_0 \) a decreasing function. We denote by \( n \) the minimal number of steps such that \( b_0 \geq b \). Let \( v_0 \in \overline{B}_\alpha(\rho_0) \) and \( v = u(\cdot; v_0) \).

Step 4) of the previous proof yields that \( \|v(t) - u(t)\|_\alpha \leq 2M_0 \|v_0 - u_0\|_\alpha \) for all \( t \in [0, b_1] \). If \( \|u_0 - v_0\|_\alpha \leq (2M_0)^{-1} \), we thus conclude
\[
\|v(b_1)\|_\alpha \leq \|v(b_1) - u(b_1)\|_\alpha + \|u(b_1)\|_\alpha \leq 1 + \|u(b_1)\|_\alpha = \rho_1.
\]

Using the mild solution starting at \( v(b_1) \), we can extend the solution \( v \) in time. Inductively, one concludes that the estimate \( \|u_0 - v_0\|_\alpha \leq (2M_0)^{-n} \) implies that \( \|v(b_j)\|_\alpha \leq \rho_j \) for all \( j \in \{0, 1, \ldots, n\} \), \( b(v_0) \geq b_0 \geq b \), and \( \|v(t) - u(t)\|_\alpha \leq (2M_0)^n \|u_0 - v_0\|_\alpha \) for all \( t \in [0, b] \).

(c) Suppose that \( b(u_0) < \infty \) and \( \|u(t)\|_\alpha \leq C \) for all \( t \in [0, b(u_0)) \). As observed in the previous step, for each \( t \in [0, b(u_0)) \) the solution \( u \) can be extended to \([t, t + b_0(C + 1)]\). This fact contradicts the definition of \( b(u_0) \).

Observe that the assumptions still hold if we replace \( \alpha \) by \( \beta \in (\alpha, 1) \). Due to the uniqueness and the embeddings in Proposition 2.1, the solution \( u \) thus belongs to \( C([0, b(u_0)), X_\gamma) \) for all \( \gamma \in (0, 1) \). We stress that \( X_\alpha \) is the adequate norm to describe the behavior of the solutions to (3.1): They are continuous in the \( X_\alpha \)-norm up to 0 and this norm gives the Lipschitz continuity and blow up condition in assertions (b) and (c).

The regularity of mild and classical solutions to (3.1) is studied in great detail in Chapter 7 of [Lu1], also for \( u_0 \in X \) under additional restrictions on \( F \). The above proofs completely break down if \( \alpha = 1 \), i.e., when the nonlinearity has the same order as the linear part. Under certain additional assumptions, one
can also develop a theory on wellposedness and asymptotic behavior for such problems, which is similar to the semilinear case discussed here. This is done in Chapters 8 and 9 of [Lu1] based on the results on maximal regularity mentioned at the end of the previous chapter; see also the remarks in the next examples.

Since the data of equation (3.1) are not known exactly in applications, it is very important to know that the solution depends continuously on the system operators, where we restrict ourselves to $F$ for simplicity. When discussing the positivity of reaction diffusion systems we will actually use a very special case of this continuous dependance (whose proof would not be much simpler). For this and other purposes, we need the singular Gronwall inequality. Let $J$ also contain $0, 0 \leq \varphi \in C(J), \alpha \in [0,1)$ and $a, b \geq 0$. Assume that

$$\varphi(t) \leq a + b \int_0^t (t-s)^{-\alpha} \varphi(s) \, ds$$

holds for all $t \in J$. Then there is a constant $c_0 > 0$ such that

$$\varphi(t) \leq a + ab_0 t^{1-\alpha} e^{c(\alpha)b/t(1-\alpha)t}$$

(3.4)

holds for all $t \in J$, where $c(\alpha) := 2(\Gamma(1-\alpha))^{1/\alpha}$, see Theorem II.3.3.1 in [Am].

**Proposition 3.3.** Let $A$ generate the analytic $C_0$-semigroup $T(\cdot)$ on $X$, $\alpha \in [0,1)$, $u_0, v_0 \in X_\alpha$ and $F, G : X_\alpha \to X$ be locally Lipschitz. Let $u$ solve (3.1) and $v$ solve (3.1) with nonlinearity $G$ and initial value $v_0$. Fix $b \in (0, b(u_0))$. Then there are constants $\delta = \delta(b) > 0$, $\rho = \rho(b) > 0$ and $c = c(b) > 0$ such that if $\|u_0 - v_0\|_\alpha \leq \rho$ and $\|F(u(t)) - G(u(t))\|_0 \leq \delta$ for all $t \in [0, b]$, then the solution $v$ exists on $[0, b]$ and $\|u(t) - v(t)\|_\alpha \leq c(\delta + \|u_0 - v_0\|_\alpha)$ for all $t \in [0, b]$.

**Proof.** Let $r > 0$ and $b \in (0, b(u_0))$ be given, $L$ be the Lipschitz constant of $G$ on the bounded set $\bigcup\{B_\alpha(u(t), r) \mid 0 \leq t \leq b\}$. Take $\rho \in (0, r)$ and $v_0 \in B_\alpha(u_0, \rho)$. We set $M_0 = \sup_{t \in [0, b]} \|T(t)\|_{B(X_\alpha)}$ and $M_1 = \sup_{t \in [0, b]} \|L(\cdot) T(t)\|_{B(X_\alpha)}$ which are finite by Proposition 2.12. Let $b^*$ be the supremum of all $\beta \in [0, b]$ such that $v(t)$ exists and $\|v(t) - u(t)\|_\alpha \leq r$ for all $t \in [0, \beta]$. Using the mild formulation of both evolution equations, we obtain

$$u(t) - v(t) = T(t)(u_0 - v_0) + \int_0^t T(t-s)(F(u(s)) - G(u(s))) \, ds$$

$$+ \int_0^t T(t-s)(G(u(s)) - G(v(s))) \, ds,$$

$$\|u(t) - v(t)\|_\alpha \leq M_0 \|u_0 - v_0\|_\alpha + M_1 \delta \int_0^t (t-s)^{-\alpha} \, ds$$

$$+ M_1 L \int_0^t (t-s)^{-\alpha} \|u(s) - v(s)\|_\alpha \, ds$$

$$\leq M_0 \|u_0 - v_0\|_\alpha + \frac{\delta M_1 b^{1-\alpha}}{1-\alpha} + M_1 L \int_0^t (t-s)^{-\alpha} \|u(s) - v(s)\|_\alpha \, ds$$

for all $t \in [0, b^*)$. The inequality (3.4) thus yields

$$\|u(t) - v(t)\|_\alpha \leq c(\delta + \|u_0 - v_0\|_\alpha) \leq c(\delta + \rho),$$

(3.5)
for all \( t \in [0,b^*) \), where \( c \) does not depend on \( t \), \( \delta \) or \( \rho \). Fixing sufficiently small \( \delta \) and \( \rho \), we conclude that \( \|u(t) - v(t)\|_{\alpha} \leq r/2 \) for all \( t \in [0,b^*) \), implying that \( b^* = b \) and the estimate (3.5) holds for all \( t \in [0,b] \). \( \square \)

In the next example we give an introduction to the \( L^p \)-approach to reaction diffusion system, whereas in Example 5.5 of [EE] we discussed a single reaction diffusion equation in a supnorm setting.

**Example 3.4. Reaction Diffusion Systems.** We first recall reaction equations without diffusion. As a simple example, we consider the chemical reaction \( A + 2B \rightleftharpoons C \), where one mol of the substance \( A \) reacts with 2 mols of \( B \) to one mol of the product \( C \), which in turn can decompose into one mol of \( A \) and two mols of \( B \). Let \( a(t) \), \( b(t) \) and \( c(t) \) be the concentrations at time \( t \geq 0 \) of the species \( A \), \( B \) and \( C \), respectively. Roughly speaking, the two reactions take place with a ‘probability’ proportional to the products of \( a(t) \) \( b(t) \) \( b(t) \) and \( c(t) \) of the concentrations, where we denote the proportionality constants by \( k_+ \) and \( k_- \), respectively. Each concentration then increases and decreases according to the two reactions, where the rate is given by the ‘probability’ times the number of mols needed of the respective substance. We arrive at the system

\[
\begin{align*}
a'(t) &= -k_+ a(t)b(t)^2 + k_- c(t), \quad t \geq 0, \\
b'(t) &= -2k_+ a(t)b(t)^2 + 2k_- c(t), \quad t \geq 0, \\
c'(t) &= k_+ a(t)b(t)^2 - k_- c(t), \quad t \geq 0, \\
a(0) &= a_0, \quad b(0) = b_0, \quad c(0) = c_0,
\end{align*}
\]

with positive initial concentrations \( a_0 \), \( b_0 \) and \( c_0 \). We write \( f(a(t), b(t), c(t)) \) for the vector of the reaction terms on the right hand side. This problem has unique local positive solutions due to the Picard–Lindelöf theorem and the positivity criterion. Since \( a' + c' = 0 \) and \( b' + 2c' = 0 \), we have \( a(t) + c(t) = a_0 + c_0 \) and \( b(t) + 2c(t) = b_0 + 2c_0 \) as long as the solutions exist. Thanks to the positivity, the solutions thus stay bounded on their existence interval, so that they exist for all \( t \in \mathbb{R}^+ \). These facts hold in much greater generality, see e.g. Sections 31 and 32 of [PSZ].

In a reaction–diffusion system one takes into account that the concentrations of the species may differ at different points of the container \( U \) which is an open and bounded subset of \( \mathbb{R}^d \) with \( \partial U \subset C^2 \) and outer unit normal \( \nu \). For given \( m \) species we thus consider concentration densities \( u(t,x) = (u_1(t,x), \ldots, u_m(t,x)) \) at every time \( t \geq 0 \) and spatial point \( x \in U \). We assume that at each \( x \) a reaction–convection term \( f(u(t,x), \nabla u(t,x)) \) acts. Later we will focus on pure reaction terms \( f(u(t,x)) \) depending only on the concentrations \( u(t,x) \) as in the ode above. If spatial gradients of \( u(t,x) \) are involved, we also have (possibly nonlinear) convective effects. The function \( f : \mathbb{C}^{m+d_m} \to \mathbb{C}^m \) (or later \( f : \mathbb{C}^m \to \mathbb{C}^m \)) is given and assumed to be locally Lipschitz. We further assume that species move in the container due to ‘homogeneous’ and ‘isotropic’ diffusion with constants \( a_1, \ldots, a_m > 0 \), resulting in diffusion terms \( a_j \Delta u_j(t,x) \).

We assume that the species do not move through the boundary \( \partial U \). It can be seen that this behaviour is described by the Neumann boundary condition \( \partial_u u_j(t,x) = 0 \) saying that in normal direction at the boundary the concentration
are locally Lipschitz. Take does not change. Summing up, we arrive at the system

\[ \partial_t u_j(t, x) = a_j \Delta u_j(t, x) + f_j(u(t, x), \nabla u(t, x)), \quad t > 0, \ x \in U, \ j \in \{1, \ldots, m\}, \]

\[ \partial_v u_j(t, x) = 0, \quad t > 0, \ x \in \partial U, \ j \in \{1, \ldots, m\}, \]  

\[ u_j(0, x) = u_{j,0}(x), \quad x \in U, \ j \in \{1, \ldots, m\}, \]  

for given initial distributions \( u_{j,0} \geq 0 \). Recall from Example 1.3, that the Neumann Laplacian \( \Delta_N \) with domain \( D(\Delta_N) = \{ v \in W^2_p(U) \mid \partial_v v = 0 \} \) generates a contractive, positive, analytic \( C_0 \)-semigroup \( S(\cdot) \) on \( L^p(U) \) for \( p \in (1, \infty) \). We now set \( E = L^p(U)^m \), \( 0 \leq u_0 = (u_{1,0}, \ldots, u_{m,0}) \in E \),

\[ A = \begin{pmatrix} a_1 \Delta_N & 0 & 0 & \cdots & 0 \\ 0 & a_2 \Delta_N & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_m \Delta_N \end{pmatrix}, \quad D(A) = D(\Delta_m)^m =: E_1, \]  

and \( [F(v)](x) = f(v(x), \nabla v(x)) \) for \( v \in W^1_p(U)^m \) and \( x \in U \). Positivity in \( E \) means that each component of \( u \in E \) is positive a.e.. It is easy to see that \( A \) generates the contractive analytic \( C_0 \)-semigroup

\[ T(t) = \begin{pmatrix} S(a_1 t) & 0 & 0 & \cdots & 0 \\ 0 & S(a_2 t) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & S(a_m t) \end{pmatrix}, \quad t \geq 0, \]

on \( E \). We write \( E_\alpha = D_A(\alpha, p) \) for \( \alpha \in (0, 1) \). Due to Proposition 2.12 the numbers \( M_0 = \sup_{t \geq 0} \| T(t) \|_{B(E_0)} \) and \( M_1 = \sup_{t \geq 0} \| t \|_{B(E_0)} \) are finite.

One could also treat more complicated diffusion phenomena. In the linear case, heterogeneous and anisotropic diffusion is described by the term \( \text{div}(a \nabla u_j) \) for a coefficient function \( a \) on \( U \) taking values in the symmetric positive definite matrices, which could even depend on time. If the diffusion coefficients \( a = a(u) \) also depend on the solution \( u \) itself, one has ‘quasilinear diffusion’ which can be treated by more sophisticated methods. In such problems, interactions between species can lead to nondiagonal diffusion terms.

We look for a framework in which \( F \) becomes locally Lipschitz. We first let \( v, w \in C^1(\overline{U})^m \) with \( C^1 \)-norm less or equal \( r \). Denoting by \( L(r) \) the Lipschitz constant of \( f \) on the maxnorm ball of radius \( r \), we estimate

\[
|F(v)(x) - F(w)(x)| \leq L(r) \max\{|v(x) - w(x)|_\infty, |
\nabla v(x) - \nabla w(x)|_\infty\} 
\]

\[
\leq L(r) \|v - w\|_{C^1}.
\]

This means that \( F : C^1(\overline{U})^m \to C(\overline{U})^m \), and thus \( F : C^1(\overline{U})^m \to L^p(U)^m \), are locally Lipschitz. Take \( p > d \) and \( \alpha \in \left( \frac{1}{2} + \frac{d}{2p}, 1 \right) \), so that \( 2\alpha - \frac{d}{p} > 1 \). Example 2.10 and the fractional Sobolev embedding theorem (see Theorem 4.6.1 in [Tr]) then imply that

\[ E_\alpha \hookrightarrow W^{2\alpha}_p(U)^m \hookrightarrow C^1(\overline{U})^m. \]  

(3.8)
As a result, \( F : E_\alpha \to E \) is locally Lipschitz. If we have a pure reaction term \( f(u) \), then by the same arguments we see that \( F : E_\alpha \to E \) is locally Lipschitz already if \( p > d/2 \) and \( \alpha \in (\frac{d}{2p}, 1) \).

We can thus write \((3.6)\) as the semilinear parabolic problem \((3.1)\) on \( E \), and apply Theorem 3.2. For \( u_0 \in E_\alpha \) it yields unique solutions \( u \in C([0, b(u_0)), E_\alpha) \cap C^1((0, b(u_0)), E) \) of \((3.1)\) such that \( u(t) \in D(\Delta_N)^m = D(A) \) for all \( t \in (0, b(u_0)) \). By the embedding \((3.8)\), we have \( u \in C([0, b(u_0)), C^1(U)^m) \). Such a solution satisfies the first line of \((3.6)\) for a.e. \( x \in U \) and the second line for all \( x \in U \). Observe that \( u \) is also the only solution of \((3.6)\) in this sense. Hence, the regularity properties hold for all \( p \in (d, \infty) \) and \( \alpha \in (\frac{1}{2} + \frac{d}{2p}, 1) \). If \( u_0 \in D(A) \), we can include \( t = 0 \) here.

In the following we restrict ourselves to pure reaction equations with \( F(u)(x) = f(u(x)) \) for a function \( f : \mathbb{C}^m \to \mathbb{C}^m \) being locally Lipschitz. The next proposition describes various properties of \((3.6)\) in this case.

**Proposition 3.5.** Let \( f : \mathbb{C}^m \to \mathbb{C}^m \) be locally Lipschitz, \( F(u) = f \circ u \), \( d < p < \infty \) and \( \alpha \in (\frac{d}{2p}, 1 - \frac{d}{2p}) \), \( u_0 \in D_A(\alpha, p) \), and \( A \) be given by \((3.7)\). Then the following assertions hold.

(a) \((3.6)\) has a unique solution \( u \in C([0, b(u_0)), E) \cap C^1((0, b(u_0)), E) \) such that \( u(t) \in D(\Delta_N)^m = D(A) \) for all \( t \in (0, b(u_0)) \). It holds \( u \in C([0, b(u_0)), C^1(U)^m) \). If \( u_0 \in D(A) \), then we can replace the interval \((0, b(u_0))\) by \([0, b(u_0)]\) and we have \( \partial_t u, \Delta u \in C((0, b(u_0)) \times \overline{U})^m \) for all \( j = 1, \ldots, m \). Assertion (b) of Theorem 3.2 holds analogously.

(b) If \( b(u_0) < \infty \), then \( \limsup_{t \to b(u_0) \wedge 0} \| u(t) \| = \infty \). If \( b(u_0) = \infty \) and \( u \) is bounded on \( \mathbb{R}^+ \times \overline{U} \), the orbit \( \{ u(t) | t \geq 0 \} \) is relatively compact in \( E_\alpha \).

(c) Let \( f(\mathbb{R}^m) \subseteq \mathbb{R}^m \) and \( u_0 \geq 0 \). Then \( u \) also takes real values. If also the positivity condition

\[
f_k(r_1, \ldots, r_{k-1}, 0, r_{k+1}, \ldots, r_m) \geq 0 \quad \text{for all} \quad r_j \geq 0, \; j, k \in \{1, \ldots, m\}
\]

holds, then \( u(t) \geq 0 \) for all \( t \in [0, b(u_0)) \).

**Proof.** (a) The first part follows from Theorem 3.2 except for the additional regularity result. To show it, we recall that \( F(u) \in C^{1-\alpha}([0, b], E) \) for any \( b \in (0, b(u_0)) \) due to Proposition 2.12 and Theorem 2.13 combined with \((3.2)\). Corollary 4.3 of [Lu1] then implies that \( \partial_t u = u' : [\varepsilon, b] \to D_A(1-\alpha, \infty) \) is bounded for each \( \varepsilon \in (0, b) \). Let \( 0 < \varepsilon \leq s \leq t \leq b \) and \( \beta \in (\frac{d}{2p}, 1-\alpha) \). The reiteration theorem combined with the interpolation estimate (see Corollaries 1.24 and 1.17 in [Lu2]) yields

\[
\|u'(t) - u'(s)\|_\beta \lesssim \|u'(t) - u'(s)\|_1^{\frac{\beta}{1-\alpha}} \|u'(t) - u'(s)\|_0^{1-\frac{\beta}{1-\alpha}} \lesssim \|u'(t) - u'(s)\|_0^{1-\frac{\beta}{1-\alpha}},
\]

where the right hand side tends to 0 as \( t - s \to 0 \). We deduce that \( \partial_t u \in C([\varepsilon, b], E_\beta) \hookrightarrow C([\varepsilon, b], C(\overline{U})^m) \), using also the fractional Sobolev embedding Theorem 4.6.1 in [Tr], Example 2.10 and \( 2\beta - \frac{p}{d} > 0 \), cf. \((3.8)\). Hence, \( \partial_t u \in C((0, b(u_0)) \times \overline{U})^m \) and the same holds for \( \Delta u \) due to \((3.6)\) and \( F(u) \in C([0, b(u_0)) \times \overline{U})^m \).
(b) Let \( b(u_0) < \infty \) and \( |u(t,x)|_\infty \leq R \) on \([0, b(u_0)) \times \overline{U}\). It follows that
\[
|f(u(t,x))| \leq |f(u(t,x)) - f(0)| + |f(0)| \leq L(R)|u(t,x)| + |f(0)| \leq RL(R) + |f(0)|
\]
for all \((t,x) \in [0, b(u_0)) \times \overline{U}\), and thus
\[
\|F(u(t))\|_E \leq \text{vol}(U) \hat{\Delta} (RL(R) + |f(0)|) =: C.
\]
Employing Proposition 2.12, we can then control \(u\) in the mild formula (3.2) by
\[
\|u(t)\|_\alpha \leq \|T(t)u_0\|_\alpha + \int_0^t \|T(t-s)\|_{\mathcal{B}(E,E,\alpha)} \|F(u(s))\|_0 \, ds
\]
\[
\leq M_0 \|u_0\|_\alpha + CM_1 \int_0^t \|t-s\|^{-\alpha} \, ds = M_0 \|u_0\|_\alpha + \frac{CM_1}{1-\alpha} b(u_0)^{1-\alpha}
\]
for all \(t \in [0, b(u_0))\). The blow-up condition in Theorem 3.2 implies that this cannot be true, and thus \(b(u_0) = \infty\).

Next, let \( b(u_0) = \infty \) and \( u \) be bounded by \( R \) on \( \mathbb{R}^+ \times \overline{U}\). Using the shifted mild formula (3.3), we conclude in a similar way that
\[
\|u(t)\|_\beta \leq \|T(1)u(t-1)\|_\beta + \int_{t-1}^t \|T(t-s)\|_{\mathcal{B}(E,E,\beta)} \|F(u(s))\|_0 \, ds
\]
\[
\leq M_1 \|u(t-1)\|_0 + \frac{CM_1}{1-\beta} \leq CM_1 \text{vol}(U)^{\frac{1}{\beta}} + \frac{CM_1}{1-\beta}
\]
for all \(t \geq 1\) and \(\beta \in (\alpha, 1)\). Proposition 2.11 now shows the second part of (b), since \(E_1\) is compactly embedded into \(E\) and \(\{u(t)|0 \leq t \leq 1\}\) is compact in \(E_\alpha\) by continuity.

(c) First of all we observe that if \(f\) leaves invariant \(\mathbb{R}^n\), one can replace in the framework of (3.6) throughout the scalar field \(\mathbb{C}\) by \(\mathbb{R}\), and thus obtain real valued solutions for real initial functions. (We only use \(\mathbb{C}\) to construct the analytic semigroup \(S(\cdot)\) generated by \(\Delta_N\). But since it is positive, it leaves invariant real valued functions.) For \(0 \leq u_0 \in E_\alpha\) and \(\varepsilon \in (0, \omega_0(A)^{-1})\), we set \(u_{0,\varepsilon} = \varepsilon^{-1}R(\varepsilon^{-1}, A)u_0 + \varepsilon \mathbb{1} \in D(A)\) and \(F_\varepsilon(v) = F(v) + \varepsilon \mathbb{1}\). Note that \(u_{0,\varepsilon} > 0\) since the resolvent is positive. Let \(u_\varepsilon\) solve (3.6) for \(u_{0,\varepsilon}\) and \(F_\varepsilon\). Proposition 2.3 implies that \(\varepsilon^{-1}R(\varepsilon^{-1}, A)\) is the Yosida approximation for \(A\) on \(E_\alpha\), and hence \(u_{0,\varepsilon}\) tends to \(u_0\) in \(E_\alpha\) as \(\varepsilon \to 0\). Fixing an arbitrary \(b \in (0, b(u_0))\), Proposition 3.3 then shows that \(u_\varepsilon(t)\) exists for \(\varepsilon \in (0, \varepsilon_0]\) and some \(\varepsilon_0 = \varepsilon_0(b) > 0\) and that \(u_\varepsilon(t)\) tends to \(u(t)\) in \(E_\alpha\) as \(\varepsilon \to 0\) for all \(t \in [0, b]\). It thus suffices to show that \(u_\varepsilon(t) > 0\) for all \(t \in [0, b]\) and \(\varepsilon \in (0, \varepsilon_0]\).

Suppose that this were not the case. Since \(u_\varepsilon(0) > 0\), there would exist a \(t_0 > 0\), \(x_0 \in \overline{U}\) and \(k \in \{1, \ldots, m\}\) such that \(v(t_0, x_0) = 0\), \(u_\varepsilon(t, x) > 0\) and \(u_\varepsilon(t_0, x) \geq 0\) for all \(t \in (0, t_0)\) and \(x \in \overline{U}\), where we put \(v = (u_\varepsilon)_k\). Hence, \(\partial_t v(t_0, x_0) \leq 0\) and \(v(t_0, x_0) = 0\) is a minimum of the function \(x \mapsto v(t_0, x)\) on \(\overline{U}\). Moreover, the condition (3.9) shows that \(f(u_\varepsilon(t_0, x_0)) \geq 0\). Thus the differential equation in (3.6) implies that
\[
ak \Delta v(t_0, x_0) = \partial_t v(t_0, x_0) - f(u_\varepsilon(t_0, x_0)) - \varepsilon \leq -\varepsilon < 0.
\]
Note that then \(\Delta v(t_0, x) \leq 0\) for \(x \in \overline{U}\) which are close to \(x_0\). If \(x_0 \in U\), Proposition 3.1.10 of [Lul] says that \(\Delta v(t_0, x_0) \geq 0\) which is impossible. We can thus assume that \(x_0 \in \partial U\) and \(v(t_0, x) > 0\) for all \(x \in U\). Since \(\partial U \in C^2\), we
find a ball $B = B(y, \rho) \subseteq U$ such that $x_0 \in \partial B$. Then $\nu(x_0)$ is proportional to $|x_0 - y|$. We apply Hopf’s Lemma 1.2 to the function $w = -v(t_0)$ and obtain that $\partial_n v(t_0, x_0) < 0$, contradicting the boundary condition in (3.6). Consequently, $u_x(t) > 0$ for all $t \in (0, b]$ and $\varepsilon \in (0, \varepsilon_0]$, as needed. (We remark that in this argument we used that $\partial_n v$ and $\Delta v$ are continuous on $(0, b(u_0)) \times \overline{U}$, as shown in assertion (a).)

□

We next show the weak maximum parabolic principle in our regularity framework. This result and slight variants are used below in the examples to obtain sup norm estimates on solutions.

**Proposition 3.6.** Let $U \subseteq \mathbb{R}^d$ be bounded and open with $\partial U \in C^2$, $a > 0$ and let the real valued function $v$ belong to $C([0, T] \times \overline{U}) \cap C^1((0, T], C(\overline{U})) \cap C((0, T], W^2_p(U))$ for all $p \in (1, \infty)$ such that $\Delta v \in C((0, T] \times \overline{U})$. Assume that $\partial_t v - a \Delta v \leq 0$ on $(0, T] \times \overline{U}$ and that $\partial_t v = 0$ on $(0, T] \times \partial U$. It then holds

$$
\max_{(t, x) \in [0, T] \times \overline{U}} v(t, x) \leq \max_{x \in \overline{U}} v(0, x).
$$

**Proof.** We first note that the maxima in the assertion exist. Considering the functions $v_\varepsilon = v - \varepsilon t I$ and letting $\varepsilon \to 0^+$ if needed, we can assume that $\partial_t v - a \Delta v < 0$ on $(0, T] \times \overline{U}$. Suppose there were $\delta > 0$, $t_0 \in (0, T]$, $x_0 \in \overline{U}$ such that $v(t_0, x_0) \geq \delta + \max_{\overline{U}} v(0) =: M(\delta)$. We then define

$$
t_1 = \sup\{t \in (0, T] \mid \max_{x \in \overline{U}} v(s, x) < M(\delta) \text{ for all } s \in [0, t] \}.
$$

Observe that $t_1 \in (0, t_0]$ and that there is an $x_1 \in \overline{U}$ with $v(t_1, x_1) = M(\delta)$. As a result, $\partial_t v(t_1, x_1) \geq 0$ and $v(t_1, x_1)$ is a maximum of $v(t_1, \cdot)$ on $\overline{U}$. As in the proof of Proposition 3.5(c), we then arrive at a contradiction implying the assertion. □

**Example 3.7. Reaction diffusion systems.** We continue to work in the framework of Example 3.4, concentrating now on the question of global existence in a simple situation. We assume that $p > d$, $p > 3/2$ if $d = 1$, that there are $m = 2$ different species denoted by $u$ and $v$ with initial values $0 \leq (u_0, v_0) \in E_{\alpha}$, and that $f(u, v) = (u^k v^l, -u^k v^l)$ for some $k, l \in \mathbb{N}_0$. Then the solution exists on a maximal existence interval $[0, b_0)$ and is positive (since (3.9) holds). We suppose that $b_0 < \infty$.

We first proceed as in the ode system in Example 3.4 and deduce that

$$
\partial_t (u + v) = a_1 \Delta_N (u + v) + \frac{a_2 - a_1}{a_1} a_1 \Delta_N v + u^k v^l - u^k v^l,
$$

$$
u(t) + v(t) = S(a_1 t)(u_0 + v_0) + \frac{a_2 - a_1}{a_1} a_1 \Delta_N \int_0^t S(a_1 (t - s)) v(s) \, ds
$$

for $t \in [0, b_0)$. It holds $a_1 \neq a_2$ since we have two different species, and thus $u + v$ satisfies an equation with an inhomogeneity $g = (a_2 - a_1) \Delta_N v$ which has probably not a fixed sign. In the second equation above we see that the first summand is bounded on $\mathbb{R}_+ \times \overline{U}$ since the semigroup $S(\cdot)$ generated by $\Delta_N$ is bounded on $L^p(U)$, and thus on $D_{\Delta_N}(\alpha, p) \hookrightarrow C(\overline{U})$, cf. (3.8). The second term is defined since $v$ takes values in an interpolation space of $\Delta_N$ (cf.
Proposition 2.12). However, the norm of \(v\) in these spaces could blow up as \(t \to b_0\). Which uniform bounds do we know about \(v\)?

Since \(\partial_t v - a_2 \Delta_N v = -u^b v^d \leq 0\) by (3.6), Propositions 3.6 and 3.5 show that \(\|v\|_\infty \leq \|v_0\|_\infty\) if \((u_0, v_0) \in D(A)\). We can extend this estimate to initial values in \(E_\alpha\) using Yosida approximations as in the proof of Proposition 3.5(c). As a result, we obtain that

\[
\|v\|_{L^q([0,b_0),L^p(U))} \leq b_0^{\frac{1}{2}} \text{vol}(U)^{\frac{1}{2}} \|v_0\|_\infty.
\]

for all \(q \in (1, \infty)\). Unfortunately, the theory presented so far does not allow to use this global bound due to the presence of \(a_1 \Delta_N\) in front of the integral.

Here the deeper theory of \textit{maximal regularity} of type \(L^q\) helps. It says that for certain classes of Banach spaces \(X\) and generators \(B\) of analytic semigroups \(R(\cdot)\) on \(X\) (including \(\Delta_N\) on \(L^r(U)\) for all \(r \in (1, \infty)\)) the function \(w(t) = \int_0^t R(t-s)g(s) \, ds\) takes values in \(D(B)\) for a.e. \(t \geq 0\) and that

\[
\|Bw\|_{L^q([0,b_0),X)} \lesssim_{b,q} \|g\|_{L^r([0,b_0),X)}
\]

for every \(g \in L^q([0,b_0),X)\). (See e.g. Theorem 8.2 in [DHP] or Section 7 in [KW] for this result and these two works for a treatment of this theory.) We recall that \(\|BR(\cdot)x\| \lesssim_{b,q} \|x\|_{1-1/q,q}\) by Proposition 2.6. With \(B = a_1 \Delta_N\) and \(X = L^q(U)\) these facts imply that \(u + v\), and thus \(u\) due to positivity, are bounded in \(L^q([0,b_0),L^q(U))\) for every \(q \in (1, \infty)\). Taking \(q = kp\) and using that \(v\) is uniformly bounded, we deduce that \(u^b v^d \in L^p([0,b_0),L^p(U))\).

We next employ the equation \(\partial_t u = a_1 \Delta_N u + u^b v^d\) and again maximal regularity to conclude that \(\Delta_N u \in L^p([0,b_0),L^p(U))\). The equation then implies that \(\partial_t u \in L^p([0,b_0),L^p(U))\). Finally, we can use the interpolative embedding

\[
L^p([0,b_0),[D(\Delta_N)]) \cap W^{1,1}_p([0,b_0),L^p(U))
\]

\[
\hookrightarrow C_b([0,b_0),D(\Delta_N) (1 - \frac{1}{p},p)) \hookrightarrow C_b([0,b_0),W^{\frac{2}{p}}((0,b_0),C(U)),
\]

see Theorem III.4.10.2 [Am], Example 2.10 and the fractional Sobolev embedding Theorem 4.6.1 in [Tr] with \(2 - \frac{2}{p} > \frac{d}{p}\). (Here we use that \(p > 3/2\) if \(d = 1\).) Summing up, the solution \((u,v)\) is bounded on \([0,b_0) \times U\) so that Proposition 3.5(b) yields \(b_0 = \infty\).

One can show global existence also for (3.6) in the case of reactions \(A + B \equiv C\) with a more refined version of the above \(L^p\) approach. For the seemingly similar case \(A + B \equiv C + D\) global existence is only known for the spatial dimension \(d = 2\). We refer to the survey articles [Pi] and [Pr] for this and related results. The paper [Pr] also develops a theory for quasilinear diffusion equations based on maximal regularity of type \(L^q\).

2. Convergence to equilibria

We first characterize the stationary solutions of (3.1).

**Lemma 3.8.** Let \(A\) generate the analytic \(C_0\)-semigroup \(T(\cdot)\) on \(X\), \(\alpha \in [0,1)\), and \(F : X_\alpha \to X\) be locally Lipschitz. A vector \(u_* \in X\) is an equilibrium of (3.1) if and only if \(u_* \in D(A)\) and \(Au_* = -F(u_*)\).
Proof. If \( u_\ast \in D(A) \) and \( Au_\ast = -F(u_\ast) \), then the function \( u(t) = u_\ast \) clearly solves (3.1) for all \( t \geq 0 \). Conversely, if (3.1) has a stationary solution \( u(t) = u_\ast \) for \( t \geq 0 \), then the mild formula (3.2) yields
\[
\frac{1}{t} (T(t)u_\ast - u_\ast) = -\frac{1}{t} \int_0^t T(t-s)F(u_\ast) \, ds \longrightarrow -F(u_\ast)
\]
as \( t \to 0 \), and hence \( u_\ast \in D(A) \) with \( Au_\ast = -F(u_\ast) \). \( \Box \)

The reaction diffusion system (3.6) possesses the spatially constant equilibrium \( u_\ast = r_\ast 1 \) if there is an \( r_\ast \in \mathbb{R}^d \) such that \( f(r_\ast) = 0 \), i.e., (3.6) inherits the equilibria of the corresponding ode \( y' = f(y) \) (since we have chosen Neumann boundary conditions). The construction of other, spatially heterogeneous equilibria is part of the theory of semilinear elliptic equations. More generally, for every initial value \( u_0 = r_0 1 \) the system (3.6) has the solution \( u(t) = r(t) 1 \) where \( r' = f(r) \) and \( r(0) = r_0 \in \mathbb{R}^m \), so that the reaction ode is contained in the reaction diffusion system.

In the applications one cannot exactly prescribe an equilibrium \( u_\ast \) as an initial value. Thus it is important that small deviations in the initial value lead to small deviations in the solutions for all \( t \geq 0 \). This property is called stability. In this section we treat the slightly different, but closely related topic of convergence to \( u_\ast \). Here are two basic results, both due to Lyapunov in the ode case: a local one using the spectrum of the linearization and a global one using Lyapunov functions. The first result is based on the idea that near an equilibrium \( u_\ast \) the problem \( u'(t) = Au(t) + F(u(t)) \) is very close to the linear one \( w'(t) = Aw(t) + F'(u_\ast)w(t) \).

**Theorem 3.9.** [Principle of linearized stability] Let \( A \) generate the analytic \( C_0 \)-semigroup \( T(\cdot) \) on \( X \), \( \alpha \in [0,1) \), and \( F : X_\alpha \rightarrow X \) be locally Lipschitz. Assume that \( u_\ast \) is an equilibrium of (3.1) such that \( F \) is differentiable at \( u_\ast \) and \( s(A + F'(u_\ast)) < 0 \). Take \( \gamma \in (0,-s(A + F'(u_\ast))) \). Then there are constants \( c, \rho > 0 \) such that for all \( u_0 \in B_\alpha(u_\ast, \rho) \) we have \( b(u_0) = \infty \) and
\[
\|u(t) - u_\ast\|_\alpha \leq ce^{-\gamma t} \|u_0 - u_\ast\|_\alpha
\]
holds for all \( t \geq 0 \), where \( u \) solves (3.1).

Proof. Theorem 3.8 of [EE] and Remark 2.9 imply that \( B := A + F'(u_\ast) \) with domain \( D(A) \) generates an analytic \( C_0 \)-semigroup \( S(\cdot) \) on \( X \). Due to Corollary 4.14 in [EE] and the assumption, we have \( \omega_0(B) = s(B) < -\gamma < 0 \) and hence there are \( M \geq 1 \) and \( \delta' \in (\gamma, -s(B)) \) such that \( \|S(t)\| \leq Me^{-\delta't} \) for all \( t \geq 0 \). We write \( X_0^B \) for the interpolation spaces of \( B \) and take \( \delta \in (\gamma, \delta') \). Proposition 2.3 then yields that \( \|S(t)\|_{B(X_0^B)} \leq Me^{-\delta't} \leq Me^{-\delta t} \) for all \( t \geq 0 \). Moreover, we have \( \|S(t)\|_{B(X,X_0^B)} \leq ct^{-\alpha} \leq c\delta t^{-\alpha}e^{-\delta t} \) for all \( t \in (0,1] \), due to Proposition 2.12. For \( t \geq 1 \), we further deduce \( \|S(t)\|_{B(X,X_0^B)} \leq \|S(1)\|_{B(X,X_0^B)} \|S(t-1)\| \leq \alpha e^{-\delta t} - \alpha c^t e^{-\delta t} \). To transfer these estimates to \( X_\alpha \), we note that \( \|x\|_B \leq c\|x\|_A \) for all \( x \in D(B) = D(A) \) by our assumptions, and thus \( I : [D(A)] \rightarrow [D(B)] \) is an isomorphism due to the open mapping theorem. Interpolation (see Theorem 2.7) then shows that \( I : X_\alpha \rightarrow X_\alpha^B \) is also
isomorphic, resulting in
\[ \| S(t) \|_{\mathcal{B}(\mathcal{X}, \mathcal{X}_a)} \leq M_0 e^{-\delta t} \quad \text{and} \quad \| S(t) \|_{\mathcal{B}(\mathcal{X}, \mathcal{X}_a)} \leq M_1 t^{-\alpha} e^{-\delta t} \]
for all \( t > 0 \) and some constants \( M_0 \geq 1 \) and \( M_1 > 0 \).

Since \( Au_x = -F(u_x) \), the function \( v = u - u_x \) with initial value \( v(0) = u_0 - u_x =: v_0 \) satisfies the equation
\[
v'(t) = u'(t) = Au(t) + F(u(t)) - Au_x - F(u_x)
= Bv(t) + F(u_x + v(t)) - F(u_x) - F'(u_x)(v(t)) =: Bv(t) + G(v(t)) \quad (3.10)
\]
for all \( t \in [0, b(u_0)) \). We can fix an \( \varepsilon > 0 \) such that
\[
c_0 M_1^\alpha t^{1-\alpha} \exp(c(\alpha)M_1^{1-\alpha} \varepsilon^{1-\alpha} t) e^{(\gamma-\delta)t} \leq \frac{1}{2}
\]
holds for all \( t \geq 0 \), where the constants \( c_0 \) and \( c(\alpha) \) are taken from (3.4).

Because \( F \) is differentiable at \( u_x \), there is an \( r > 0 \) such that \( \| G(x) \|_\alpha \leq \varepsilon \| x \|_\alpha \) for all \( x \in \overline{B}_\alpha(r) \). To use this estimate, we have to restrict ourselves to times \( t \geq 0 \) such that \( \| v(t) \|_\alpha \leq r \). We first set \( \rho = (2M_0)^{-1} r \in (0, r) \) and take \( u_0 \in \overline{B}_\alpha(u_x, \rho) \) so that \( \| v_0 \|_\alpha \leq \rho < r \). We then introduce the number
\[
\tau = \sup \{ t \in (0, b(u_0)) \mid \| v(s) \|_\alpha \leq r \quad \text{for all} \quad s \in [0, t] \} \in (0, b(u_0)].
\]
The equation (3.10) and the above estimates thus yield
\[
\| v(t) \|_\alpha \leq \| S(t)v_0 \|_\alpha + \int_0^t \| S(t-s)G(v(s)) \|_\alpha \, ds
\leq M_0 e^{-\delta t} \| v_0 \|_\alpha + \varepsilon M_1 \int_0^t (t-s)^{-\alpha} e^{-\delta(t-s)} \| v(s) \|_\alpha \, ds,
\]
e\delta t \| v(t) \|_\alpha \leq M_0 \| v_0 \|_\alpha + \varepsilon M_1 \int_0^t (t-s)^{-\alpha} e^{\delta s} \| v(s) \|_\alpha \, ds
\]
for all \( t \in [0, \tau) \). The singular Gronwall inequality (3.4) then implies
\[
e^{\delta t} \| v(t) \|_\alpha \leq M_0 \| v_0 \|_\alpha \left(1 + c_0 M_1^\alpha \varepsilon^{1-\alpha} \exp(c(\alpha)M_1^{1-\alpha} \varepsilon^{1-\alpha} t)\right),
\]
\[
\| v(t) \|_\alpha \leq M_0 e^{-\gamma t} \| v_0 \|_\alpha \left(1 + c_0 M_1^\alpha \varepsilon^{1-\alpha} \exp(c(\alpha)M_1^{1-\alpha} \varepsilon^{1-\alpha} t) e^{(\gamma-\delta)t}\right)
\]
\[
\leq \frac{3}{4} M_0 e^{-\gamma t} \| v_0 \|_\alpha \leq \frac{3}{4} r,
\]
for all \( t \in [0, \tau) \), where we also use our choice of \( \varepsilon \) and \( \rho \). If \( \tau < b(u_0) \), we would obtain \( \| v(\tau) \|_\alpha \leq 3r/4 < r \) by continuity, contradicting the definition of \( \tau \). Hence, \( \tau = b(u_0) \) so that \( \| v(t) \|_\alpha \) is bounded on \( [0, b(u_0)) \). Theorem 3.2 then implies \( b(u_0) = \infty \). The assertion now follows from the above estimate.

In view of our terminology in the linear case, one can call the property established above ‘local exponential stability’. We note that refinements of the above estimate give a description of the neighborhood of an equilibrium depending on the spectrum of \( A + F'(u_x) \), see e.g. Chapter 9 of [Lu1].

**Example 3.10. Reaction diffusion systems.** We continue to work in the framework of Example 3.4 with \( m = 2 \), \( d = 3 \), and \( f \in C^1(\mathbb{R}^2, \mathbb{R}^2) \). We
assume that \( f(r_*, s_*) = 0 \) for some \( (r_*, s_*) \in \mathbb{R}_+^2 \), and consider the equilibrium \( (u_*, v_*) = (r_*, s_*)1 \). We write
\[
f'(r_*, s_*) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} =: C.
\]
As in Example 5.5 of [EE], one sees that \( F \in C^1(C(U)^2) \) with derivative given by \( F'(u_*, v_*)(u, v)(x) = C(u(x), v(x)) \). Since \( E_\alpha \to C(U)^2 \) by (3.8) and \( C(U)^2 \to E \), we conclude that \( F \in C^1(E_\alpha, E) \) with the same formula for \( F'(u_*, v_*) \); i.e.,
\[
B(u, v) := [A + F'((u_*, v_*))](u, v) = \begin{pmatrix} a_1 \Delta_N + c_{11} I & c_{12} I \\ c_{21} I & a_2 \Delta_N + c_{22} I \end{pmatrix} (u, v).
\]
If \( s(B) < 0 \), then Theorem 3.9 shows that all solutions starting close to \((u_*, v_*)\) converge to \((u_*, v_*)\) in \( E_\alpha \) exponentially as \( t \to \infty \).

We want to obtain some information about \( s(B) \). We first note that \( D(B) \subseteq W^2_p(U)^2 \) is compactly embedded into \( E = L(U)^2 \) by the Rellich–Kondrachov Theorem 3.28 in [ST]. Thus, \( B \) has pure point spectrum by Remark 2.14 and Theorem 2.16 of [ST]. To use the powerful \( L^2 \)-theory, we consider the restriction \( B_2 \) of \( B \) to \( L^2(U)^2 \) with domain \( D(B_2) = \{(u, v) \in W^2_p(U)^2 | \partial_p u = \partial_p v = 0\} \). As for \( B_2 \), we have \( \sigma(B_2) = \sigma_p(B_2) \). Since \( p > 2 \), it holds \( D(B) \subseteq D(B_2) \) and hence each eigenvector of \( B \) is one of \( B_2 \) for the same eigenvalue, i.e., \( \sigma(B) \subseteq \sigma(B_2) \). Conversely, if \( w = (u_1, u_2) \in D(B_2) \) satisfies \( \lambda w = B_2 w \), then Sobolev’s embedding theorem yields \( w \in C(U)^2 \subseteq L^p(U)^2 \) because of \( 2 - d/p = 2 - 3/2 > 0 \), see Theorem 3.25 of [ST]. Further, \( \partial_p u_j = 0 \) and \( \Delta u_j - u_j = a_j^{-1}(\lambda u_j - c_{1j} u_1 - c_{2j} u_2) - u_j =: \varphi_j \in L^p(U) \) for \( j = 1, 2 \). Since \( \Delta N - I \) is surjective on \( L^p(U) \), there is a function \( v_j \in W^2_p(U) \subseteq L^2(U) \) such that \( \partial_p v_j = 0 \) and \( \Delta v_j - v_j = \varphi_j \). From the injectivity of \( \Delta N - I \) on \( L^2(U) \), it follows that \( u_j = v_j \) for \( j = 1, 2 \), and so \( w \) is an eigenfunction for \( B \) for the eigenvalue \( \lambda \). (We note that the spectrum of restrictions of generators to subspaces is systematically studied in Section IV.2.b of [EN]. See also Lemma 4.4 below.)

Arguing as in Example 4.8 of [ST] for the Dirichlet Laplacian, we see that \( \Delta_N \) is self adjoint on \( L^2(U) \). The spectral theorem in the compact case (see Theorem 4.15 in [ST]) thus yields an orthonormal basis of eigenfunctions \( e_n \) for eigenvalues \( \mu_n \) of \( \Delta_N \). For \( (u, v) \in D(B_2) \), we can now use the orthonormal series \( u = \sum_{n \geq 0} \alpha_n e_n \) and \( v = \sum_{n \geq 0} \beta_n e_n \) with the coefficients \( \alpha_n = (u|e_n) \) and \( \beta_n = (v|e_n) \). It further holds \( \Delta_N u = \sum_{n \geq 0} \alpha_n \mu_n e_n \) and \( \Delta_N v = \sum_{n \geq 0} \beta_n \mu_n e_n \). Hence, \( B_2(u, v) = \lambda(u, v) \) is equivalent to
\[
\sum_{n=0}^{\infty} \left( \begin{array}{c} a_{1n} \alpha_n e_n \\ a_{2n} \beta_n e_n \end{array} \right) = \sum_{n=0}^{\infty} \lambda \left( \begin{array}{c} \alpha_n e_n \\ \beta_n e_n \end{array} \right).
\]
Since the functions \( e_n \) are orthonormal, this equation holds if and only if
\[
M_n \left( \begin{array}{c} \alpha_n \\ \beta_n \end{array} \right) := \begin{pmatrix} a_{1n} + c_{11} & c_{12} \\ c_{21} & a_{2n} + c_{22} \end{pmatrix} \left( \begin{array}{c} \alpha_n \\ \beta_n \end{array} \right) = \lambda \left( \begin{array}{c} \alpha_n \\ \beta_n \end{array} \right)
\]
holds for all \( n \in \mathbb{N}_0 \). This means that an eigenvalue \( \lambda \) of some \( M_n \) with eigenvector \( (\alpha_n, \beta_n) \) is an eigenvalue of \( B_2 \) with eigenfunction \( (\alpha_n e_n, \beta_n e_n) \). Conversely, any eigenfunction \( w = (u, v) \) for an eigenvalue of \( B_2 \) gives an
There also exist ‘backward’ versions. \(\sigma\) eigenvalue of \(M\) for those \(n\) such that \((\alpha_n, \beta_n) \neq 0\). As a consequence, \(\sigma(B) = \sigma_p(B_2) = \bigcup_{n \in \mathbb{N}_0} \sigma(M_n)\) and thus \(s(B) < 0\) if and only if \(s(M_n) < 0\) for all \(n \in \mathbb{N}_0\) and only if \(\text{tr} \ M_n < 0\) and \(\det M_n > 0\) for all \(n \in \mathbb{N}_0\).

We discuss this result a bit. Since \(\mu_0 = 0\), we have \(M_0 = C = f'(r_s, s_s)\) so that \(s(C) < 0\) is a necessary condition for the local exponential stability of the reaction diffusion equation; i.e., \((r_s, s_s)\) must be locally exponential stable for the ode describing the pure reaction. If this is the case, we have \(\text{tr} \ C = c_{11} + c_{12} < 0\) and hence \(\text{tr} \ M_n = (a_1 + a_2)\mu_n + c_{11} + c_{12} < 0\) since \(\mu_n \leq 0\). Moreover, \(\mu_n \to -\infty\) as \(n \to \infty\) (see Theorem 4.15 in [ST]) so that \(\det M_n > 0\) for all large \(n\). However, for small \(n \geq 1\) it can happen that

\[
0 > \det M_n = a_1a_2\mu_n^2 + \mu_n(a_1c_{22} + a_2c_{11}) + \det C,
\]

despite \(\det C > 0\) and \(a_1a_2\mu_n^2 > 0\) since \(\mu_n < 0\). This could occur if \(\det C\) is very close to 0, \(a_1\) is very small and \(c_{11}\) is rather big, compared to \(\mu_1\), say. (Typically, diffusion coefficients are small.) In this case one has ‘diffusion induced instability’.

It would be nice to know that the solutions converge to \(u_*\) for a ‘larger’ set of initial values (e.g., all positive ones). To obtain a result in this direction, we need two more tools: Lyapunov functions and omega limit sets.

**Definition 3.11.** Let \(D \subseteq X_\alpha\). A map \(\Phi \in C(D, \mathbb{R})\) is called Lyapunov function for (3.1) on \(D\) if the function \(\varphi_u(t) = \Phi(u(t))\) decreases for every solution \(u\) of (3.1) as long as \(u(t) \in D\) and \(t \in [0, b(u_0))\). The Lyapunov function is strict if \(\varphi_u\) strictly decreases except for stationary solutions \(u\).

A Lyapunov function \(\Phi\) allows to detect invariant sets for (3.1) and global solutions, if \(\Phi\) blows up at \(\partial D\) and for large \(\|x\|_\alpha\).

**Proposition 3.12.** Let \(A\) generate an analytic \(C_0\)–semigroup on \(X\), \(\alpha \in [0, 1)\), and \(F : X_\alpha \to X\) be locally Lipschitz. Let \(D \subseteq X_\alpha\) be open and \(\Phi \in C(D, \mathbb{R})\) be a Lyapunov function for (3.1). Then the following assertions hold.

(a) If \(\Phi(x) \to \infty\) as \(x \to \partial D\) in \(X_\alpha\) (if \(\partial D \neq \emptyset\)), then \(D\) is invariant\(^1\), i.e., for every \(u_0 \in D\), we have \(u(t; u_0) \in D\) for all \(t \in [0, b(u_0))\).

(b) If \(D\) is invariant and \(\Phi(x) \to \infty\) as \(\|x\|_\alpha \to \infty\) for \(x \in D\), then \(b(u_0) = \infty\) for all \(u_0 \in D\).

**Proof.** Let \(u_0 \in D\). Then we have \(u(t) \in \{x \in D : \Phi(x) \leq \Phi(u_0)\}\) as long as \(u(t)\) stays in \(D\). In assertion (a), \(u\) thus cannot approach \(\partial D\) and hence \(u(t) \in D\) for all \(t \in [0, b(u_0))\). The condition in (b) then implies that \(\|u(t)\|_\alpha\) is bounded on \([0, b(u_0))\) so that \(b(u_0) = \infty\) due to Theorem 3.2. \(\square\)

**Proposition 3.13.** Let \(A\) generate the analytic \(C_0\)–semigroup \(T(\cdot)\) on \(X\), \(\alpha \in [0, 1)\), and \(F : X_\alpha \to X\) be locally Lipschitz. Let \(u = u(\cdot; u_0)\) be a solution of (3.1) on \(\mathbb{R}_+\) such that the orbit \(\gamma(u_0) := \{u(t) \mid t \geq 0\}\) is relatively compact in \(X_\alpha\). Then the omega limit set

\[
\omega(u_0) := \{x \in X_\alpha : \exists n_j \to \infty : u(t_{n_j}) \to x\ \text{in}\ X_\alpha \ \text{as}\ j \to \infty\}
\]

\(^1\)Here and below we restrict ourselves to ‘forward concepts’, i.e., those for times \(t \geq 0\). There also exist ‘backward’ versions.
is nonempty, compact and connected in $X_\alpha$, and invariant for (3.1). We further have $d_\alpha(u(t),\omega(u_0)) := d_{X_\alpha}(u(t),\omega(u_0)) \to 0$ as $t \to \infty$. Finally, if $v_0 \in \omega(u_0)$, then $b(v_0) = \infty$ and there is a classical solution $v$ of (3.1) on $\mathbb{R}$ with $v(0) = v_0$ and $v(t) \in \omega(u_0)$ for all $t \in \mathbb{R}$. In particular, $v_0 \in D(A)$.

**Proof.** Since $\gamma(u_0)$ is compact in $X_\alpha$, we have $\omega(u_0) \neq \emptyset$. Let $x_n \in \omega(u_0)$ tend to $x$ in $X_\alpha$ as $n \to \infty$. For each $n \in \mathbb{N}$ we choose $t_n \geq n$ such that $\|u(t_n) - x_n\|_\alpha \leq 1/n$. Hence, $\|x - u(t_n)\|_\alpha \leq \|x - x_n\|_\alpha + 1/n$ tends to 0 as $n \to \infty$. Thus, $x \in \omega(u_0)$ and $\omega(u_0)$ is closed in $X_\alpha$. The asserted compactness of $\omega(u_0) \subseteq \gamma(u_0)^\alpha$ now follows from its closedness in $X_\alpha$.

To show the connectedness of $\omega(u_0)$, we assume that there were nonempty disjoint subsets $\omega_j$ of $\omega(u_0)$ such that $\omega_j = O_j \cap \omega(u_0)$ for open sets $O_j \subseteq X_\alpha$ for $j = 1, 2$ and $\omega_1 \cap \omega_2 = \omega(u_0)$. Writing $\omega_j = \omega(u_0) \cap (X_\alpha \setminus O_j)$ for $i \neq j$, we see that $\omega_j$ is closed, hence compact, in $X_\alpha$ for $j = 1, 2$. It follows that $\|x_1 - x_2\|_\alpha \geq \delta > 0$ for some $\delta > 0$ and all $x_j \in \omega_j$. Take any $x \in \omega_1$ and $y \in \omega_2$. There are $s_n, t_n \to \infty$ such that $s_n < t_n < s_{n+1}$, $\|u(s_n) - x\|_\alpha < \delta/3$ and $\|v(t_n) - y\|_\alpha < \delta/3$ for all sufficiently large $n$. By continuity, we can thus fund an $r_n \in (s_n, t_n)$ such that $d_\alpha(u(r_n), \omega_1) = \delta/3$ and $d_\alpha(u(r_n), \omega_2) \geq 2\delta/3$. But for a subsequence the states $u(r_{n_j})$ converges to $\omega(u_0)$ in $X_\alpha$ as $j \to \infty$, which is impossible. Consequently, $\omega(u_0)$ is connected.

Suppose that $d_\alpha(u(t_n), \omega(u_0)) \geq \delta > 0$ for a sequence $t_n \to \infty$. Then there is a subsequence such that $u(t_{n_j})$ converges to some $x \in \omega(U_0)$ in $X_\alpha$, which is a contradiction.

Let $v_0 \in \omega(u_0)$. Then there are $t_n \to \infty$ such that $u(t_n) \to v_0$ in $X_\alpha$ as $n \to \infty$. The continuous dependence on initial data and the uniqueness of (3.2) (see Theorem 3.2) imply that

$$u(t; v_0) = \lim_{n \to \infty} u(t; u(t_n; u_0)) = \lim_{n \to \infty} u(t + t_n; u_0)$$

in $X_\alpha$, and thus $u(t; v_0) \in \omega(u_0)$ for all $t \in [0, b(v_0))$. As a compact set, $\omega(u_0)$ is bounded in $X_\alpha$. Theorem 3.2(c) now yields $b(v_0) = \infty$.

Inductively, we obtain $v_l \in \omega(u_0)$ and subsequences $(t_{\nu_m(k)})_k \subseteq (t_{\nu_{m-1}(k)})_k \subseteq (t_n)_n$ such that $u(t_{\nu_m(k)} - l; u_0)$ converges to $v_l$ in $X_\alpha$ as $k \to \infty$, for all $l = 0, 1, \ldots, m$ and $m \in \mathbb{N}$. We set $v^{(m)} = u(\cdot; v_m)$ on $\mathbb{R}_+$ and observe that

$$v^{(m)}(m-l) = \lim_{k \to \infty} u(m-l + t_{\nu_m(k)} - m; u_0) = v_l,$$

holds for all $l = 0, 1, \ldots, m$ and $m \in \mathbb{N}$. The uniqueness of (3.1) implies that $v^{(m+1)}(t) = v^{(m)}(t-1)$ for all $t \geq 1$, and hence we can define a classical solution of (3.1) on $\mathbb{R}$ by setting $v(t) = v^{(m)}(t + m)$ for all $t \geq -m$ and $m \in \mathbb{N}$. In particular, $v_0 = v(0) \in D(A)$. Since $v(-m) = v_m \in \omega(u_0)$ for all $m \in \mathbb{N}$, the (forward) invariance of $\omega(u_0)$ implies that $v(t) \in \omega(u_0)$ for all $t \in \mathbb{R}$. □

This result (except for the very last assertion) holds in a much more general setting, see Chapter 4 of [He]. The proof of the above result indicates again that one should describe the behavior of (3.1) in the norm of $X_\alpha$. The compactness assumption in the above proposition is crucial, of course, as seen by the solution $e^t$ of the ode $u'(t) = u(t)$.
Our second convergence theorem is based on the above proposition and the observation that the omega limit set only contains equilibria if the problem possesses a strict Lyapunov function.

**Theorem 3.14.** Let $A$ generate an analytic $C_0$–semigroup on $X$, $\alpha \in [0,1)$, and $F : X_\alpha \to X$ be locally Lipschitz. Let $D \subseteq X_\alpha$ and $\Phi \in C(D,\mathbb{R})$ be a strict Lyapunov function for (3.1). Assume that $u_0 \in D$ satisfies $b(u_0) = \infty$ and that $\gamma(u_0)^\alpha$ is compact in $X_\alpha$ and belongs to $D$. Then $\omega(u_0) \subseteq \mathcal{E}_D$ and $d_\alpha(u(t),\mathcal{E}_D)$ tends to 0 as $t \to \infty$, where $\mathcal{E}_D = \{u_* \in D \cap D(A) \mid Au_* = -F(u_*)\}$ is the set of equilibria in $D$. If $\mathcal{E}_D$ is discrete, $u$ converges to an $u_* \in \mathcal{E}_D$.

**Proof.** Since $\varphi_u = \Phi \circ u$ decreases and $\Phi$ is bounded on the compact set $\gamma(u_0)^\alpha$, the function $\varphi_u(t)$ converges to some $\ell \in \mathbb{R}$. Take any $x \in \omega(u_0)$ and $t_n \to \infty$ such that $u(t_n) \to x$ in $X_\alpha$. Then, $x \in D$ and $\Phi(x) = \lim_{n \to \infty} \varphi_u(t_n) = \ell$ which means that $\Phi$ is constant on $\omega(u_0)$. The orbit $u(\cdot;x)$ stays in $\omega(u_0)$ by Proposition 3.13 so that $x$ must belong to $\mathcal{E}_D$ because $\Phi$ is strict. The assertions now follow from Proposition 3.13.

There is a variant of this theorem without the strictness assumption, called LaSalle’s invariance principle, see Theorem 4.3.4 of [He]. In some situations one can also show convergence to an equilibrium if $\mathcal{E}_D$ is not discrete, see Section 8.8 and 10.3 in [PW] for such results in an ode setting.

Despite its surprisingly elementary proof, the above theorem is very powerful. In our reaction diffusion equations one obtains compactness of the orbit if it is uniformly bounded on $\mathbb{R}_+$. For (strict) Lyapunov functions there are at least some candidates as the one used in the next example.

**Example 3.15. Predator-prey model.** Again we work in the framework of Example 3.4 with $m = 2$ assuming that $U$ is connected. We consider the ‘reaction term’

$$f(u,v) = \begin{pmatrix} 1 - \lambda_1 u - v \\ \mu - \lambda_2 v + u \end{pmatrix}$$

with $\lambda_1, \lambda_2 > 0$ and $\mu \in \mathbb{R}$, describing the (normalized) interaction between the prey species $u$ and the predators $v$. Let $0 \leq u_0, v_0 \in E_\alpha \to C(\overline{U})^2$. Since the positivity condition (3.9) holds, there is a unique positive solution $(u(t),v(t))$ of (3.1) with the above $f$ on a maximal existence interval $[0,b_0]$.

We first show that $u \leq \kappa := \max\{\|u_0\|_\infty, 1/\lambda_1\}$ starting with $0 \leq (u_0,v_0) \in D(A)$. Suppose that there are $(t_0,x_0) \in (0,b_0) \times \overline{U}$ and $\delta > 0$ such that $u(t_0,x_0) \geq \delta + \kappa$. Since $u(t_0,x_0) > 1/\lambda_1$, it follows that

$$\partial_t u(t_0,x_0) - a_1 \Delta u(t_0,x_0) \leq u(t_0,x_0) - \lambda_1 u(t_0,x_0)^2 < 0$$

for any such $(t_0,x_0)$. As in the proof of the parabolic maximum principle Proposition 3.6, we obtain a contradiction so that $u \leq \kappa$. We can extend this estimate to all $0 \leq (u_0,v_0) \in E_\alpha$ using the approximations $0 \leq nR(n,A)(u_0,v_0) \in D(A)$. It then follows that $(\mu - \lambda_2 v + u) v \leq (\mu + \kappa) v - \lambda_2 v^2$. We can now proceed in the same way as for $u$, arriving at the global boundedness of $v$. (If even $\mu + \kappa \leq 0$, one can simply use Proposition 3.6.) Proposition 3.5 thus yields that $b_0 = \infty$ and that the orbit is relatively compact in $E_\alpha$. 

39
Below we will also need that the solution is strictly positive if \( u_0 \neq 0 \) and \( v_0 \neq 0 \). To this aim, we take first initial data \( 0 \leq (u_0, v_0) \in D(A) \) with \( u_0 \neq 0 \) and \( v_0 \neq 0 \). Set \( \omega = \lambda_1 \|u\|_\infty + \|v\|_\infty - 1 \) for the given solution. The rescaled function \( \varphi(t) = e^{\omega t}u(t) \) then satisfies the inequality
\[
\partial_t \varphi - a_1 \Delta \varphi = \varphi(\omega + 1 - \lambda_1 u - v) \geq 0
\]
on \( \mathbb{R}_+ \times \overline{U} \). If \( u(t_0, x_0) = 0 \) for some \( (t_0, x_0) \in (0, \infty) \times \overline{U} \), then \( \varphi = 0 \) on \( [0, t_0] \times \overline{U} \) due to the strong parabolic maximum principle (see e.g. Theorems 5 and 6 in Chapter 3 of [PWe]²), contradicting \( u_0 \neq 0 \). In the same way one sees that \( v(t) > 0 \) for all \( t > 0 \). As a result, for each \( 0 < a < b \) there is an \( \varepsilon > 0 \) such that \( u, v \geq \varepsilon \) on \( [a, b] \times \overline{U} \). By approximation in \( E_\alpha \), we infer that \( (u(t), v(t)) > 0 \) on \( \overline{U} \) for all \( t > 0 \) and all \( 0 \leq (u_0, v_0) \in E_\alpha \) with \( u_0 \neq 0 \) and \( v_0 \neq 0 \).

We only consider strictly positive equilibria. It is not difficult to see that \( f \) has a strictly positive zero \( (r_*, s_*) \) if and only if \( \lambda_2 > \mu > -1/\lambda_1 \), and then
\[
(r_*, s_*) = \frac{1}{1 + \lambda_1 \lambda_2} (\lambda_2 - \mu, 1 + \mu \lambda_1).
\]
This condition means that the internal reproduction coefficient \( \mu \) of the predators is neither too negative nor bigger than the internal ‘damping’ coefficient \( \lambda_2 \) reflecting the competition between the predators. We thus study the equilibrium \( (u_*, v_*) = (r_*, s_*) \) of (3.6). Take \( 0 \leq (u_0, v_0) \in E_\alpha \) with \( u_0 \neq 0 \) and \( v_0 \neq 0 \), and let \( (u, v) \) be the corresponding solution.

We introduce the set \( D = \{ w \in E_\alpha \mid w > 0 \text{ on } \overline{U} \} \) and the functional
\[
\Phi(w) = \int_U ((w_1 - r_* \ln w_1) + (w_2 - s_* \ln w_2)) dx = \int_U \Psi(w) dx.
\]
Because of \( E_\alpha \hookrightarrow C(\overline{U})^2 \), we have \( \Psi \in C^1(D, C(\overline{U})) \) with derivative \( \Psi'(w)[\overline{w}] = (1 - r_* w_1)/w_1 + (1 - s_* w_2)/w_2 \) for all \( \overline{w} \in E_\alpha \). The integral is just a linear functional on \( C(\overline{U}) \) so that \( \Phi \in C^1(D, \mathbb{R}) \). Since \( (u(t), v(t)) \in D \) for all \( t > 0 \), the chain rule and (3.6) yield
\[
\frac{d}{dt} \Phi(u, v) = \int_U \left( (1 - r_* u) u' + (1 - s_* v) v' \right) dx
\]
\[
= \int_U \left( (1 - r_* u) a_1 \Delta u + (1 - s_* v) a_2 \Delta v \right) dx
\]
\[
+ \int_U \left( (u - u_*)(1 - \lambda_1 u - v) + (v - v_*)(\mu - \lambda_2 v + u) \right) dx,
\]
omitting the variables \( (t, x) \in (0, \infty) \times U \). We denote the integrals in the last two lines by \( J_1 \) and \( J_2 \), respectively. An integration by parts further shows that
\[
J_1 = - \int_U \left( \frac{a_1 r_* |\nabla u|^2}{u^2} + \frac{a_2 s_* |\nabla v|^2}{v^2} \right) dx \leq 0.
\]

²To apply these results, one needs the connectness of \( U \). The proofs given in [PWe] can be extended to the present situation.
Using \( 1 = \lambda_1 r_* + s_* \) and \( \mu = \lambda_2 s_* - r_\ast \), we compute

\[
J_2 = \int_U \left( (u - u_\ast)(\lambda_1 u_\ast + v_* - \lambda_1 u - v) + (v - v_\ast)(\lambda_2 v_* - u_\ast - \lambda_2 v + u) \right) dx
\]

\[
= - \int_U \left( \lambda_1 (u - u_\ast)^2 + \lambda_2 (v - v_\ast)^2 \right) dx \leq 0.
\]

Summing up, we arrive at

\[
\Phi(u(t), v(t)) \leq \Phi(u(s), v(s)) - \int_s^t \int_U \left( \lambda_1 (u(\tau) - u_\ast)^2 + \lambda_2 (v(\tau) - v_\ast)^2 \right) dx d\tau.
\]

for all \( t \geq s > 0 \). As a consequence, \( \Phi(u, v) \) decays along each orbit with nonzero positive initial data, and if it is constant on \([s, t]\), it must hold \( u = u_* \) and \( v = v_* \) on \([s, t]\), and thus on \( \mathbb{R}_+ \). In particular, \((u_\ast, v_\ast)\) is the only equilibrium in \( D \) and \( \Phi \) is a strict Lyapunov function on \( D \).

Let \( \hat{w}_0 = (\hat{u}_0, \hat{v}_0) \in \omega((u_0, v_0)) \). We have \( \hat{w}_0 \geq 0 \) as a uniform limit of positive functions. Fatou’s lemma further implies that

\[
\Phi(\hat{w}_0) = \liminf_{t \to \infty} \Phi(u(t), v(t)) \leq \Phi(u_0, v_0) < \infty
\]

so that \( \hat{u}_0(x) > 0 \) and \( \hat{v}_0(x) > 0 \) must hold for a.e. \( x \in U \). Recall from Proposition 3.13, that \( \hat{w}_0 = \hat{w}(0) \) for a solution \( \hat{w} \) of (3.1) on \( \mathbb{R} \) belonging to \( \omega((u_0, v_0)) \). In particular, \( \hat{u}(-1)(x) > 0 \) and \( \hat{v}(-1)(x) > 0 \) for a.e. \( x \in U \), and hence \( \hat{w}_0 > 0 \) on \( U \) by the strict positivity. This means that also the closure of \( \gamma((u_0, v_0)) \) in \( E_\alpha \) belongs to \( D \).

Theorem 3.14 now implies that the solution \((u(t), v(t))\) converges to \((u_\ast, v_\ast)\) in \( E_\alpha \) as \( t \to \infty \) if \( 0 \leq (u_0, v_0) \in E_\alpha \) with \( u_0 \neq 0 \) and \( v_0 \neq 0 \). (Observe that the decay of the \( L^1 \) type quantity \( \Phi(u(t), v(t)) \) already implies the convergence of \((u(t), v(t))\) to the equilibrium in all norms \( \| \cdot \|_\alpha \)).
The nonlinear Schrödinger equation

In this chapter we investigate the nonlinear Schrödinger equation
\[ i \partial_t u(t, x) = -\Delta u(t, x) + \mu |u(t, x)|^{\alpha-1}u(t, x), \quad t \in J, \ x \in \mathbb{R}^d, \]
\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}^d, \]
where \( J \) is a nontrivial interval containing 0. Equivalently, one can write
\[ \partial_t u(t, x) = i\Delta u(t, x) - i\mu |u(t, x)|^{\alpha-1}u(t, x). \]
Throughout it is assumed that \( \mu \in \{-1, 1\} \) and \( 1 < \alpha < \frac{d+2}{(d-2)+} \).

The results below can be extended to more general nonlinearities, see [Ca], but the model equation (4.1) already gives a very good insight in the field. The borderline case \( \alpha = \frac{d+2}{(d-2)+} \) for \( d \geq 3 \) is much more difficult and beyond the scope of these lectures. An extended survey is given in [Ta]. If \( \mu = 1 \) one has the (in some sense simpler) defocussing case and for \( \mu = -1 \) the focussing case.

The nonlinear Schrödinger equation (and its variants) appears in quantum field theory, e.g., in the study of so called Bose-Einstein condensates. It is also used to describe (approximately) the amplitudes of wave packages in nonlinear optics, see [MN] or [SS]. Natural numbers \( \alpha \) play a significant role when nonlinear material laws are given by power series. Due to symmetry constraints one then often considers odd \( \alpha \). Observe that for the space dimension \( d = 3 \) we can allow for \( \alpha = 3 \), but \( \alpha = 5 \) is precisely the borderline case.

In this chapter we write \( W^k_p = W^k_p(\mathbb{R}^d) \) for \( k \in \mathbb{N}_0 \) and \( p \in [1, \infty] \), and similarly for other function spaces on \( \mathbb{R}^d \). We often drop the domain \( \mathbb{R}^d \) in integrals over \( \mathbb{R}^d \). The norm on \( W^k_p \) is denoted by \( \|v\|_{k,p} \), where we put \( W^0_p := L^p \) and \( \|v\|_{0,p} := \|v\|_p \).

A \( W^2_2 \)-solution of (4.1) is a map \( u \in C^1(J, L^2) \cap C(J, W^2_2) \) satisfying (4.1).

1. Basic properties and the linear problem

We start with a few (more or less explicit) special solutions of the differential equation in (4.1).

1) Plane waves. Given a wave vector \( \xi \in \mathbb{R}^d \setminus \{0\} \) we look for a \( \phi: \mathbb{R} \rightarrow \mathbb{C} \) such that the function \( u(t, x) := \phi(t)e^{i\xi \cdot x} \) solves (4.1) with \( u_0(x) = \phi(0)e^{i\xi \cdot x} \). It holds \( \partial_t u(t, x) = \phi'(t)e^{i\xi \cdot x}, \ \partial_k u(t, x) = i\xi_k u(t, x) \) and \( \Delta u(t, x) = -|\xi|^2 u(t, x) \) for \( (t, x) \in \mathbb{R}^{1+d} \). Since \( |u| = |\phi| \), the map \( u \) satisfies (4.1) if and only if
\[ \phi'(t) = -i(|\xi|^2 + \mu |\phi(t)|^{\alpha-1}) \phi(t). \]
This scalar ordinary differential equation can be solved leading to
\[ u(t, x) = a e^{\xi x} e^{-|\xi|^2 t} e^{-i |\xi|^2 t} \]
where \( a := \phi(0) \) and \( |a| \) is the amplitude. Here the factor \( e^{-|\xi|^2 t} \) describes oscillations in time arising from \( i\Delta \), whereas the oscillation \( e^{-i |\xi|^2 t} \) is determined by the nonlinearity. If \( \mu = 1 \), the oscillations have the same sign and thus are increased. If \( \mu = -1 \), the nonlinearity decreases (or even annihilates) the oscillation. This partly explains the terminology of the two cases.

2) STANDING WAVES. We look for a solution of (4.1) of the form \( u(t, x) = e^{-i\omega t} \psi(x) \) for any \( \omega \in \mathbb{R} \) and \( \psi \in W^2_2 \). This \( \psi \) solves (4.1) if and only if
\[
i \partial_t u(t, x) = i\omega e^{-i\omega t} \psi(x) = -e^{-i\omega t} \Delta \psi(x) + \mu |\psi(x)|^{\alpha-1} e^{-i\omega t} \psi(x),
\]
where
\[
\omega < 1 \quad \text{and} \quad \mu > 0.
\]
This scalar ordinary differential equation can be solved leading to
\[
\psi(x) = \frac{\sqrt{2}}{\cosh x}, \quad x \in \mathbb{R}.
\]

3) A SOLUTION WITH BLOW-UP. Let \( \mu = -1, \alpha = 1 + \frac{4}{d} \) and \( \omega < 0 \). One can even prove that \( \psi > 0 \) and that \( \psi, \nabla \psi \) decay exponentially. (See §8.1 of [Ca] and the references given there.) For \( d = 1, \alpha = 3 \) and \( \omega = -1 \) one has the explicit solution
\[
\psi(x) = \frac{\sqrt{2}}{\cosh x}, \quad x \in \mathbb{R}.
\]

As a result, this solution explodes in \( W^2_2 \) as \( t \to 0^- \) though it stays bounded in \( L^2 \) and the initial value \( u(-1) \) at, say, \( t = -1 \) behaves very well.

The basic tools to study the longterm behavior of solutions are the conservation laws for the \( L^2 \)-norm and the ‘energy’. Let \( u \) be a \( W^2_2 \)-solution of (4.1).

A) CONSERVATION OF THE \( L^2 \)-NORM. Equation (4.1) and an integration by parts yield
\[
\frac{d}{dt} \|u(t)\|^2 \leq \frac{d}{dt} \int u(t) \overline{u(t)} \, dx = 2 \Re \int (\partial_t u(t)) \overline{u(t)} \, dx
\]
\[
= 2 \Re i \int (\Delta u(t) - \mu |u(t)|^{\alpha-1} u(t)) \overline{u(t)} \, dx
\]
\[
= 2 \Im \int (||\nabla u(t)||^2 + \mu |u(t)|^{\alpha+1}) \, dx = 0,
\]

44
\[ \|u(t)\|_2 = \|u(t_0)\|_2 \]  
(4.2)

for all \( t, t_0 \in J \).

B) Conservation of energy. Sobolev’s embedding theorem says that \( W^1_2 \hookrightarrow L^{\alpha+1} \) since \( 1 - d/2 \geq -d/(\alpha + 1) \) because of \( 1 < \alpha < (d+2)/(d-2)_+ \). (Here we can allow for equality if \( d \neq 2 \).) Thus the energy

\[ E(v) := \frac{1}{2} \int |\nabla v|^2 \, dx + \frac{\mu}{\alpha + 1} \int |v|^{\alpha+1} \, dx = \frac{1}{2} \| \nabla v \|_2^2 + \frac{\mu}{\alpha + 1} \| v \|_{1+\alpha}^{1+\alpha} \]

is defined for \( v \in W^1_2 \). The following lemma further yields that \( E \in C^1(W^1_2) \) with the derivative

\[ E'(v)w = \int \text{Re}(\nabla v \cdot \nabla \overline{w}) \, dx + \mu \int \text{Re}(|v|^{\alpha-1} v \overline{w}) \, dx \]

for \( v, w \in W^1_2 \). We assume that \( u \in C^1(J, W^1_2) \hookrightarrow C^1(J, L^{\alpha+1}) \), in addition. The chain rule, an integration by parts and (4.1) then imply

\[
\frac{d}{dt} E(u(t)) = \int \text{Re}(\nabla u(t) \cdot \nabla \overline{\partial_t u(t)}) \, dx + \mu \int \text{Re}(|u(t)|^{\alpha-1} u(t) \overline{\partial_t u(t)}) \, dx \\
= \text{Re} \left( -\Delta u(t) + \mu |u(t)|^{\alpha-1} u(t) \partial_t u(t) \right) \\
= \text{Re} \left( i \partial_t u(t) \overline{\partial_t u(t)} \right) \, dx = 0,
\]

for all \( t, t_0 \in J \). We point out that \( E(v)^{1/2} + \| v \|_2 \) dominates the \( W^1_2 \)-norm if \( \mu = 1 \). In this respect the defocussing case is easier to treat.

**Lemma 4.1.** Let \( \alpha \geq 1 \). The map \( \Phi(v) = \frac{1}{\alpha+1} \| v \|_{\alpha+1}^{\alpha+1} \) belongs to \( C^1(L^{\alpha+1}) \) and its derivative is given by

\[ \Phi'(v)w = \int_{\mathbb{R}^d} |v|^{\alpha-1} \text{Re}(v \overline{w}) \, dx = \int_{\mathbb{R}^d} |v|^{\alpha-1} v \overline{w} \, dx, \quad v, w \in L^{\alpha+1}. \]

**Proof.** The result is easy to show for \( \alpha = 1 \) so that we consider \( \alpha > 1 \). Let \( (r, s), (\rho, \sigma) \in \mathbb{R}^2 \) and \( z = r + is, \zeta = \rho + i\sigma \). It holds

\[ \nabla |z|^{\alpha+1} = (\alpha + 1) |z|^{\alpha-1} (r, s) \]

for the gradient in \( \mathbb{R}^2 \). Since the real scalar product \( (r, s) \cdot (\rho, \sigma) = r\rho + s\sigma \) is equal to \( \text{Re}(z \overline{\zeta}) = \text{Re}(z \overline{\zeta}) \), we obtain

\[ |v(x) + w(x)|^{\alpha+1} - |v(x)|^{\alpha+1} - (1 + \alpha) |v(x)|^{\alpha-1} \text{Re}(v(x)\overline{w(x)}) \]

\[ = \int_0^1 \frac{d}{d\tau} |v(x) + \tau w(x)|^{\alpha+1} \, d\tau - (1 + \alpha) |v(x)|^{\alpha-1} \text{Re}(v(x)\overline{w(x)}) \]

\[ = (1 + \alpha) \int_0^1 \left( |v(x) + \tau w(x)|^{\alpha-1} \text{Re}[(v(x) + \tau w(x))\overline{w(x)}] \right. \]

\[ - |v(x)|^{\alpha-1} \text{Re}(v(x)\overline{w(x)}) \right) \, d\tau \]

\[ = (1 + \alpha) \int_0^1 \left( |v(x) + \tau w(x)|^{\alpha-1} - |v(x)|^{\alpha-1} \right) \text{Re}(v(x)\overline{w(x)}) \, d\tau \]

45
for $v, w \in L^{\alpha+1}$ and $x \in \mathbb{R}^d$. Setting $F(v)w = \int |v|^{\alpha-1} \text{Re}(v \overline{w}) \, dx$, we conclude

$$|\Phi(v + w) - \Phi(v) - F(v)w| \leq \int_0^1 \int |v + \tau w|^{\alpha-1} - |v|^{\alpha-1} \, |vw| \, dx \, d\tau$$

$$+ \int(|v| + |w|)^{\alpha-1} |w|^2 \, dx =: J_1 + J_2.$$ 

Hölder’s inequality with exponents $(\alpha + 1)/\alpha - 1$ and $(\alpha + 1)/2$ now yields

$$J_2 \leq \left( \int (|v| + |w|)^{\alpha+1} \, dx \right)^{\frac{\alpha-1}{\alpha+1}} \left[ \int |w|^{\alpha+1} \, dx \right]^{\frac{\alpha}{\alpha+1}} \leq (\|v\|_{\alpha+1} + \|w\|_{\alpha+1})^{\alpha-1} \|w\|_{\alpha+1}^2,$$

so that $J_2/\|w\|_{\alpha+1} \to 0$ as $\|w\|_{\alpha+1} \to 0$. Using again Hölder’s inequality in the $x$-integral, we further deduce

$$J_1 \leq \|v\|_{\alpha+1} \|w\|_{\alpha+1} \int_0^1 \left[ \int |v + \tau w|^{\alpha-1} - |v|^{\alpha-1} \right]^{\alpha+1} \frac{\alpha+1}{\alpha+1} \, d\tau.$$

Suppose that there are $w_n$ tending to 0 in $L^{\alpha+1}$ such that the corresponding integral $J(n)$ on the right–hand side of the above inequality does not converge to 0 as $n \to \infty$. Then there is a subsequence and a function $g \in L^{\alpha+1}$ such that $J(n_j) \neq 0$ and $w_{n_j} \to 0$ a.e. as $j \to \infty$ and $|w_{n_j}| \leq g$ for all $j \in \mathbb{N}$. The integrand of the $x$-integral in $J(n_j)$ then converges to 0 a.e. and it is dominated by $c((|v| + g)^{\alpha+1} + |v|^{\alpha+1}) \in L^1$. By Lebesgue’s theorem, $J(n_j)$ tends to 0 as $j \to \infty$. This contradiction yields that also $J_1/\|w\|_{\alpha+1} \to 0$ as $\|w\|_{\alpha+1} \to 0$. Consequently, $\Phi$ is differentiable at each $v \in L^{\alpha+1}$ with the derivative $\Phi'(v) = F(v)$. The continuity of $\Phi'$ is shown similarly. \(\square\)

Note that this lemma and its proof still work if we replace $\mathbb{R}^d$ by any other measure space. There are no conservation laws involving second space derivatives. We thus need a blow up condition in terms of $W^2_2$ in order to exploit the above conservation laws so that we should work with solutions of (4.1) being continuous only in $W^1_2$. We first develop the linear theory in this framework.

**The free Schrödinger group.** The Laplace operator $\Delta$ with domain $W^2_2$ is self adjoint in $L^2$ due to Example 4.8 in [ST]; i.e., $i\Delta$ is skew adjoint. Stone’s theorem (see Theorem 1.36 in [EE]) thus shows that $i\Delta$ generates a unitary $C_0$-group $T(\cdot)$ on $L^2$, which is called the free Schrödinger group. By the next result, this group looks like the diffusion semigroup with ‘imaginary time’ $it$. The resulting representation formula implies that $T(t)v \in C^\infty(\mathbb{R}^d)$ if $v \in L^2$ has compact support. However, there is no smoothing effect in the full space $L^2$ since $T(t)$ is bijective.

**Lemma 4.2.** The free Schrödinger group is given by

$$T(t)v(x) = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{it\frac{|x-y|^2}{4}} v(y) \, dy =: (K_t * v)(x)$$

for all $t \in \mathbb{R} \setminus \{0\}$, $v \in L^1 \cap L^2$, and a.e. $x \in \mathbb{R}^d$. 46
Proof. We can assume that \( v \in C_\infty \) since we can approximate a given \( v \in L^1 \cap L^2 \) by functions \( v_n \in C_\infty \) simultaneously in \( L^1 \) and in \( L^2 \). The orbit transform \( F : f \mapsto \hat{f} \) is a unitary operator on \( L^2 \) and a bijection on the Schwartz space \( \mathcal{S}_d \), see Theorem 5.11 and Proposition 5.10 of [FA14]. Observing that \( F([u'(t) - u(t)]/(t' - t)) = (\hat{u}(t') - \hat{u}(t))/(t' - t) \), we conclude that \( \hat{u} \in C^1(\mathbb{R}, L^2) \) with \( \frac{d}{dt} \hat{u} = F(u') \). Treating the Laplacian as in Example 4.8 of [ST], we thus obtain
\[
\frac{d}{dt} \hat{u}(t) = F(i\Delta u(t)) = -i|\xi|^2 \hat{u}(t).
\]
Solving this ordinary differential equation for fixed \( \xi \in \mathbb{R}^d \), we arrive at
\[
\hat{u}(t, \xi) = e^{-it|\xi|^2} \hat{\gamma}(\xi) = e^{-it|\xi|^2} \hat{\gamma}(\xi)
\]
for all \( t \in \mathbb{R} \) and \( \xi \in \mathbb{R}^d \), where \( \gamma_t \in C_b \) and \( \hat{\gamma} \in \mathcal{S}_d \). As a result, \( u(t) = F^{-1}(\hat{\gamma} \hat{\gamma}) \). Since \( \gamma_t \) is not the Fourier transform of an \( L^1 \)-function, we cannot directly apply the convolution formula in Theorem 5.11 of [FA14]. Instead we consider the regularization \( \gamma_{\varepsilon} = \gamma_{t+\varepsilon} \in \mathcal{S}_d \) for \( \varepsilon > 0 \). Since \( |m| \leq 1 \) and \( m_t \) converges pointwise to \( \gamma_t \), Lebesgue’s theorem and Theorem 5.11 of [FA14] imply that
\[
\hat{u}(t) = \lim_{\varepsilon \to 0} F^{-1}(m_{\varepsilon}(t)) \hat{\gamma} = \lim_{\varepsilon \to 0} (2\pi)^{-\frac{d}{2}} \int e^{ix \xi} e^{-it|\xi|^2} e^{-i\varepsilon|\xi|^2} d\xi = \prod_{k=1}^{d} \frac{1}{\sqrt{2\pi}} \int e^{ix_k \xi_k - (it+\varepsilon)\xi_k^2} d\xi_k
\]
for all \( t \neq 0 \) and \( x \in \mathbb{R}^d \). By means of complex contour integrals one can show
\[
\int_{\mathbb{R}} e^{-(it+\varepsilon)^2} e^{ix_k s} ds = \sqrt{\frac{\pi}{it+\varepsilon}} e^{\frac{-s^2}{4(it+\varepsilon)}}
\]
see Exercise 2.26 in [Ta], so that
\[
u(t,x) = \frac{1}{(4\pi)^{d/2}} \lim_{\varepsilon \to 0} \int \frac{1}{(it+\varepsilon)^{d/2}} e^{-\frac{|y|^2}{4(it+\varepsilon)}} \nu(y) dy.
\]
For fixed \( t \neq 0 \) and \( x \in \mathbb{R}^d \), Lebesgue’s theorem allows to let \( \varepsilon \to 0 \) in the integral since \( \nu \in C_{\infty} \). The assertion follows. \( \square \)

This representation formula allows to establish the dispersive behavior of \( T(t) \). The first part of the next corollary says that \( T(t) \) flattens initial data which become bounded immediately and then tend to 0 in all \( L^p \)-norms for \( p > 2 \) as \( t \to \infty \). Since the \( L^2 \)-norm is preserved, local concentrations of \( T(t)v \) must be pushed towards infinity in \( \mathbb{R}^d \).

Corollary 4.3. (a) \( T(t) \) extends from \( L^1 \cap L^2 \) to an operator in \( \mathcal{B}(L^p, L^p') \) for all \( p \in [1, 2] \) and \( t \in \mathbb{R} \setminus \{0\} \), having norms less or equal \((4\pi|t|)^{d(1/2 - 1/2)}\).

(b) \( T(\cdot) \) leaves invariant the space \( W^k_2 \) for each \( k \in \mathbb{N} \) and its restriction \( T_k(\cdot) = T(\cdot) \) yields a unitary \( C_0 \)-group on \( W^k_2 \) generated by the restriction \( i\Delta_k \) of \( i\Delta \) with domain \( D(\Delta_k) = \{u \in W^k_2 : \Delta u \in W^k_2\} \). We further have
\[ \partial^\beta T(t)v = T(t)\partial^\beta v \text{ for all } v \in W_p^k, \ 1 \leq p < \infty, \ t \in \mathbb{R}, \ k \in \mathbb{N} \text{ and multi indices with } 0 \leq |\beta| \leq k. \]

**Proof.** (a) By the above lemma, \( T(t) \) maps \((L^1 \cap L^2, \| \cdot \|_1)\) into \( L^\infty \) with norm less or equal \((4\pi|t|)^{-d/2}\). Moreover, it has norm 1 as an operator on \( L^2 \). Let \( p \in [1, 2] \). The Riesz–Thorin interpolation theorem now shows that we can extend \( T(t) \) to an operator from \( L^p \) to \( L^p' \) with norm less or equal \((4\pi|t|)^{(d/2-2/d)p'}\) \(= (4\pi|t|)^{d/2-2/d}\), see p.45 of [Lu2].

(b) Let \( v \in C^\infty, \ k \in \{1, \ldots, d\}, \ t \neq 0 \) and \( x \in \mathbb{R}^d \). One then obtains

\[ (\partial_k T(t)v)(x) = \partial_k \int K_t(y)(x-y) dy = \int K_t(y)(\partial_k v)(x-y) dy = (T(t)\partial_k v)(x). \]

Since the weak derivative is a closed operator in \( L^p' \) and \( C^\infty \) is dense in \( W_p^1 \), we conclude that \( \partial_k T(t)v = T(t)\partial_k v \) exists in \( L^p' \) for all \( v \in W_p^1 \). Induction yields the last assertion. This fact implies that \( T(t) \) is isometric and strongly continuous on \( W^1_p \) for \( t \in \mathbb{R} \) (because of \( \| u \|_{K^2_{1,2}} = \sum_{|\beta| \leq k} \| \partial^\beta u \|_2^2 \)). Since \( T(t) \) is still a group on \( W^k_p \) and thus bijective, Proposition 5.52 of [FA] further shows that the restriction of \( T(t) \) to \( W^k_p \) is unitary. The representation of its generator follows from the next general result from semigroup theory. \( \square \)

**Lemma 4.4.** Let \( S(t) \) be a \( C_0 \)-semigroup on Banach space \( X \) with generator \( A \) and let \( Y \) be a Banach space which is embedded into \( X \). Assume that \( S(t)Y \subset Y \) for all \( t \geq 0 \) and that the restrictions \( S_Y(t) \) yield a \( C_0 \)-semigroup on \( Y \). Then its generator is the restriction \( A_Y \) of \( A \) to \( D(A_Y) = \{ y \in D(A) \cap Y \mid Ay \in Y \} \).

(We call \( A_Y \) the part of \( A \) in \( Y \).) If we also assume that \([D(A)] \hookrightarrow Y\), then \( \sigma(A) = \sigma(A_Y) \) and the resolvent of \( A_Y \) is the restriction of that of \( A \).

**Proof.** Let \( B \) be the generator of \( S_Y(\cdot) \). If \( y \in D(B) \), then \( \frac{1}{t}(S(t)y - y) \) converges to \( By \) in \( Y \), and hence in \( X \). As a result, \( B \subset A_Y \). Let \( y \in D(A_Y) \). For some \( \lambda \) larger than the growth bounds of \( S(\cdot) \) and \( S_Y(\cdot) \), we put \( z = \lambda y - A_Y y = \lambda y - Ay \in Y \). Since

\[ y = R(\lambda, A)z = \int_0^\infty e^{-\lambda t} S(t)z \ dt = R(\lambda, B)z, \]

we deduce that \( y = R(\lambda, B)z \in D(B) \) and \( A_Y = B \).

Next, assume that \([D(A)] \hookrightarrow Y\). Let \( \lambda \in \rho(A) \). Because of \( R(\lambda, A)Y \subset D(A) \subset Y \), the restriction \( R_\lambda := R(\lambda, A)|Y \) belongs to \( B(Y) \). We further have \( (\lambda I - A)y = y \) for \( y \in Y \), so that \( R_\lambda Y \subset D(A_Y) \) and \( (\lambda I - A_Y)R_\lambda = I \).

Since also \( R_\lambda((\lambda I - A_Y)z) = R(\lambda, A)(\lambda I - A)z = z \) for \( z \in D(A_Y) \), we conclude that \( \lambda \in \rho(A_Y) \) and \( R(\lambda, A_Y) = R(\lambda, A)|Y \). Conversely, if \( \lambda \in \rho(A_Y) \), we apply the same argument to the spaces \([D(A_Y)] \hookrightarrow [D(A)] \hookrightarrow Y\) inferring that \( \rho(A_Y) \subset \rho(A_1) \), where \( A_1 \) is the part of \( A \) in \( D(A) \) which is the same as the part of \( A_Y \) in \( D(A) \). Finally, \( A_1 \) is similar to \( A \) via \( R(\omega, A) : X \rightarrow [D(A)] \) for \( \omega \in \rho(A) \), so that \( \rho(A_1) = \rho(A) \). (See an exercise in Spectral Theory.) \( \square \)

We note that we can apply the above lemma and the arguments of Corollary 4.3(b) also to conclude that the Laplacians \( \Delta \) in \( X = L^2 \) and \( Y = W^k_2 \) have the same spectrum (namely \( \mathbb{R} \) by Example 4.8 in [ST]).
In order to consider solutions of (4.1) taking values in \( W^1 \), we need to extend the Laplacian to this space. To that purpose, we define the negative Sobolev spaces on \( \mathbb{R}^d \) by
\[
W^{-k}_p(\mathbb{R}^d) := W^k_p(\mathbb{R}^d)^* = B(W^k_p(\mathbb{R}^d), C)
\]
for \( k \in \mathbb{N} \) and \( p \in (1, \infty) \). We write \( \langle v, \varphi \rangle = \varphi(v) \) for \( \varphi \in W^{-k}_p \) and \( v \in W^k_p \).

The norm on \( W^{-k}_p \) is denoted by \( \| \varphi \|_{-k,p} \). If \( k - d/p' > 0 \), we have \( W^k_p \hookrightarrow C_0 \) by Sobolev’s embedding, and thus point evaluations \( \varphi : u \mapsto u(x_0) \) belong to \( W^{-k}_p \) in this case. Let \( 1 \leq p < 2d/(d-2)_+ \). We then have \( W^1 \hookrightarrow L^p \) and hence
\[
L^p \hookrightarrow W^{-1}_2 \quad \text{with} \quad \langle v, \varphi_g \rangle = \int vg \, dx \quad \text{for} \quad g \in L^p' \text{,} \quad v \in W^1.
\]
(For \( d \neq 2 \), we could allow for \( p = 2d/(d-2)_+ \) here.) Since \( 2 < \alpha + 1 < 2d/(d-2)_+ \), the spaces \( L^2 \) and \( L^{(1+\alpha)_+} \) are thus embedded into \( W^{-1}_2 \) and functions from these spaces act on \( W^1 \) by integration. (Observe that \( (1 + \alpha)_+ = (1 + \alpha)/\alpha \).)

We will identify \( L^p \) in this way with a subspace of \( W^{-1}_2 \). But it is important to note that we do not identify \( W^k \) with its dual \( W^{-k}_p \) for \( k \in \mathbb{N} \) though it is a Hilbert space. The corresponding Riesz isomorphism is different from that of \( L^2 \), and we already identify \( L^2 \) and \( (L^2)^* \) as usual. Nevertheless, \( W^1 \) is a reflexive Banach space, and in this way we consider \( W^1 \) as the dual space of \( W^{-1}_2 \). We need the following lemma about dual semigroups, cf. p.87/88 in [EE].

**Lemma 4.5.** Let \( S(\cdot) \) be a \( C_0 \)-semigroup on a reflexive Banach space \( X \) with generator \( A \). Then the adjoints \( S(t)^* \), \( t \geq 0 \), yield a \( C_0 \)-semigroup on \( X^* \) which is generated by \( A^* \).

**Proof.** It is clear that \( S(0)^* = I \) and \( S(t)^* S(s)^* = S(t+s)^* \) for \( t, s \geq 0 \). We define the subspace \( X_0^* = \{ x^* \in X^* \mid S(t)^* x^* \to x^* \text{ as } t \to 0^+ \} \). Observing that \( \| S(t)^* \| = \| S(t) \| \) is bounded by a constant \( M \) for \( t \in [0, 1] \), it can be seen that \( X_0^* \) is closed in \( X^* \). Moreover, \( S(s)^* X_0^* \subset X_0^* \) for all \( s \geq 0 \) since \( S(t)^* S(s)^* = S(s)^* S(t)^* \). Lemma 1.5 in [EE] now shows that \( S(\cdot)^* \) is a \( C_0 \)-semigroup on \( X_0^* \).

To show that \( X_0^* = X^* \), let \( x^* \in D(A^*) \). Using Lemma 1.12 of [EE], we estimate
\[
| \langle x, S(t)^* x^* - x^* \rangle | = | \langle S(t)x - x, x^* \rangle | = \left| \left\langle A \int_0^t S(s)x \, ds, x^* \right\rangle \right|
\]
for all \( x \in X \) and \( t \in [0, 1] \). Taking the supremum over \( x \) with \( \| x \| \leq 1 \), it follows that \( x^* \in X_0^* \), and hence the closure of \( D(A^*) \) is contained in \( X_0^* \). Further, let \( \text{gr}(A) = \{ (x, Ax) \mid x \in D(A) \} \subset X \times X \) be the graph of \( A \). It is closed since \( A \) is closed, and thus \( \text{gr}(A) = \{(x, A^*x) \mid x \in D(A^*) \} \). On the other hand, the definition of an adjoint yields
\[
\text{gr}(A)^\perp = \{ (y^*, x^*) \in X^* \times X^* \mid \langle x, y^* \rangle = -\langle Ax, x^* \rangle, \quad \forall x \in D(A) \} = \{ (-A^*x^*, x^*) \mid x^* \in D(A^*) \} =: \text{gr}'(-A^*).
\]
Let \( y \in X \) satisfy \( 0 = \langle y, x^* \rangle \) for all \( x^* \in D(A^*) \). Then we have \( (0, y) \in \perp \text{gr}'(-A^*) = \text{gr}(A) \) which implies that \( y = 0 \). Since \( X \) is reflexive, this means that \( D(A^*) \) is dense in \( X^* \) and therefore \( X_0 = X^* \).

Let \( B \) generate \( S(\cdot)^* \) on \( X^* \). Let \( x^* \in D(A^*) \). We compute as above

\[
\langle x, S(t)x^* - x^* \rangle = \left\langle A \int_0^t S(s)x \, ds, x^* \right\rangle = \left\langle \int_0^t S(s)x, A^*x^* \right\rangle = \int_0^t \langle S(s)x, A^*x^* \rangle \, ds = \left\langle x, \int_0^t S(s)^*A^*x^* \, ds \right\rangle
\]

for every \( x \in X \). The strong continuity of \( S(\cdot)^* \) thus yields

\[
\frac{1}{t}(S(t)x^* - x^*) = \frac{1}{t} \int_0^t S(s)^*A^*x^* \, ds \to A^*x^*
\]
as \( t \to 0^+ \), so that \( A^* \subset B \). Since \( \sigma(B) \) and \( \sigma(A^*) = \sigma(A) \) are contained in left halfplanes, \( B \) and \( A^* \) possess a common point in their resolvent sets so that \( B = A^* \).

In the context of this lemma, we remark that on nonreflexive \( X \) the adjoint operators \( S(t)^* \in \mathcal{B}(X^*) \) still form a semigroup such that \( t \to \langle x, S(t)^*x^* \rangle \) is continuous for all \( x \in X \) and \( x^* \in X^* \), but this semigroup may fail to be strongly continuous. This happens, e.g., for the left translation semigroup \( S(t)f = f(\cdot + t) \) on \( X = L^1(\mathbb{R}) \) where \( S(t)^* \) is the right translation on \( L^\infty(\mathbb{R}) \), see Example 1.6 in [EE]

We now apply the above lemma to the free Schrödinger group \( T_1(\cdot) \) on \( W_2^1 \) with its generator \( i\Delta_1 \) as obtained in Corollary 4.3. The adjoints yield a \( C_0 \)-group on \( W_2^{-1} \) with generator \( (i\Delta_1)^* = i\Delta_1^* \). It is unitary since it still consists of contractions, see the proof of Corollary 1.35 of [EE]. The next lemma gives more precise information about this group and its generator.

**Lemma 4.6.** The unitary \( C_0 \)-group \( T_1(\cdot)^* \) on \( W_2^{-1} \) extends the free Schrödinger group \( T(\cdot) \) on \( L^2 \). Its generator \( i\Delta_1^* \) is given by the weak Laplacian, i.e.,

\[
D(\Delta_1^*) = \{ \varphi \in W_2^1 \mid \exists \psi \in W_2^{-1} : \langle v, \psi \rangle = -\int \nabla v \cdot \nabla \varphi \, dx, \; \forall v \in W_2^2 \},
\]

\[
\Delta_1^* \varphi = \psi.
\]

(We thus write \( T(t) \) instead of \( T_1(t)^* \) and \( \Delta \) instead of \( \Delta_1 \).)

**Proof.** For \( v \in W_2^1 \cap C_c, \varphi \in C_c \hookrightarrow W_2^{-1} \) and \( t \neq 0 \), Lemma 4.2 and Fubini’s theorem yield

\[
\langle T_1(t)v, \varphi \rangle = \int \int K_t(x-y)v(y)\varphi(x) \, dy \, dx = \int \int K_t(y-x)\varphi(x)v(y) \, dx \, dy = \langle v, T(t)\varphi \rangle.
\]

By approximation, we deduce \( \langle T_1(t)v, \varphi \rangle = \langle v, T(t)\varphi \rangle \) for all \( v \in W_2^2 \) and \( \varphi \in L^2 \) which means that \( T_1(t)^*L^2 = T(t) \).

For a fixed \( \varphi \in W_2^2 \) and all \( v \in W_2^1 \), we set \( \psi(v) = -\int \nabla v \cdot \nabla \varphi \, dx \). Clearly, \( \psi \in (W_2^2)^* = W_2^{-1} \) with norm less or equal \( \| \cdot \|_{1,2} \). So we can define \( \Delta \in \)
\( \mathcal{B}(W^1_2, W^{-1}_2) \) by \( \Delta \varphi = \psi \). For \( v \in D(\Delta_1) \subset W^2_2 \), an integration by parts yields
\[
\langle \Delta v, \varphi \rangle = \int \varphi \Delta v \, dx = -\int \nabla v \cdot \nabla \varphi \, dx = \langle v, \Delta \varphi \rangle;
\]
i.e., \( \Delta \subset \Delta^*_1 \). On the other hand, the Lax–Milgram lemma (see Theorem 1.40 of [EE]) gives for each \( \psi \in (W^2_2)^* \) a function \( \varphi \in W^2_2 \) such that
\[
a(v, \varphi) := \langle v, \varphi \rangle + \langle \nabla v, \nabla \varphi \rangle = \psi(v)
\]
for all \( v \in W^1_2 \) and the scalar products of \( L^2 \) and \( (L^2)^d \). This means that \( I - \Delta : W^1_2 \to W^{-1}_2 \) is surjective. Finally, Theorem 1.24 and Example 4.8 of [ST] and Lemma 4.4 imply that the spectrum of \( \Delta^*_1 \) is the same as Laplacian on \( L^2 \), namely \( \sigma(\Delta^*_1) = \mathbb{R}_- \). Consequently, \( \Delta^*_1 = \Delta \). \( \square \)

Since \( \alpha + 1 < 2d/(d - 2)_+ \), we have \( W^1_2 \hookrightarrow L^{\alpha + 1} \) so that \( |v|^{\alpha - 1} v \) belongs to \( L^{(1+\alpha)/\alpha} \) for \( v \in W^1_2 \). As observed after (4.4), it follows that \( |v|^\alpha v \in W^2_2 \) and that \( |v|^\alpha v \) acts on \( W^1_2 \) via integration.\(^1\)

**Definition 4.7.** A function \( u \in C^1(J,W^1_2) \cap C(J,W^2_2) \) satisfying (4.1) in \( W^2_2 \) is called a \( W^2_2 \)-solution of (4.1). This means that
\[
\langle v, i\partial_t u(t) \rangle = \langle v, -\Delta u(t) + \mu |u(t)|^{\alpha - 1} u(t) \rangle = \int (\nabla u(t) \cdot \nabla v + \mu |u(t)|^{\alpha - 1} u(t)v) \, dx
\]
for all \( v \in W^1_2 \) and \( t \in J \).

When solving (4.1), we want to proceed as in Chapter 3. However, we now cannot use the interpolation spaces of \( \Delta \) since the linear semigroup \( T(\cdot) \) does not improve regularity in the full state space \( L^2 \) or \( W^1_2 \). On the other hand, we have a special nonlinearity which can be controlled by suitable \( L^p \) norms. Corollary 4.3 already says that \( T(t) \) improves integrability, though the corresponding norms blow up as \( t \to 0 \). Using also Lebesgue spaces in time, one can describe this dispersive behavior in a more convenient way, and it is possible to deal also with inhomogeneities. The next theorem is crucial for the following sections.

We refer to p.47 of [EE] for a few remarks about Banach–space valued integration. The basic results (up to Fubini’s theorem) are analogous to the scalar–valued case. Moreover, for an interval \( J \subset \mathbb{R} \) and \( 1 \leq p, q < \infty \) we have
\[
L^q(J) \subset L^p(J)
\]
via
\[
\langle f, g \rangle_{L^q(J),L^p(J)} = \int_J \langle f(t), g(t) \rangle_{L^p} \, dt
\]
for \( f \in L^q(J, L^p) \) and \( g \in L^q(J, L^p) \), see Theorems 8.20.3 and 8.20.5 in [Ed]. The space \( L^q(J, L^p) \) is thus reflexive for \( p, q \in (1, \infty) \). One can show that \( L^p(J, L^p) \) is separable if \( p, q \in (1, \infty) \), using the density of simple functions \( u : J \to L^p \).

Let \( k \in \mathbb{N} \). By means of cutoffs and mollifiers over \( \mathbb{R}^d \), we can approximate \( f \in L^q(J, L^p) \) by \( g \in L^q(J, W^k \cap L^1) \).\(^2\) We can also approximate \( g \in L^q(J, W^k \cap L^1) \) by \( h \in C_c(J, W^k \cap L^1) \) as in the scalar case. (See Proposition 4.13 in [FA].) As a result, \( C_c(J, W^k \cap L^1) \) is dense in \( L^q(J, L^p) \).

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\(^1\)Again, one could allow for \( \alpha = (d + 2)/(d - 2)_+ \) for \( d \neq 2 \).

\(^2\)The intersection of Banach spaces \( X \) and \( Y \) is a Banach space for the norm \( \|x\|_X + \|y\|_Y \).
Exponents \((q,p)\) are called **admissible** (for the Schrödinger equation) if

\[
2 \leq p,q \leq \infty, \quad \frac{2}{q} + \frac{d}{p} = \frac{d}{2} \quad \text{and} \quad (q,p) \neq (2,\infty) \quad \text{if} \quad d = 2.
\]

Observe that \((\infty,2)\) is admissible (this will be the trivial case) and \((2,2d/(d-2))\) for \(d \geq 3\) is admissible (the endpoint case). The other admissible exponents \((q,p)\) belong to \((2,\infty) \times (2,2d/(d-2))\). In particular, \(L^{q'} \hookrightarrow W_2^{-1}\) for admissible \((q,p)\) due to (4.4). For a locally integrable \(f : \mathbb{R} \rightarrow W_2^{-1}\), we set

\[
(T_{*+}f)(t) = \int_0^t T(t-s)f(s)\,ds, \quad t \in \mathbb{R},
\]

where the integral is defined in \(W_2^{-1}\).

**Theorem 4.8 (Strichartz’ estimates).** Let \((q,p)\) and \((\overline{q},\overline{p})\) be admissible, \(k \in \mathbb{N}_0\), \(\varphi \in W_k^2\), \(J \subset \mathbb{R}\), and \(f \in L^\overline{q}(J,W_k^\overline{p})\). Then \(T_{*+}f(t)\) exists in \(W_k^p\) for a.e. \(t \in J\) and there is a constant \(c > 0\) (independent of \(\varphi\), \(f\) and \(J\)) such that

(a) \(|T(\cdot)\varphi|_{L^p(J,W_k^p)} \leq c\|\varphi\|_{k,2},

(b) \|T_{*+}f\|_{L^p(J,W_k^p)} \leq c\|f\|_{L^{\overline{q}'}(J,W_\overline{p}^\overline{q})}.

If \(q = \infty\) and \(p = 2\), we can replace \(L^\infty\) by \(C_0\).

Compared to the \(L^2\)-setting, one gains space integrability from \(p = 2\) to \(p > 2\), but one loses time integrability from \(q = \infty\) to \(q < \infty\). Moreover, in (b) the exponents on the right-hand side are smaller than 2, whereas they are larger than 2 on the left-hand side. Part (a) is wrong for non-admissible \((q,p)\), whereas part (b) is true for some non-admissible exponents, see §2.4 of [Ca]. The theorem was proved by various authors in the case \(q > 2\), \(\overline{q} > 2\) starting with Strichartz in the seventies, see §2.3 of [Ca]. The much more difficult endpoint case \(q = 2\), \(\overline{q} = 2\) was established by Keel and Tao in [KT].

For \(q > 2\), the **homogeneous** Strichartz’ estimate (a) follows without many difficulties from Corollary 4.3. The same is true for the **inhomogeneous** estimate (b) if \((q,p) = (\overline{q},\overline{p})\) and \(q > 2\) or if \((q,p) = (\infty,2)\) and \(\overline{q} > 2\). Since we just need these two cases later on, we only prove them.

**Proof of 4.8** if \(q,\overline{q} > 2\) and either \((q,p) = (\overline{q},\overline{p})\) or \((q,p) = (\infty,2)\).

Using Corollary 4.3, we can reduce the case \(k \in \mathbb{N}\) to the case \(k = 0\). For part (a) this is clear. For part (b), let \(k = 1\) and take \(m \in \mathbb{N}\) such that \(W_2^m \hookrightarrow W_1^1\). We approximate \(f \in L^\overline{q}(J,W_\overline{p}^\overline{q})\) by functions \(g \in C_c(J,W_2^m \cap L^{p'})\) in \(L^\overline{q}(J,W_\overline{p}^1)\). In Step 1 below we see that then (a subsequence of)

\[
\int_0^t T(t-s)\partial_{x_i} f(s)\,ds \text{ converges to } \int_0^t T(t-s)\partial_{x_i} g(s)\,ds \text{ in } L^p\]

for \(j \in \{1,\ldots,d\}\), \(i = 0,1\), and a.e. \(t \in J\). Since we further have \(T_{*+}(\partial_{x_i} g) = \partial_{x_i} T_{*+} g\) and \(T_{*+} g(t) \in W_2^m \hookrightarrow W_1^1\), it follows that \(T_{*+} f(t) \in W_1^1\) for a.e. \(t \in J\) and \(T_{*+}(\partial_{x_i} f) = \partial_{x_i} T_{*+} f\). Higher order derivatives are treated by induction. By restriction and extension by 0, the result for general \(J\) follows from the result for \(J = \mathbb{R}\). Moreover, the theorem is clear for \((q,p) = (\infty,2) = (\overline{q},\overline{p})\) since \(T(\cdot)\) is a unitary \(C_0\)-group on \(L^2\). So let \(\varphi \in L^2\), \(f \in L^\overline{q}'(\mathbb{R},L^p)\) and \(q \in (2,\infty)\).

1) Let \(g \in C_c(\mathbb{R},W_2^k \cap L^{p'})\), where \(k \in \mathbb{N}\) is chosen such that \(W_2^k \hookrightarrow L^p\). The map \((t,s) \mapsto T(t-s)g(s)\) is continuous from \(\mathbb{R}^2\) to \(W_2^k \hookrightarrow L^p\). Approximating
Corollary 4.3 further yields

$$\|T(t-s)f(s)\|_p \leq c|t-s|^{\frac{d}{p} - \frac{d}{q'}} \|f(s)\|_{q'} = c|t-s|^{\frac{d}{p} - \frac{d}{q}} \|f(s)\|_p$$

for $t > s \geq 0$. Here and below the constants only depend on the exponents and $d$. The Hardy–Littlewood–Sobolev inequality (see Examples 2 and 3 in Sections IX.4 of [RS]) then implies that

$$\|T \ast_+ f\|_{L^q, L^{q'}} = \left[ \int_\mathbb{R} \left( \int_0^t \|T(t-s)f(s)\|_p ds \right)^{q' \frac{q}{q'} \frac{d}{p}} \right]^{\frac{1}{q'}} \leq c \left[ \int_\mathbb{R} \left( \int_0^t |t-s|^{\frac{d}{p} - \frac{d}{q'}} \|f(s)\|_{q'} ds \right)^{q' \frac{q}{q'} \frac{d}{p}} \right]^{\frac{1}{q'}} \leq c \|f\|_{L^r, L^{r'}}.$$

(Here we need that the exponents are admissible and $1 < q' < q < \infty$.) It actually follows from Fubini’s theorem and the second estimate above that $(T \ast_+ f)(t)$ is defined in $L^p$ for a.e. $t \in \mathbb{R}$ and that $T \ast_+ f : \mathbb{R} \to L^p$ is strongly measurable, cf. the text before Theorem 2.14 in [FA]. Observe that in the same way we can prove the above estimate for the usual convolution $T \ast f$.

2) To show (a), we first consider $g \in C_c(\mathbb{R}, L^2 \cap L^{q'})$. Formula (4.5) and Step 1) yield

$$J := \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} T(t-s)g(s) ds \overline{g(t)} dx dt = \langle T \ast g, \overline{g} \rangle_{L^q, L^{q'}}$$

$$|J| \leq \|T \ast g\|_{L^q, L^{q'}} \|g\|_{L^r, L^{r'}} \leq c \|g\|_{L^r, L^{r'}}^2.$$

From Fubini’s theorem we further deduce

$$J = \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} (T(t-s)g(s)|g(t)|)_{L^2} ds dt = \int_\mathbb{R} \int_\mathbb{R} (T(-s)g(s)|T(-t)g(t)|)_{L^2} ds dt$$

$$= \left( \int_\mathbb{R} T(-s)g(s) ds \right) \left( \int_\mathbb{R} T(-s)g(s) ds \right)_{L^2} = \left\| \int_\mathbb{R} T(-s)g(s) ds \right\|_2^2$$

where all integrals are just $L^2$–valued Riemann integrals. We have thus shown

$$\left\| \int_\mathbb{R} T(-t)g(t) dt \right\|_2 \leq c \|g\|_{L^r, L^{r'}}$$

(4.6)

for $g \in C_c(\mathbb{R}, L^2 \cap L^{q'})$. This estimate gives

$$|\langle T \cdot \varphi, \overline{g} \rangle_{L^q, L^{q'}}| = \left| \int_\mathbb{R} (T(t)\varphi|g(t)|)_{L^2} dt \right| = \left| \int_\mathbb{R} (\varphi|T(-t)g(t)|)_{L^2} dt \right|$$

$$= \left\| \varphi \right\|_2 \left\| g \right\|_{L^{r'}, L^{r'}}.$$
where we also used (4.5) and that the duality pairings $L^2 \times L^2$ and $L^p \times L^{p'}$ coincide on the intersection of the spaces. Since $C_c(\mathbb{R}, L^2 \cap L^{p'})$ is dense in $L^q(\mathbb{R}, L^p)$, it follows from (4.5) that

$$\|T(\cdot)\varphi\|_{L^q(\mathbb{R}, L^p)} = \sup_{\|g\|_{L^{p'}(\mathbb{R}, L^q)}} |\langle T(\cdot)\varphi, \overline{g} \rangle|_{L^q(\mathbb{R}, L^p)} \leq c \|\varphi\|_2.$$  

3) To prove (b) for the exponents $(\infty, 2)$ and $(\overline{p}, \overline{q}) := (p, q)$, we set $f_t = \mathbb{1}_{[0,t]} f$ for $t > 0$ and $f_t = \mathbb{1}_{[t,0]} f$ for $t < 0$. First let $f \in C_c(\mathbb{R}, L^2 \cap L^{p'})$. Using that $T(t)$ is isometric on $L^2$ and inequality (4.6), we compute

$$\|T * f\|_{C_b(\mathbb{R}, L^2)} = \sup_{t \in \mathbb{R}} \left\| \int_0^t T(t)T(-s)f_t(s)\, ds \right\|_2 = \sup_{t \in \mathbb{R}} \left\| \int_0^t T(-s)f_t(s)\, ds \right\|_2 \leq \sup_{t \in \mathbb{R}} c \|f_t\|_{L^p(\mathbb{R}, L^{p'})} = c \|f\|_{L^p(\mathbb{R}, L^{p'})}.$$

If we approximate the given $f$ in $L^q(\mathbb{R}, L^{p'})$ by $f_n \in C_c(\mathbb{R}, L^2 \cap L^{p'})$, then the above estimate shows that $(T * f_n)(t)$ converges to a function $u$ in $C_b(\mathbb{R}, L^2)$. On the other hand, Step 1) implies that a subsequence $(T * f_n)(t)$ tends to $(T * f)(t)$ in $L^p$ for a.e. $t \in \mathbb{R}$. As a result, $T * f$ belongs to $C_b(\mathbb{R}, L^2)$ and (b) also holds in the present case.

\[ \square \]

2. Local wellposedness

In this section we establish the local wellposedness theory of the semilinear problem (4.1). The strategy of the proofs goes back to Kato. It is similar to the approach in Section 3.1 in the parabolic case. However, the smoothing effect of analytic semigroups is now replaced by Strichartz’ estimates, and many of the arguments are more sophisticated. There is also a local wellposedness theory in the critical case $\alpha = (d + 2)/(d - 2)$ if $d \geq 3$, see Theorem 4.5.1 in [Ca]. But here one obtains a much less convenient blow up condition, and one needs the endpoint case of Strichartz’ estimates.

**Theorem 4.9.** Let $1 < \alpha < \frac{d + 2}{2d/(d - 2)}$, $r > 0$ and $u_0 \in W^1_2$ with $\|u_0\|_{L^r} \leq r$. Take $q > 2$ with $\frac{2}{q} + \frac{d}{\alpha + 1} = \frac{d}{2}$. Then the following assertions hold.

(a) There are numbers $0 < b_0(r) < b_\pm(u_0) \leq \infty$ and a unique $W^1_2$-solution $u = u(\cdot; u_0)$ of (4.1) on $J(u_0) := (-b_-(u_0), b_+(u_0))$.

(b) Let $[a, b] \subset J(u_0)$. Then $u \in L^q([a, b], W^1_{\alpha + 1})$.

(c) Let $0 \in [a, b] \subset J(u_0)$ and $\varphi_n \to u_0$ in $W^1_2$. Then there is an index $n_0 \in \mathbb{N}$ such that $[a, b] \subset J(\varphi_n)$ for all $n \geq n_0$ and $u(\cdot; \varphi_n) \to u$ in $C([a, b], W^1_2) \cap L^q([a, b], W^1_{\alpha + 1})$ as $n \to \infty$.

(d) If $b_\pm(u_0) < \infty$, then $\limsup_{t \to \pm b_\pm(u_0)} \|u(t)\|_{L^r} = \infty$.

**Proof.** We only consider dimensions $d \geq 2$. The easier case $d = 1$ can be treated similarly. Let $p := \alpha + 1 \in (2, 2d/(d - 2))$, $q > 2$ with $\frac{2}{q} + \frac{d}{p} = \frac{d}{2}$, and $F(v) := -i v |v|^{\alpha - 1} v$ for $v : \mathbb{R}^d \to \mathbb{C}$. In particular, $(q, p)$ is admissible. Without further notice, we use Sobolev’s embedding $W^1_2 \hookrightarrow L^p$.

**Step 1: Preparations.** We put $\phi(z) = |z|^{\alpha - 1} z = (x^2 + y^2)^{(\alpha - 1)/2}(x, y)$ for $z = (x, y) \in \mathbb{R}^2 \cong \mathbb{C}$. It easy to check that $|\phi'(z)| \leq c |z|^{\alpha - 1}$ for a constant
\(c > 0\) and all \(z \in \mathbb{C}\). It follows that
\[
|\phi(z) - \phi(w)| \leq \sup_{t \in [0,1]} |\phi'(1-t)z + tw| |z - w| \leq c(|z| + |w|)^{1-1} |z - w|
\]
for all \(z, w \in \mathbb{C}\). Let \(\varphi, \psi \in L^p\). Using Hölder’s inequality with exponents
\[
\frac{1}{p} = \frac{\alpha}{\alpha + 1} = \frac{\alpha - 1}{\alpha + 1} + \frac{1}{\alpha + 1} + \frac{1}{p},
\]
we then deduce
\[
\|F(\varphi) - F(\psi)\|_p \leq c \left(\|\varphi\|^{\alpha-1} + \|\psi\|^{\alpha-1}\right) \|\varphi - \psi\|_p
\]
\[
\leq c \left(\|\varphi\|^{\alpha-1} \|\varphi\|^{\alpha - 1} + \|\psi\|^{\alpha-1} \|\varphi\|^{\alpha - 1}\right) \|\varphi - \psi\|_p
\]
\[
= c \left(\|\varphi\|^{\alpha-1} + \|\psi\|^{\alpha-1}\right) \|\varphi - \psi\|_p. \tag{4.7}
\]
For \(\varphi \in W^2_2\) and \(k \in \{1, \ldots, d\}\), we see in a similar way that
\[
\|\partial_k F(\varphi)\|_p = \|\varphi' \partial_k \varphi\|_p \leq c \|\varphi\|^{\alpha-1} \|\partial_k \varphi\|_p \leq c \|\varphi\|^{\alpha-1} \|\partial_k \varphi\|_p. \tag{4.8}
\]
Here the constants \(c > 0\) only depend on \(\alpha\) and \(d\).

For \(T > 0\) and \(k = 0, 1\) we define the spaces for the fixed point argument by
\[
E_k = E_k(T) = L^\infty((-T,T), W^k_2) \cap L^q((-T,T), W^k_1) \quad \text{with norm}
\|
\| u \|_{k,T} = \max\{\|u\|_{L^\infty((-T,T), W^k_2)}, \|u\|_{L^q((-T,T), W^k_1)}\}
\}
\]
Analogously, we define \(E_k(J)\) for intervals \(J \subset \mathbb{R}\). For \(R > 0\) we introduce the set \(\Sigma = \Sigma(R, T) = \overline{E_0(T)}(0, R)\) and endow it with the metric \(\|u - v\|_{0,T}\). We claim that \(\Sigma\) is complete (though the ball is defined by a stronger norm!).

**Proof.** Let \((u_n)\) be a Cauchy sequence in \(\Sigma\). There is a limit \(u\) of \(u_n\) in \(E_0(T)\). We have to show that \(u \in \Sigma\). As indicated after (4.5) we know that \(L^\infty((-T,T), W^2_2) = (L^1((-T,T), W^{-1}_2))^*\) and that \(L^q((-T,T), W^1_1)\) is reflexive with dual \(L^q((-T,T), W^{-1}_1)\). The Banach–Alaoglu theorem (see Theorems 5.38 and 5.40 in [FA1]) then gives a subsequence as well as \(v \in L^\infty((-T,T), W^2_1)\) and \(w \in L^q((-T,T), W^1_1)\) such that \(u_n\) converges weakly* to \(v\) in \(L^\infty((-T,T), W^2_2)\) and weakly to \(w\) in \(L^q((-T,T), W^1_1)\) as \(j \to \infty\). As weak* and weak limits, \(v\) and \(w\) have norms less or equal to \(R\) in the respective norms. Moreover, \(u_n\) tends weakly* to \(u\) and \(v\) in \(L^\infty((-T,T), L^2)\) as well as weakly to \(u\) and \(w\) in \(L^q((-T,T), L^p)\). Since weak* and weak limits are unique, it follows that \(u = v = w\). Consequently, \(u\) belongs to \(E_1(T)\) and \(\|u\|_{1,T} \leq R\).

Let \(u, v \in \Sigma\) and \(J = (-T,T)\). Observe that then \(\|w(t)\|_p \leq c\|w(t)\|_{1,2} \leq cR\) for all \(t \in J\) and \(w = u, v\). Estimates (4.7), (4.8) and Hölder’s inequality yield
\[
\|F(u) - F(v)\|_{L^q(J,L^{p'})} \leq c \left[\int_{-T}^T \left(\|u(t)\|^{\alpha-1}_p + \|v(t)\|^{\alpha-1}_p\right)^q \|u(t) - v(t)\|_q^2 \, dt\right]^{\frac{1}{q}}
\]
\[
\leq cR^{\alpha-1} \|u - v\|_{L^q(J,L^p)}, \tag{4.9}
\]
\[
\|F(u) - F(v)\|_{L^{q'}(J,L^{p})} \leq cT^{\frac{q}{q'} - \frac{1}{q}} \|F(u) - F(v)\|_{L^q(J,L^{p'})}
\]
\[
\leq cR^{\alpha-1} T^{\frac{1}{q'} - \frac{1}{q}} \|u - v\|_{L^q(J,L^p)}, \tag{4.10}
\]

55
Due to (4.9) and (4.4), we have
\[ q \]
\[ \text{is defined in } q \]
\[ \text{and} \]
\[ (\text{4.10}) \]
\[ \text{combined with Theorem 4.8} \]
\[ \text{yield } \Phi(\cdot) \]
\[ \text{on intervals } [a,b] \]
\[ \text{C} \]
\[ \rightarrow \]
\[ \text{Therefore, } \Phi : \Sigma \]
\[ \text{where the constants } c \]
\[ \text{for } t,s \]
\[ \text{C} \]
\[ \rightarrow \]
\[ \text{R} \]
\[ \text{and} \]
\[ u \]
\[ \text{for constants only depending on } \alpha \text{ and } d. \]
\[ \text{We point out that } \frac{1}{q'} - \frac{1}{q} > 0. \]

**Step 2: Existence.** Let \( u, v \in \Sigma(T, R) \) for some \( T, R > 0 \) to be fixed below, and \( u_0 \in W^1_2 \) with \( \|u_0\|_{1,2} \leq r \). Using the free Schrödinger group \( T(\cdot) \), we define the map
\[ [\Phi_{u_0}(u)](t) = [\Phi(u)](t) := T(t)u_0 + \int_0^t T(t-s)F(u(s)) \, ds, \quad t \in J = [-T, T]. \]

Due to (4.9) and (4.4), we have \( F(u) \in L^{q'}(J, W^{-1}_2) \) so that the above integral is defined in \( W^{-1}_2 \). In the following we apply Strichartz’ estimates with the exponents \((q, p)\) and \((2, \infty)\) on the left–hand side, using on the right–hand side of part (b) in both cases the exponents \((q', p')\). The inequalities (4.12) and (4.10) combined with Theorem 4.8 thus yield \( \Phi(u) \in E_1(T) \cap C([-T, T], W^1_2) \) and
\[ ||\Phi u||_{1, 1, T} \leq c (\|u_0\|_{1, 2} + ||F(u)||_{L^{q'}(J, W^{-1}_2)}) \leq c_1 \|u_0\|_{1, 2} + c_2 R^\alpha T^\frac{1}{q'} - \frac{1}{q}, \]  \tag{4.13}
\[ ||\Phi(u - v)||_{0, 0, T} \leq c ||F(u) - F(v)||_{L^{q'}(J, L^p)} \leq c_3 R^\alpha T^\frac{1}{q'} - \frac{1}{q} \|u - v\|_{0, 0, T}, \]  \tag{4.14}
where the constants \( c_j > 0 \) do not depend on \( u, v, u_0, T \) or \( R \). We choose \( R = 2c_1r \) and \( b_0(r) > 0 \) with
\[ b_0(r)T^\frac{1}{q'} - \frac{1}{q} R^\alpha - 1 \leq \min\{(2c_2)^{-1}, (2c_3)^{-1}\} \]

For each \( T \in (0, b_0(r)] \), it follows that
\[ ||\Phi u||_{1, 1, T} \leq \frac{R}{2} + \frac{R}{2} = R \quad \text{and} \quad ||\Phi(u - v)||_{0, 0, T} \leq \frac{1}{2} ||u - v||_{0, 0, T}. \]
Therefore, \( \Phi : \Sigma \to \Sigma \) has a unique fixed point \( u = \Phi(u) \) belonging to \( E_1(T) \cap C([-T, T], W^1_2) \). Estimate (4.7) further yields
\[ ||F(u(t)) - F(u(s))||_{p'} \leq c (\|u(t)||^p_{p'} + \|u(s)||^p_{p'} ||u(t) - u(s)||_{p} \leq c R^\alpha - 1 \|u(t) - u(s)||_{1, 2} \]
for \( t, s \in [-T, T] \). Since \( L^p' \hookrightarrow W^{-1}_2 \) by (4.4), it follows that \( F(u) \in C([-T, T], W^{-1}_2) \). Lemma 2.8 of [EE] and \( u \in C([-T, T], W^1_2) \) now imply that \( u \) is a \( W^1_2 \)-solution of (4.1).

**Step 3: Blow-up condition and uniqueness.** Consider \( W^1_2 \)-solutions \( u \) and \( v \) of (4.1) on intervals \([a, b]\) and \([b, c]\), respectively, satisfying \( u(b) = v(b) \). These functions can be glued together to a \( W^1_2 \)-solution on \([a, c]\) since \( u'(b) = i\Delta u(b) + F(u(b)) = v'(b) \in W^{-1}_2 \). Let \( u \) and \( v \) be \( W^1_2 \)-solutions of (4.1) on an interval \([a, b]\) with the same initial value \( u(a) = v(a) \). If they do not coincide, there are
$t_0 \geq a$ and $t_n \to t_0^+$ such that $u(t_0) = v(t_0)$ and $u(t_n) \neq v(t_n)$ for all $n \in \mathbb{N}$. Set $r' = \|u(t_0)\|_{1,2}$ and $R' = (2c_1 + 1)r'$. For a sufficiently small $T \in (0, b_0(r'))$ the functions $u(\cdot + t_0)$ and $v(\cdot + t_0)$ belong to $\Sigma(R', T)$. So Step 2 yields that $u(\cdot + t_0) = v(\cdot + t_0)$ on $[0, T]$ which is impossible. The case that $u(b) = v(b)$ is treated in the same way. Hence, solutions are unique.

We define $b_+(u_0)$, resp. $b_-(u_0)$, as the supremum of $b \geq b_0(\|u_0\|_{1,2})$ such that there is a $W_2^1$–solution of (4.1) on $[0, b]$ resp. on $[-b, 0]$. Since we can restart the problem with initial value $v(b_0(\|u_0\|_{1,2}))$, Step 2 yields that $b_+(u_0) > b_0(\|u_0\|_{1,2})$. Similarly, if $b_+(u_0)$ were finite and $\|u(t)\|_{1,2}$ is bounded by some $r_1$ for $t \in [0, b_+(u_0))$, we can find a $t_1 < b_+(u_0)$ such that $t_1 + b_0(r_1) > b_+(u_0)$ which contradicts the definition on $b_+(u_0)$. Negative times are treated similarly. We have thus shown assertions (a) and (d).

For any $t_0 \in J(u_0)$ Step 2 gives a $\delta > 0$ such that on $J' = [t_0 - \delta, t_0 + \delta]$ the solution $u$ is given as the fixed point constructed above. Hence, $u \in E_1(J')$ and by compactness statement (b) follows.

**Step 4: Continuous dependence.** Let $0 \in [a, b] \subset J(u_0)$ and $\varphi_n \to u_0$ in $W_2^1$ as $n \to \infty$. We set $u_n = u(\cdot; \varphi_n)$ for $n \in \mathbb{N}$ and $C_0 = \sup_{t \in [0, b]} \|u(t)\|_{1,2}$. Let $R_0 = 4c_1C_0$. There is an $n_0 \in \mathbb{N}$ such that $\|\varphi_n\|_{1,2} \leq 2C_0$ for all $n \geq n_0$. Step 2 with $R_0$ instead of $R$ then yields that $b_+(\varphi_n) \geq b_0(2C_0) =: b_0$ and that $u = \Phi_{u_0}(u)$ and $u_n = \Phi_{\varphi_n}(u_n)$ on $[-b_0, b_0]$. Moreover,

$$\sup_{t \in [-b_0, b_0]} \|u(t)\|_{1,2} \leq R_0, \sup_{t \in [-b_0, b_0]} \|u_n(t)\|_{1,2} \leq R_0$$

for all $n \geq n_0$. At first, Strichartz’ estimate in Theorem 4.8(a) and the strict contractivity of $\Phi_{u_0}$ imply

$$\|u - u_n\|_{0, b_0} \leq \|\Phi_{u_0}(u) - \Phi_{u_0}(u_n)\|_{0, b_0} + \|\Phi_{u_0}(u_n) - \Phi_{\varphi_n}(u_n)\|_{0, b_0} \leq \frac{1}{2} \|u - u_n\|_{0, b_0} + c\|u_0 - \varphi_n\|_2,$$

$$\|u - u_n\|_{0, b_0} \leq 2c\|u_0 - \varphi_n\|_2$$

for all $n \geq n_0$. After passing to a subsequence, we conclude that $u_n(t) \to u(t)$ in $L^p$ for a.e. $t \in [-b_0, b_0]$ as $n \to \infty$. Let $T \in (0, b_0)$ and $t \in J := [-T, T]$. Using also (4.15), as in (4.7)–(4.10) we compute that

$$\|\nabla F(u_n(t)) - \nabla F(u(t))\| \leq \|\phi'(u_n(t))\|_p \|\nabla(u_n(t) - u(t))\|_p + \|\phi'(u_n(t)) - \phi'(u(t))\| \|\nabla(u(t))\|_p$$

$$\leq c\|u_n(t)\|^p_{L^p} \|\nabla(u_n(t) - u(t))\|_p + \|\phi'(u_n(t)) - \phi'(u(t))\| \|\nabla(u(t))\|_p \leq cR_0^{-1} \|u_n(t) - u(t)\|_p + \|\phi'(u_n(t)) - \phi'(u(t))\| \|\nabla(u(t))\|_p \leq cR_0^{-1} \|u_n(t) - u(t)\|_p + \|\phi'(u_n(t)) - \phi'(u(t))\| \|\nabla(u(t))\|_p$$

$$\|\nabla F(u_n(t)) - \nabla F(u(t))\|_{L^p(J')},$$

$$\|\nabla F(u_n(t)) - \nabla F(u(t))\|_{L^p(J')},$$

$$\|\nabla F(u_n(t)) - \nabla F(u(t))\|_{L^p(J')},$$

$$\|\nabla F(u_n(t)) - \nabla F(u(t))\|_{L^p(J')},$$

To exploit this estimate, we digress a little bit. Let $v_n \to v$ in $L^p$ and suppose that $\phi'(v_n)$ does not converge to $\phi'(v)$ in $L^r$, where $r = \frac{q+1}{\alpha+1}$. Then there is
a subsequence such that \( \|\phi'(v_{n_m}) - \phi'(v)\|_r \geq \delta > 0 \) for all \( m \) and \( v_{n_m} \) tends to \( v \) a.e. as \( m \to \infty \) and \( |v_{n_m}| \leq w \) for all \( m \in \mathbb{N} \) and a function \( w \in L^p \). It follows that \( \phi'(v_{n_m}) \) tends to \( \phi'(v) \) a.e. and that it is bounded by \( cw^{a-1} \in L^r \).

Lebesgue’s theorem leads to a contradiction.

This means that \( \|\phi'(u_n(t)) - \phi'(u(t))\|_r \) tends to 0 for a.e. \( t \in J \). Moreover, it is bounded by \( c(\|u_n(t)\|_p^{-1} + \|u(t)\|_p^{-1}) \leq cR_0^{-1} \) for all \( t \in J \) by (4.15), where \( c \) does not depend on \( n \) or \( t \). Since \( u \in L^q(J, W^1_p) \hookrightarrow L^q(J, W^1_p) \), the last summand in (4.17) tends to 0 as \( n \to \infty \) by dominated convergence. We now use that \( u_n - u = T(\cdot)(\varphi_n - u_0) + T^+_*(F(u_n) - F(u)) \). Choosing a small \( T = T_0 > 0 \) only depending on \( R_0 \), we deduce from Strichartz’ estimate and inequalities (4.16) and (4.17) that

\[
\|u - u_n\|_{1,T_0} \leq c\|u_0 - \varphi_n\|_{1,2} + \|\phi'(u_n(\cdot)) - \phi'(u(\cdot))\|_{\frac{p+1}{p-1}} + \|\nabla u(\cdot)\|_p \|L^q(J_0)
\]

where the right–hand side tends to 0 as \( n \to \infty \) and \( J_0 := [-T_0, T_0] \). Because \( T_0 \) only depends on \( R_0 \) and \( u_n(\pm T_0) \to u(\pm T_0) \) in \( W^2_2 \) as \( n \to \infty \), we can establish assertion (c) by finitely many iterations.

In the derivation of the conservation laws in Section 1 we needed \( W^2_2 \)-solutions. Of course, such solutions can only exist if \( u_0 \in W^2_2 \). By a refinement of the above proof we will show that the solution of (4.1) preserves the initial regularity on its full existence interval.

To this aim, we employ the vector–valued Sobolev space \( W^1_p(J, X) \) for \( p \in [1, \infty] \), an open interval \( J \subseteq \mathbb{R} \) and a Banach space \( X \). One defines the weak derivative \( u' \) of \( u : J \to X \) as usual by means of test functions \( \varphi \in C^\infty_c(J) \). The (Banach) space \( W^1_p(J, X) \) is then introduced as in the scalar case. It is known that \( u \in W^1_p(J, X) \) if and only if \( u \in L^p(J, X) \) and there is a \( v \in L^p(J, X) \) such that \( u(t) = u(a) + \int_a^t v(s) \mathrm{d}s \) for all \( t, a \in J \) (where we can choose a continuous representative of \( u \)). In this case, \( v = u' \) and \( u'(t) \) is equal to the pointwise derivative for a.e. \( t \in J \). Let \( X \) be reflexive. Then it can be shown that \( W^1_p(J, X) \) is isometrically isomorphic to the space Lipschitz continuous functions \( u : [-T, T] \to X \). Moreover, \( W^1_p(J, X) \) is reflexive if also \( p \in (1, \infty) \).

Here one can identify \( W^1_p(J, X) \) with \( L^p(J, X)^2 \) via \( u \mapsto (u, u') \), and thus the duality is given by

\[
\langle u, (v, w) \rangle = \int_J \langle u(t), v(t) \rangle_X \mathrm{d}t + \int_J \langle u'(t), w(t) \rangle_X \mathrm{d}t
\]

for \( (v, w) \in L^q(J, X^*)^2 \). (See §III.1.1+2 in [Am] and §1.2 in [ABHN].)

**Proposition 4.10.** Let \( 1 < a < \frac{d+2}{(d-2)+} \) and \( u_0 \in W^2_2 \). Then the solution obtained in Theorem 4.9 is a \( W^2_2 \)-solution on \( J(u_0) \).

**Proof.** Let \( u \) be the solution obtained in Theorem 4.9. Take any \( [a, b] \subset J(u_0) \) containing 0 and set \( r = \max_{t \in [a, b]} \|u(t)\|_{1,2} \). We want to show the asserted regularity at first in a neighborhood of 0 by a refinement of the fixed point argument given in the proof of Theorem 4.9. Throughout we use the setting and the notation of this proof. For \( T > 0 \) we define the space

\[
E(T) = E_1(T) \cap W^1_q((-T, T), L^p) \cap W^1_{ac}((-T, T), L^2)
\]

with the norm
\[\|u\| = \max\{\|u\|_{1,T}, \|\partial_t u\|_{0,T}\}.\]

We fix \(R = 2c_1r\), where the constant \(c_1\) is taken from (4.13). For \(R_1 > 0\) and \(u_0 \in W_2^2\), we further introduce the set
\[\Theta = \Theta(R_1, T) = \{u \in \mathcal{E}(T) \mid u(0) = u_0, \|u\|_{1,T} \leq R, \|\partial_t u\|_{0,T} \leq R_1\} \]

We again endow \(\Theta\) with the metric \(\|u - v\|_{0,T}\). To show that \(\Theta\) is complete, take a Cauchy sequence \((u_n)\) in \(\Theta\). In Step 1) of the proof of Theorem 4.9 we have seen that \(u_n\) converges in \(E_0(T)\) to a function \(u \in E_1(T)\) with \(\|u\|_{1,T} \leq R\), as \(n \to \infty\). Since the functions \(u_n : [-T, T] \to L^2\) converge in \(L^\infty((-T, T), L^2)\) and are uniformly Lipschitz with bound \(R_1\), we conclude that \(u_n\) tends to \(u\) in \(C_b([-T, T], L^2)\) as \(n \to \infty\), that \(u(0) = u_0\) and that \(u : [-T, T] \to L^2\) is Lipschitz with bound \(R_1\). Further, after passing to a subsequence, \(u_n\) converges weakly in \(W^1_q((-T, T), L^p)\) to a function \(v \in W^1_q((-T, T), L^p)\) as \(n \to \infty\). In particular, \(u_n\) and \(\partial_t u_n\) converge weakly in \(L^q((-T, T), L^p)\) to \(v\) and \(\partial_t v\), respectively. We thus obtain \(u = v, u, v \in \mathcal{E}(T)\) and \(\|\partial_t u\|_{L^q((-T, T), L^p)} \leq R_1\). Summing up, \(u \in \Theta\) and \(\Theta\) is complete.

Let \(u, v \in \Theta\). For fixed \(u_0 \in W_2^2\), we define again
\[[\Phi_{u_0}(u)](t) = [\Phi(u)](t) := T(t)u_0 + \int_0^t T(t-s)F(u(s))\,ds, \quad t \in J = [-T, T],\]
\[= T(t)u_0 + \int_0^t T(s)F(u(t-s))\,ds.\]

We repeat from (4.14) and (4.13) that \(\Phi(u) \in E_1(T) \cap C([-T, T], W_2^1)\) and
\[\|\Phi(u - v)\|_{0,T} \leq c_3 R^\alpha T^{1 - \frac{1}{q}} \|u - v\|_{0,T}, \quad (4.18)\]
\[\|\Phi u\|_{1,T} \leq c_1 \|u_0\|_{1,2} + c_2 R^{3-1} T^{\frac{1}{q}} - \frac{1}{q} R \quad (4.19)\]
for constants \(c_j > 0\) not depending on \(u, v, u_0, T, R\) and \(R_1\).

One can show that \(F \in C^1(L^p, L^p)\) with \(F'(u)v = -i\rho\phi'(u)v\) as we treated the term \(J_1\) at the end of proof of Lemma 4.1. Using (scalar) a mollifier, one further checks that \(C^1([-T, T], L^p)\) is dense in \(W^1_q((-T, T), L^p)\). Let \(v \in C^1([-T, T], L^p)\). Because of \(L^p \hookrightarrow W_2^{-1}\), the derivative
\[
\frac{d}{dt} \int_0^t T(s)F(v(t-s))\,ds = T(t)F(u_0) + \int_0^t T(s)F'(v(t-s))\partial_t v(t-s)\,ds
\]
\[= T(t)F(u_0) + \int_0^t T(t-s)F'(v(s))\partial_t v(s)\,ds\]
thus exists in \(W_2^{-1}\). We approximate \(u\) in \(W^1_q((-T, T), L^p) \hookrightarrow C([-T, T], L^p)\) by \(v \in C^1([-T, T], L^p)\). Due to Strichartz’ inequality, the right–hand side of the above identity converges in \(L^2\) uniformly in \(t\) as \(v \to u\). The same holds for the integral on the left–hand side. We can thus differentiate the integral term of \(\Phi\) and obtain
\[\partial_t[\Phi(u)](t) = T(t)[i\Delta u_0 + F(u_0)] + \int_0^t T(t-s)F'(u(s))\partial_t u(s)\,ds\]
for all \( t \in J \). It holds \( \|F'(v)w\|_{p'} \leq c\|v\|_{p'}^{\alpha-1}\|w\|_p \leq c\|v\|_{1,2}^{\alpha-1}\|w\|_p \) for \( w \in L^p \) and \( v \in W^2_2 \).

Strichartz’ estimate thus allows to estimate the integral term in \( E_0(T) \) by

\[
c R^{\alpha-1} \|\partial_t u\|_{L^p(J,L^p)} \leq c R^{\alpha-1} T^{\frac{1}{p} - \frac{1}{q}} \|\partial_t u\|_{L^q(J,L^p)} \leq c R^{\alpha-1} T^{\frac{1}{p} - \frac{1}{q}} R_1.
\]

We further observe that \( W^2_2 \hookrightarrow L^{2\alpha} \) since \( \alpha < d/(d - 4)_+ \). Hence, \( \|F(u_0)\|_2 = \|u_0\|_{2,0} \leq c\|u_0\|_{2,2} \), and Strichartz’ estimate yields that

\[
\|\partial_t \Phi(u)\|_{0,T} \leq c_4 \|u_0\|_{2,2} + \|u_0\|_{2,2}^{\alpha} + R^{\alpha-1} T^{\frac{1}{p} - \frac{1}{q}} R_1),
\]

where the constant does not depend on \( u, u_0, T, R \) or \( R_1 \). Moreover, \( \partial_t \Phi(u) \) belongs to \( C([-T,T], L^2) \). We next choose \( T_0 > 0 \) such that

\[
c_j R^{\alpha-1} T_0^{\frac{1}{q} - \frac{1}{p}} \leq \frac{1}{2}
\]

for \( j = 2, 3, 4 \). Recall that \( R = 2c_1 \max_{\in [a,b]} \|u(t)\|_{1,2} \) and thus \( T_0 \) only depends on \( a, b, \alpha, \) and \( d \). We then set \( R_1 = 2c_4 \|u_0\|_{2,2} + \|u_0\|_{2,2}^{\alpha} \). The inequalities (4.18), (4.19) and (4.20) now imply that \( \Phi : \Theta \to \Theta \) is a strict contraction. We thus obtain a fixed point \( v \in \Theta \subset \Theta(T_0) \) with \( v \in C(J_0,W^2_2) \cap C^1(J_0,L^2) \), where \( J_0 = [-T_0, T_0] \). Since \( R \) was chosen as in Step 2 of the proof of Theorem 4.9 and \( T_0 \leq b_0(\|u_0\|_{1,2}) \), we have \( \Theta \subset \Sigma \) and thus \( u = v \in C^1(J_0,L^2) \).

We still have to show that \( u \in C(J_0,W^2_2) \). To that purpose we note that

\[
u - \Delta u = u + i\partial_t u - u \partial_\Phi(u) = f + g,
\]

where \( f = u + i\partial_t u \in C(J_0,L^2) \) and \( g = -u \partial_\Phi(u) \in C(J_0,L^p/\alpha) \) since \( u \in C(J_0,L^p) \) and \( p' = p/\alpha \). Recall that \( I - \Delta : W^2_2 \to L^p \) is invertible for every \( r \in (1, \infty) \), see Example 2.18 of [EE]. Hence, \( (I - \Delta)^{-1}f \in C(J_0,W^2_2) \) and \( (I - \Delta)^{-1}g \in C(J_0,W^2_2) \). Sobolev’s embedding thus yields \( u \in C(J_0,L^{r_1}) \),

with \( r_1 = \frac{dp}{2\alpha - dp} =: \gamma \) if \( d\alpha > 2p \) and any \( r_1 < \infty \) otherwise. Note that \( \gamma > 1 \) if \( d\alpha > 2p \) since \( p = \alpha + 1 \) and \( \alpha < (d + 2)/(d - 2)_+ \).

This extra regularity of \( u \) implies that \( g \in C(J_0,L^{r_1/\alpha}) \). If \( r_1 \geq 2\alpha \), we infer that \( u \in C(J_0,W^2_2) \). If \( r_1 < 2\alpha \), we repeat the above argument, arriving at \( u \in C(J_0,L^{r_2}) \) with \( r_2 = \frac{dr_1}{\alpha - 2\alpha} \) if \( d\alpha > 2r_1 \) and any \( r_2 < \infty \) otherwise. Observe that \( r_2 \geq \gamma r_1 \geq \gamma^2 \) if \( d\alpha > 2r_1 \). Since \( \gamma > 1 \), in finitely many steps we reach at \( \gamma^m \geq \gamma^{m+1} \) and thus \( u \in C(J_0,W^2_2) \).

Now assume that \( [a,b] \not\subset J_0 \), where \( T_0 \not\in [a,b] \), say. Observe that one can glue together \( W^2_2 \)-solutions, cf. Step 3) of the proof of Theorem 4.9. Since \( u(T_0) \in W^2_2 \) and \( \|u(T_0)\|_{1,2} < r \), the above argument shows that \( u \) is a \( W^2_2 \)-solution on \([-T_0,2T_0]\). Iteratively, one sees that \( u \) is a \( W^2_2 \)-solution on \([a,b]\). Since this an arbitrary compact subinterval of \( J(u_0) \), the result has been established.

3. Asymptotic behavior

In this section we show global existence of solutions for \( 1 < \alpha < \frac{d+2}{(d-2)_+} \) in the defocusing case and if either \( \alpha \) or \( u_0 \) are small. Moreover, we state a result on

\[3\text{Such arguments based on self improving estimates are called ‘boot-strapping’}.\]
asymptotic stability in the defocussing case. A main ingredient of the proofs is
the energy given by
\[ E(v) = \frac{1}{2} \| \nabla v \|_2^2 + \frac{\mu}{\alpha + 1} \| v \|_{1+\alpha} \]
for \( v \in W^1_2 \). Since \( W^1_2 \hookrightarrow L^{\alpha+1} \), the map \( E : W^1_2 \rightarrow \mathbb{R} \) is locally Lipschitz.

**Theorem 4.11.** Let \( 1 < \alpha < \frac{d+2}{d-2} \) and \( u_0 \in W^1_2 \) with the corresponding
\( W^1_2 \)-solution \( u \) of (4.1) on \( J(u_0) \). We then have \( \| u(t) \|_2 = \| u_0 \|_2 \) and \( E(u(t)) = E(u_0) \) for all \( t \in J(u_0) \). It follows that \( J(u_0) = \mathbb{R} \) if \( \mu = 1 \) or if \( \alpha < 1 + \frac{4}{d} \).

**Proof.** 1) Theorem 4.9 says that \( u_0 \mapsto u(t) \) is continuous on \( W^1_2 \). It thus
suffices to show the conservation laws for \( u_0 \in W^1_2 \) where \( u \) is a \( W^1_2 \)-solution
by Proposition 4.10. In this setting, the equality \( \| u(t) \|_2 = \| u_0 \|_2 \) was already shown in (4.2).

For the energy balance, we recall that cutoff and mollification give linear
operators \( R_n : L^2 \rightarrow C^\infty_c \) which converge strongly to \( I \) both in \( L^2 \) and \( W^1_2 \).
Hence, \( u_n := R_n u \) belongs to \( C^1(J(u_0), W^k_2) \) for all \( k, n \in \mathbb{N} \). Since the operators \( R_n \) also converge on compact subsets of \( L^2 \) and \( W^1_2 \), the functions \( u_n \) tend to \( u \) in \( C^1(J, L^2) \cap C(J, W^k_2) \) for all compact intervals \( J \subset J(u_0) \). In particular,
\( E(u_n(t)) \) converges to \( E(u(t)) \) for all \( t \in J(u_0) \) as \( n \to \infty \). As in (4.3), we obtain
\[
\frac{d}{dt} E(u_n(t)) = \text{Re} \int (-\Delta u_n(t) + \mu |u_n(t)|^{\alpha-1} u_n(t)) \overline{\partial_t u_n(t)} \, dx.
\]
Here, \( \Delta u_n(t) \) and \( \partial_t u_n(t) \) tend to \( \Delta u(t) \) and \( \partial_t u(t) \) in \( L^2 \) uniformly for \( t \in J \)
as \( n \to \infty \). To treat the nonlinearity \( F(v) := \mu |v|^{\alpha-1} v \), we observe that \( W^1_2 \)
is embedded into \( L^\infty \) if \( d \leq 3 \), into any \( L^q \) with \( q \in [2, \infty) \) if \( d = 4 \), and into
\( L^r \) with \( q = 2d/(d-4) \) if \( d > 4 \). We set \( q = \infty \) in the first case. We have \( \frac{1}{2} = \frac{1}{q} + \frac{1}{r} \) with \( \frac{1}{r} = 2 \) for \( d \leq 3 \), any \( r > 2 \) if \( d = 3 \), and \( r = \frac{d}{2} \) if \( d > 4 \). Sobolev’s
embedding shows that \( W^1_2 \hookrightarrow L^{(\alpha-1)} \) for all \( d \). As in (4.7) we thus compute
\[
\| F(u_n(t)) - F(u(t)) \|_2 \leq c (\| u_n(t) \|^{\alpha-1} + \| u(t) \|^{\alpha-1}) \| u_n(t) - u(t) \|_2
\]
\[
\leq c (\| u_n(t) \|^{\alpha-1} + \| u(t) \|^{\alpha-1}) \| u_n(t) - u(t) \|_q
\]
\[
\leq c R^{\alpha-1} \| u_n(t) - u(t) \|_{1,2}
\]
for all \( n \in \mathbb{N} \) and \( t \in J \), where \( R \) is a bound on the norms of \( u_n \) and \( u \) in \( W^1_2 \).

Summing up, we see that \( \frac{d}{dt} E(u_n(t)) \) converges to
\[
\text{Re} \int (-\Delta u(t) + \mu |u(t)|^{\alpha-1} u(t)) \overline{\partial_t u(t)} \, dx = \text{Re} \int |\partial_t u(t)|^2 \, dx = 0,
\]
where we use (4.1). Hence, \( E(u(t)) = E(u_0) \) for all \( t \in J(u_0) \).

2) If \( \mu = 1 \), Step 1) yields \( \| u(t) \|_{2,2}^2 \leq 2E(u(t)) + \| u(t) \|_2^2 = 2E(u_0) + \| u_0 \|_2^2 \)
for all \( t \in J(u_0) \). The blow-up criterion in Theorem 4.9 now implies that
\( J(u_0) = \mathbb{R} \). Next, let \( 1 < \alpha < 1 + \frac{4}{d} \) and \( \mu = -1 \). We consider \( d \geq 3 \), the proof for \( d = 1, 2 \) is similar. We have
\[
\frac{1}{\alpha + 1} = 1 - \frac{\theta}{2} + \frac{\theta}{2d} - \frac{d}{2} \quad \text{for} \quad \theta = \frac{d}{2} - \frac{d}{\alpha + 1} \in (0, 1).
\]
The interpolation and Sobolev’s inequality (see (3.16) in [ST]) then yield
\[
\|v\|_{\alpha+1}^{\alpha+1} \leq \left( \|v\|_2 \|v\|_2^\theta \right)^{\alpha+1} \leq c \|v\|_2^{\alpha+1-\delta(\alpha-1)/2} \|\nabla v\|_2^{\delta(\alpha-1)/2}
\]
for all \( v \in W^1_2 \). We have \( \beta := \frac{4-\delta}{d(\alpha-1)} > 1 \) by our assumption. Young’s inequality with \( \beta \) and \( \beta' \) leads to
\[
\frac{1}{\alpha+1} \|v\|_{\alpha+1}^{\alpha+1} \leq \frac{1}{4} \|\nabla v\|_2^2 + c \|v\|_2^{\beta'(\alpha+1-\delta(\alpha-1)/2)}
\]
for a constant only depending on \( \alpha \) and \( d \). Denoting the last summand by \( k(||v||_2^2) \), we infer from Step 1) that
\[
E(u_0) = E(u(t)) = \frac{1}{2} \|\nabla u(t)\|_2^2 - \frac{1}{\alpha+1} \|u(t)\|_{}^{\alpha+1} \leq \frac{1}{4} \|\nabla u(t)\|_2^2 - k(||u(t)||_2) = \frac{1}{4} \|\nabla u(t)\|_2^2 - k(||u_0||_2)
\]
for all \( t \in J(u_0) \). Hence, \( ||u(t)||_\alpha^2 \leq 4E(u_0)+4k(||u_0||_2)+||u_0||_\alpha^2 \) for all \( t \in J(u_0) \), and again it follows that \( J(u_0) = \mathbb{R} \).

Global existence holds in the defocussing case also if \( \alpha = (d+2)/(d-2)_+ \) and \( d \geq 3 \). This deep result is far beyond the scope of these lectures, see Chapter 5 of [Ta] for an extended survey.

**Theorem 4.12.** Let \( 1 < \alpha < (d+2)/(d-2)_+ \) and \( u_0 \in W^1_2 \) with the corresponding \( W^1_2 \)-solution \( u \) of (4.1) on \( J(u_0) \). Then there are \( \rho, \gamma > 0 \) such that \( ||u_0||_\alpha \leq \rho \) implies that \( J(u_0) = \mathbb{R} \) and \( ||u(t)||_\alpha^2 \leq \gamma \) for all \( t \in \mathbb{R} \).

**Proof.** The conservation laws and Sobolev’s embedding yield
\[
\frac{1}{2} \|u(t)||_{\alpha,2}^2 = \frac{1}{2} \|u_0||_{\alpha,2}^2 + E(u(t)) - \frac{\mu}{\alpha+1} \|u(t)||_{\alpha+1}^{\alpha+1} \leq \frac{1}{2} \|u_0||_{\alpha,2}^2 + E(u_0) + c_0 ||u(t)||_{\alpha,2}^{\alpha-1} \|u(t)||_{\alpha,2}^2
\]
for all \( t \in J(u_0) \) and a constant \( c_0 > 0 \) depending only on \( \alpha \) and \( d \). We set \( \gamma = (4c_0)^{1/(\alpha-1)} \) and take any \( \rho \in (0, \gamma) \). Let \( ||u_0||_{\alpha,2} \leq \rho \). We now define
\[
t_0 := \sup\{t \in (0, b_+(u_0)) : ||u(s)||_{\alpha,2} \leq \gamma \text{ for all } s \in [0, t]\}
\]
and observe that \( t_0 \in (0, b_+(u_0)) \). The estimate (4.21) and Sobolev’s embedding next imply that
\[
\frac{1}{2} \|u(t)||_{\alpha,2}^2 \leq \frac{1}{2} \|u_0||_{\alpha,2}^2 + E(u_0) + \frac{1}{4} \|u(t)||_{\alpha,2}^2,
\]
\[
\|u(t)||_{\alpha,2}^2 \leq 2 \|u_0||_{\alpha,2}^2 + \frac{4}{\alpha+1} \|u_0||_{\alpha+1}^{\alpha+1} \leq c_1 (\rho^2 + \rho^{\alpha+1})
\]
for all \( t \in [0, t_0] \) and a constant \( c_1 > 0 \) depending only on \( \alpha \) and \( d \). We now choose \( \rho \in (0, \gamma) \) such that \( c_1 (\rho^2 + \rho^{\alpha+1}) \leq \gamma/2 \). Then (4.22) would be impossible if \( t_0 < b_0(u_0) \). We thus obtain \( t_0 = b_0(u_0) \) and so Theorem 4.9(d) gives \( b_0(u_0) = \infty \), as asserted. Similarly one treats negative times. \( \square \)

\(^4\)Such estimates are called Gagliardo–Nirenberg inequalities.
Our final result is an easy consequence of the ‘pseudo–confomal’ conservation law for (4.1), Theorem 7.2.1 in [Ca].

**Proposition 4.13.** Let $1 < \alpha < (d + 2)/(d - 2)_+$ with $\alpha \geq 1 + 4/d$, $\mu = 1$ and $u_0 \in W^1_2$ with $|x|u_0 \in L^2$ and $u$ be the corresponding $W^1_2$-solution of (4.1) on $\mathbb{R}$. Then there is a constant $c > 0$ such that

$$\|u(t)\|_{\alpha+1} \leq c|t|^{-\frac{2}{\alpha+1}} \| |x|u_0\|_{2+\frac{2}{\alpha+1}}$$

for $t \in \mathbb{R}$.

**Proof.** Let $t \in J(u_0) = \mathbb{R}$. Theorem 7.2.1 in [Ca] yields

$$\|(x + 2it\nabla)u(t)\|_2 + \frac{8t^2}{\alpha + 1} \|u(t)\|_{\alpha+1} = \|xu_0\|_2 + 4 \frac{d + 4 - \alpha d}{\alpha + 1} \int_0^t s \|u(s)\|_{\alpha+1}^2 ds.$$ 

Since the last summand is negative, the assertion follows. □

A stronger version of this result is given in Theorem 7.3.1 of [Ca].
## Bibliography


