Lecture Notes
Evolution Equations

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CHAPTER 1

Strongly continuous semigroups and their generators

Throughout we assume that \((X, \| \cdot \|_X)\) is a Banach space, where we mostly write \(\| \cdot \|\) instead of \(\| \cdot \|_X\) if no confusion is to be expected. By \(\mathcal{B}(X,Y)\) we denote the space of all bounded linear operators from \(X\) into another Banach space \(Y\), where we set \(\mathcal{B}(X) = \mathcal{B}(X,X)\). Further, \(X^*\) is the dual space of \(X\) and \(I\) the identity map on \(X\).

1.1. Basic Properties

Definition 1.1

A map

\[ T(\cdot) : \mathbb{R}_+ := [0, \infty) \to \mathcal{B}(X) \]

is called strongly continuous operator semigroup or just \(C_0\)-semigroup if the following conditions are fulfilled:

(a) \(T(0) = I\) and we have \(T(t + s) = T(t)T(s)\) for all \(t, s \geq 0\).

(b) For each \(x \in X\) the orbit defined as the map

\[ T(\cdot)x : \mathbb{R}_+ \to X; \ t \mapsto T(t)x \]

is continuous; this property is called strong continuity.

The generator \(A\) of \(T(\cdot)\) is given by setting

\[ \mathcal{D}(A) := \left\{ x \in X; \ \text{the limit} \ \lim_{t \to 0} \frac{1}{t} (T(t)x - x) \ \text{exists} \right\} \]

and defining

\[ Ax := \lim_{t \to 0} \frac{1}{t} (T(t)x - x) \]

for \(x \in \mathcal{D}(A)\).

If one replaces throughout in this definition \(\mathbb{R}_+\) by \(\mathbb{R}\) one obtains the concept of a \(C_0\)-group with generator \(A\).

Remark

(a) In the above situation, \(\mathcal{D}(A)\) is a linear subspace and \(A\) is a linear map.

(b) If \(T(\cdot)\) is a \(C_0\)-semigroup, it holds

\[ T(t)T(s) = T(t + s) = T(s + t) = T(s)T(t) \]
for all \( t, s \geq 0 \). Moreover, we have for all \( n \in \mathbb{N} \) and all \( t \geq 0 \)
\[
T(nt) = T \left( \sum_{j=1}^{n} t \right) = \prod_{j=1}^{n} T(t) = T(t)^n.
\]

If \( T(\cdot) \) is even a \( C_0 \)-group, the above properties hold for all \( s, t \in \mathbb{R} \) and thus
\[
T(t)T(-t) = T(0) = I = T(-t)T(t).
\]

Hence, there exists \( T(t)^{-1} \in \mathcal{B}(X) \) and equals \( T(t) \) for every \( t \in \mathbb{R} \).

**Example 1.2**

Let \( X = \mathbb{C}^d \), \( A \in \mathcal{B}(X) = \mathbb{C}^{d \times d} \) and set \( T(t) := e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \) for \( t \in \mathbb{R} \). It is well known that \( T(t) \) exists, satisfies (a) in Definition 1.1 and is continuously differentiable with \( \frac{d}{dt} e^{tA} = Ae^{tA} \) in \( \mathcal{B}(X) \) for all \( t \in \mathbb{R} \). Consequently, \( T(\cdot) \) is a \( C_0 \)-group with generator \( A \). Moreover, for any given \( x \in X \) the function \( u \) defined by \( u(t) := e^{tA}x \) \((t \geq 0)\) solves the ordinary differential equation
\[
u'(t) = Au(t), \ t \geq 0,
\]
with the initial condition
\[
u(0) = x.
\]

**Lemma 1.3**

Let \( T(\cdot) : \mathbb{R}_+ \rightarrow \mathcal{B}(X) \) satisfy the condition (a) in Definition 1.1 as well as \( \lim_{t \rightarrow 0} \| T(t)x \| < \infty \) for all \( x \in X \). Then there are \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that
\[
\| T(t) \| \leq Me^{\omega t}
\]
for all \( t \geq 0 \).

**Proof.** Suppose there is a null sequence \( (t_n)_n \) in \( \mathbb{R}_+ \) with the additional property that \( \lim_{n \rightarrow \infty} \| T(t_n) \| = \infty \). Then the principle of uniform boundedness implies the existence of some \( x \in X \) such that \( \| T(t_n)x \| \rightarrow \infty \) as \( n \rightarrow \infty \), contradicting the assumption. Hence, there are \( c, t_0 > 0 \) with \( \| T(\tau) \| \leq c \) for all \( \tau \in [0,t_0] \). Now let \( t \geq 0 \) be arbitrary. Then there are \( n \in \mathbb{N}_0 \) and \( \tau \in [0,t_0) \) such that \( t = nt_0 + \tau \). Setting \( \omega := \frac{\log c}{t_0} \), we obtain
\[
\| T(t) \| = \| T(\tau)T(nt_0) \| = \| T(\tau)T(t_0)^n \| \leq c \cdot c^n = e^{(n+1)t_0 \frac{\log c}{t_0}} = e^{t\omega} \cdot e^{t_0 \frac{\log c}{t_0} \log c} \leq e^{\max\{\log c,0\}} e^{t\omega},
\]
which completes the proof.

**Definition 1.4**

Let \( T(\cdot) \) be a \( C_0 \)-semigroup with generator \( A \). Then
\[
\omega_0(T) := \omega_0(A) := \inf\{\omega \in \mathbb{R} ; \exists M_\omega \geq 1 \ \forall t \geq 0 : \| T(t) \| \leq M_\omega e^{\omega t}\}
\]
is called the growth bound of \( T(\cdot) \). Note that we have \( \omega_0(T) < \infty \) due to Lemma 1.3 and the continuity of all orbits \( T(\cdot)x \) with \( x \in X \).
Remark
In general the infimum in Definition 1.4 is not a minimum. For instance, for \( X = \mathbb{C}^2 \) endowed with the 1-norm \( \| \cdot \|_1 \) and \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) we have \( T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \) so that \( \|T(t)\|_1 = t + 1 \) for \( t \geq 0 \), while \( \omega_0(T) = 0 \). As we shall see below in Example 1.6, even \( \omega_0(T) = -\infty \) is possible.

Lemma 1.5
Let \( T(\cdot) : \mathbb{R}_+ \to \mathcal{B}(X) \) be a map satisfying condition (a) in Definition 1.1. Then the following assertions are equivalent.
(a) \( T(\cdot) \) is strongly continuous (and thus a \( C_0 \)-semigroup).
(b) It holds \( \lim_{t \to 0^+} T(t)x = x \) for all \( x \in X \).
(c) There is a \( t_0 > 0 \) and a dense subspace \( D \subseteq X \) such that \( c := \sup_{0 \leq t \leq t_0} \|T(t)\| < \infty \) and \( \lim_{t \to 0^+} T(t)x = x \) for all \( x \in D \).

An analogous assertion holds in the group case.

Proof. The implication “(a)⇒(c)” is an immediate consequence of Lemma 1.3. The assertion (b) can be deduced from (c) by an application of the Banach-Steinhaus theorem (see, e.g., Corollary 3.5 in [FA]).

So it only remains to conclude (a) from (b). For this purpose consider an arbitrary \( x \in X \) and \( t > 0 \). For \( h > 0 \) the semigroup property implies
\[
\|T(t + h)x - T(t)x\| = \|T(t)(T(h)x - x)\| \leq \|T(t)\| \cdot \|T(h)x - x\|,
\]
where the right hand side of this inequality converges to 0 as \( h \) tends to 0. In the case that \( h \in (-t_0, 0] \), we note that Lemma 1.3 implies that
\[
\|T(t + h)\| \leq Me^{\omega(t + h)} \leq Me^{\omega(t)}
\]
for some constants \( M \geq 1 \) and \( \omega \in \mathbb{R} \). As a result,
\[
\|T(t + h)x - T(t)x\| \leq \|T(t + h)\| \cdot \|x - T(-h)x\| \to 0
\]
as \( h \to 0^- \), completing the proof. The final assertion is shown similarly. \( \Box \)

Remark
In the above lemma the implication “(c)⇒(a)” can fail if one omits the boundedness assumption (cf. Exercise I.5.9(4) in [E-N]).

We now examine a basic class of examples.

Example 1.6 (Translation semigroups)
(a) Let \( X = C_0(\mathbb{R}) := \{ f \in C(\mathbb{R}); f(s) \to 0 \text{ as } |s| \to \infty \} \) and \( (T(t)f)(s) = f(s + t) \) for \( t \in \mathbb{R} \), \( f \in X \) and \( s \in \mathbb{R} \). Obviously, \( T(0) = I \) and \( T(t) \) is a linear isometry so that \( \|T(t)\| = 1 \). We

\( \text{[Recall that the matrix norm induced by the 1-norm is the maximal absolute column sum of the matrix.]} \)
further obtain
\[ T(t)T(r)f = (T(r)f)(\cdot + t) = f(\cdot + (t + r)) = T(t + r)f \]
for all \( f \in X \) and \( r, t \in \mathbb{R} \). Hence, \( T(t)T(r) = T(t + r) \). Finally, for \( f \in C_c(\mathbb{R}) \) (where \( C_c(\mathbb{R}) \) is the set
\[ \{ f \in C(\mathbb{R}); \text{supp}(f) := \{ s \in \mathbb{R}; f(s) \neq 0 \} \text{ is compact} \} \]
the function \( T(t)f \) converges uniformly to \( f \) as \( t \) tends to 0 since \( f \) is uniformly continuous. To check \( C_c(\mathbb{R}) = C_0(\mathbb{R}) \), take for each \( n \in \mathbb{N} \) a function \( \varphi_n \in C(\mathbb{R}) \) such that \( \varphi_n = 1 \) on \([-n, n] \), \( 0 \leq \varphi_n \leq 1 \) and \( \text{supp} \varphi_n \subseteq \left(-n - 1, n + 1\right) \). For \( f \in C_0(\mathbb{R}) \) we then have \( \varphi_n f \in C_c(\mathbb{R}) \) and
\[ \| f - \varphi_n f \|_{\infty} = \sup_{|s| \geq n} \left| (1 - \varphi_n(s)) f(s) \right| \leq \sup_{|s| \geq n} |f(s)| \to 0 \]
as \( n \to \infty \). Now, we conclude by means of Lemma 1.5 that \( T(\cdot) \) is a \( C_0 \)-group.

The same assertions hold for \( X = L^p(\mathbb{R}) \) with \( 1 \leq p < \infty \) by similar arguments (see, e.g., Example 3.8 in [FA]).

In contrast to these results, \( T(\cdot) \) is not strongly continuous on \( X = L^\infty(\mathbb{R}) \). Indeed, consider \( f = 1_{[0,1]} \) and observe that
\[
T(t)f(s) = 1_{[0,1]}(s + t) = \begin{cases} 1, & \text{if } s + t \in [0,1] \\ 0, & \text{else} \end{cases} = 1_{[-t,1-t]}
\]
for \( s, t \in \mathbb{R} \). Thus, \( \| T(t)f - f \|_{\infty} = 1 \) for every \( t \neq 0 \).

In addition, \( T(\cdot) \) is not continuous as a \( B(X) \)-valued function for \( X \) being \( L^p(\mathbb{R}) \) (see Example 3.8 in [FA]) or \( C_0(\mathbb{R}) \). In fact, for \( X = C_0(\mathbb{R}) \) consider for each \( n \in \mathbb{N} \) functions \( f_n \in C_c(\mathbb{R}) \) with \( 0 \leq f_n \leq 1 \), \( f_n(n) = 1 \) and \( \text{supp} f_n \subseteq \left(n - \frac{1}{n}, n + \frac{1}{n}\right) \). We then have \( \text{supp} T\left(\frac{2}{n}\right) f_n \subseteq \left(n - \frac{3}{n}, n - \frac{1}{n}\right) \) for \( n \in \mathbb{N} \), which implies
\[
\| T\left(\frac{2}{n}\right) - I \| \geq \| T\left(\frac{2}{n}\right) f_n - f_n \|_{\infty} = 1
\]
for all \( n \in \mathbb{N} \).

(b) Let \( X = C_0([0,1]) := \{ f \in C([0,1]); \lim_{s \to 1} f(s) = 0 \} \) be endowed with the supnorm. For \( t > 0 \) we define
\[
(T(t)f)(s) := \begin{cases} f(s + t), & \text{if } s \in [0,1) \text{ and } s + t < 1, \\ 0, & \text{if } s \in [0,1) \text{ and } s + t \geq 1. \end{cases}
\]
Since \( f(s + t) \to 0 \) as \( s + t \to 1 \), we have \( T(t)f \in X \). Clearly, \( T(t) \) is linear and \( \| T(t) \| \leq 1 \). Note, that \( T(t) = 0 \) whenever \( t \geq 1 \). As a consequence, \( \omega_0(T) = -\infty \). In this case, one says that \( T(\cdot) \) is nilpotent. Now, let \( 0 \leq t, r \) and \( s \in [0,1) \). We then obtain
\[
T(t)T(r)f(s) = \begin{cases} (T(r)f)(s + t), & \text{if } t < 1 \text{ and } s \in [0,1-t), \\ 0, & \text{if } s \in [1-t,1), \end{cases}
\]
continuous, we have shown that
\[ f(s + t + r), \quad \text{if } r, t < 1, \quad s + t < 1, \quad s + t + r < 1, \]
\[ 0, \quad \text{else}, \]
\[ f(s + t + r), \quad \text{if } s \in [0, 1 - t - r) \text{ and } r + t < 1, \]
\[ 0, \quad \text{else}, \]
\[ = (T(t + r)f)(s). \]

Hence, \( T(\cdot) \) is a semigroup, but not a group. As in (a) one sees that \( C_c([0,1]) := \{ f \in C([0,1]); \exists b_f \in (0,1) : \supp f \subseteq [0,b_f] \} \) is a dense subspace of \( X \). For \( f \in C_c([0,1]) \) and \( t \in (0,1 - b_f) \) we compute
\[ T(t)f(s) - f(s) = \begin{cases} f(s + t) - f(s), & \text{if } s \in [0,1-t), \\ 0, & \text{if } s \in [1-t,1) \subseteq [b_f,1], \end{cases} \]
and deduce \( \lim_{t \to 0} \|T(t)f - f\|_\infty = 0 \) using the uniform continuity of \( f \). According to Lemma 1.5, \( T(\cdot) \) is a \( C_0 \)-semigroup on \( X \).

Let \( A \) be a linear operator on \( X \) with domain \( \mathfrak{D}(A) \) and let \( x \in \mathfrak{D}(A) \). We say a function \( u : \mathbb{R}_+ \to X \) solves
\[
(1.1) \quad \begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = x, \end{cases}
\]
if \( u \) is a \( \mathfrak{D}(A) \)-valued function belonging to \( C^1(\mathbb{R}_+,X) \) and fulfilling (1.1).

**Proposition 1.7**

Let \( A \) generate the \( C_0 \)-semigroup \( T(\cdot) \) and \( x \in \mathfrak{D}(A) \). Then \( T(t)x \in \mathfrak{D}(A) \), \( AT(t)x = T(t)Ax \) for all \( t \geq 0 \) and the function
\[ u : \mathbb{R}_+ \to X; \ t \mapsto T(t)x \]
is the unique solution of (1.1).

**Proof.** 1): Let \( t > 0, h > 0 \) and \( x \in \mathfrak{D}(A) \). We then have
\[ \frac{1}{h} (T(h) - I)T(t)x = T(t) \frac{1}{h} (T(h)x - x) \to T(t)Ax \]
as \( h \to 0 \). As a result, \( T(t)x \in \mathfrak{D}(A) \) and \( AT(t)x = T(t)Ax \). Next let \( 0 < h < t \). It then holds
\[ \frac{1}{-h} (T(t-h)x - T(t)x) = T(t-h) \frac{1}{h} (T(h)x - x) \to T(t)Ax \]
as \( h \to 0 \), where we have used Lemma 1.8 below (with \( S(t,h) = T(t-h) \)). Since \( T(\cdot)Ax \) is continuous, we have shown that \( T(\cdot)x \in C^1(\mathbb{R}_+,X) \) with derivative \( AT(\cdot)x \); i.e., \( u \) solves (1.1).

2): Let \( v \) be another solution of (1.1). Take \( t > 0 \) and set \( w(s) := T(t-s)v(s) \) for \( s \in [0,t] \). For \( s \in [0,t] \) and \( h \in \mathbb{R} \setminus \{0\} \) with \( s + h \in [0,t] \), we compute
\[ \frac{1}{h}(w(s+h) - w(s)) = T(t-s-h) \frac{1}{h} (v(s+h) - v(s)) - \frac{1}{-h} (T(t-s-h) - T(t-s)) v(s). \]
Using \( v(s) \in \mathcal{D}(A) \), the first step and again Lemma 1.8, we infer that \( w \) is continuously differentiable with derivative

\[
w'(s) = T(t - s)v'(s) - T(t - s)Av(s) = 0,
\]

where the last equality follows from the assumption that \( v \) solves (1.1). So for every \( x^* \in X^* \) the scalar function \( \langle w(\cdot), x^* \rangle \) is differentiable with vanishing derivative and thus constant, which leads to

\[
\langle T(t)x, x^* \rangle = \langle w(0), x^* \rangle = \langle w(t), x^* \rangle = \langle v(t), x^* \rangle
\]

for all \( t > 0 \) and \( x^* \in X^* \). The Hahn-Banach theorem now yields \( T(\cdot)x = v \) as asserted. \( \square \)

**Remark**

If \( f \in C_0(\mathbb{R}) \setminus C^1(\mathbb{R}) \), then the orbit \( T(\cdot)f \) of the translation semigroup on \( C_0(\mathbb{R}) \) is not differentiable.

**Lemma 1.8**

Let \( D = \{(t,s) \in [a,b]^2; t \geq s\} \), \( S : D \to \mathcal{B}(X) \) be strongly continuous and \( f \in C([a,b],X) \). Then the function

\[
g : D \to X; \quad (t,s) \mapsto S(t,s)f(s)
\]

is also continuous.

**Proof.** Observe that \( \sup_{(t,s)\in D} \|S(t,s)x\| \leq c \) for every \( x \in X \) by continuity. Hence, the uniform boundedness principle implies the existence of a constant \( c > 0 \) with \( \sup_{(t,s)\in D} \|S(t)\| \leq c \). For \( (t,s) \in D \) we thus obtain

\[
\|S(t',s')f(s') - S(t,s)f(s)\| \leq c\|f(s') - f(s)\| + \|(S(t',s') - S(t,s))f(s)\|
\]

where the right hand side of this inequality tends to 0 as \((t',s') \to (t,s)\). \( \square \)

**Remark**

Similarly, one shows that if \( x_n \to x \) in \( X \) and \( T_n \to T \) strongly, then \( T_n x_n \to Tx \) as \( n \to \infty \).

**Intermezzo 1: Closed operators and their spectra**

Let \( \mathcal{D}(A) \subseteq X \) be a linear subspace and \( A : \mathcal{D}(A) \to X \) be linear. The operator \( A \) is called *closed* if it holds:

If \( (x_n)_{n \in \mathbb{N}} \) is any sequence in \( \mathcal{D}(A) \) such that the limits \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} Ax_n = y \) exist in \( X \), then \( x \in \mathcal{D}(A) \) and \( Ax = y \).

(For this definition and the following results we refer to, e.g., Chapter 1 in [ST].) Note that any operator \( A \in \mathcal{B}(X) \) is closed with \( \mathcal{D}(A) = X \).

**Example 1.9**

Let \( X = C([0,1]) \) and \( Af := f' \) with \( \mathcal{D}(A) = C^1([0,1]) \). Let \( (f_n)_n \) be a sequence in \( \mathcal{D}(A) \) such that \( (f_n)_n \), respectively \( (f'_n)_n \), converges to \( f \), respectively \( g \), in \( X \). It is a well known
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fact that in this case we have $f \in C^1([0, 1])$ as well as $f' = g$, which states nothing else than that $A$ is closed. Next, consider $A_0 f := f'$ with $\mathcal{D}(A_0) := \{ f \in C^1([0, 1]); f'(0) = 0 \}$. If $(f_n)_n$ is a sequence in $\mathcal{D}(A)$ such that $(f_n)_n \to f$ and $(f'_n)_n \to g$ in $X$ as $n \to \infty$ then we obtain $f \in C^1([0, 1])$ with $f' = g$ as above. Furthermore, $f'(0) = g(0) = \lim_{n \to \infty} f'_n(0) = 0$. Consequently, also $A_0$ is closed.

Next, we define the Riemann integral for vector valued functions. Let $a < b$ be real numbers. A (tagged) partition $Z$ of an interval $[a, b]$ is a finite subset of $[a, b]$ containing both $a$ and $b$ (written in the form $Z = \{ a = t_0 < t_1 < \ldots < t_m = b \}$) together with a finite sequence $(\tau_k)_{k=1}^m$ satisfying $t_{k-1} \leq \tau_k \leq t_k$ for all $k \in \{ 1, \ldots, m \}$. The number $\Delta(Z) := \max_{k=1,\ldots,m}(t_k - t_{k-1})$ is called the mesh or norm of $Z$. For a function $g \in \mathcal{C}([a, b], X)$ we define the Riemann sum $S(g, Z)$ (of $g$ with respect to $Z$) by

$$S(g, Z) := \sum_{k=1}^m g(\tau_k)(t_k - t_{k-1}) \in X.$$ 

As for real valued functions it can be shown that for any sequence $(Z_n)_{n=1}^\infty$ of (tagged) partitions with $\lim_{n \to \infty} \Delta(Z_n) = 0$ the sequence $(S(g, Z_n))_{n=1}^\infty$ converges in $X$ and the limit $J$ does not depend on the choice of such $(Z_n)_{n=1}^\infty$; in this sense we say that $S(g, Z)$ converges in $X$ to $J$ as $\Delta(Z) \to 0$. The Riemann integral $\int_a^b g(t) \, dt$ is now defined by this limit:

$$\int_a^b g(t) \, dt := \lim_{\Delta(Z) \to 0} S(g, Z).$$

The integral has the usual properties known from the real valued case (with similar proofs) like linearity, additivity and validity of the standard estimate.

Incidentally, the same definition and results work for piecewise continuous functions.

We especially emphasize that the fundamental theorem of calculus also holds in the vector valued case (see (e) in Remark 1.10 below).

Remark 1.10

Let $A$ be a linear operator in $X$. Then the following assertions hold.

(a) The operator $A$ is closed if and only if the graph of $A$, i.e. the set

$$\text{gr}(A) := \{ (x, Ax); x \in \mathcal{D}(A) \},$$

is closed in $X \times X$ (endowed with the product topology) if and only if $\mathcal{D}(A)$ is a Banach space with respect to the graph norm $\| x \|_A := \| x \| + \| Ax \|$. We write $[\mathcal{D}(A)]$ for $(\mathcal{D}(A), \| \cdot \|_A)$.

(b) If $A$ is closed with $\mathcal{D}(A) = X$, then $A$ is even continuous (“closed graph theorem”).

(c) Let $A$ be injective and $\mathcal{D}(A^{-1}) := R(A) := \{ Ax; x \in \mathcal{D}(A) \}$. Then $A$ is closed if and only if $A^{-1}$ is closed.

(d) Let $A$ be closed and $f \in \mathcal{C}([a, b], X)$ with values in $\mathcal{D}(A)$ such that $Af \in \mathcal{C}([a, b], X)$. We then have
\[
\int_a^b f(t) \, dt \in \mathcal{D}(A) \quad \text{and} \quad A \int_a^b f(t) \, dt = \int_a^b Af(t) \, dt.
\]
An analogous result holds for piecewise continuous functions.

(e) For \( f \in \mathcal{C}([a, b], X) \) the function \([a, b] \to X; \quad t \mapsto \int_a^t f(\tau) \, d\tau\)
is differentiable with

\begin{equation}
\frac{d}{dt} \int_a^t f(\tau) \, d\tau = f(t) \quad \text{for all } t \in [a, b].
\end{equation}

For \( g \in \mathcal{C}^1([a, b], X) \) we have

\begin{equation}
\int_a^b g'(\tau) \, d\tau = g(b) - g(a).
\end{equation}

(f) Let \((f_n)_{n=1}^{\infty}\) be a sequence in \(\mathcal{C}^1(J, X)\) and \(f, g \in \mathcal{C}(J, X)\) for an interval \(J\) such that \(f_n \to f\) and \(f'_n \to g\) uniformly on \(J\) as \(n \to \infty\). Take any \(a \in J\). Formula (1.3) now gives

\[ f_n(t) = f_n(a) + \int_a^t f'_n(\tau) \, d\tau \]
for all \(t \in J\). Letting \(n \to 0\), we deduce that

\[ f(t) = f(a) + \int_a^t g(\tau) \, d\tau \]
for all \(t \in J\). Hence, \(f \in \mathcal{C}^1(J, X)\) and \(f' = g\) due to (1.2).

**Proof.** Parts (a) and (c) can easily be shown. Part (b) can be found in [ST] (Theorem 1.6). In part (f) there is nothing left to prove.

(d): Let \(f\) be as in the statement. Clearly, \(S(f, Z) \in \mathcal{D}(A)\) for any partition \(Z\) of \([a, b]\) and

\[ AS(f, Z) = \sum_{k=1}^m (Af)(\tau_k)(t_k - t_{k-1}) = S(Af, Z) \to \int_a^b Af(t) \, dt \]
as \(\Delta(Z) \to 0\) because \(Af\) is continuous. The assertion now follows from the closedness of \(A\).
(e): Let \( t \in [a, b] \) be arbitrary and \( h \neq 0 \) such that \( t + h \in [a, b] \). We can then estimate
\[
\left\| \frac{1}{h} \left( \int_a^{t+h} f(\tau) \, d\tau - \int_a^t f(\tau) \, d\tau \right) - f(t) \right\| = \left\| \frac{1}{h} \int_t^{t+h} (f(\tau) - f(t)) \, d\tau \right\| \leq \sup_{|\tau - t| \leq h} \|f(\tau) - f(t)\| \to 0
\]
as \( h \to 0 \). So we have shown (1.2). In the proof of Proposition 1.7 we have seen that a function belonging to \( C^1([a, b]) \) is constant if its derivative vanishes on the entire interval. Therefore (1.3) can be deduced from (1.2) by a standard argument from Analysis 1.

For a closed operator \( A \) we define the resolvent set
\[
\rho(A) := \{ \lambda \in \mathbb{C} ; \lambda I - A : \mathcal{D}(A) \to X \text{ is bijective} \}.
\]
For \( \lambda \in \rho(A) \) we write \( R(\lambda, A) \) for \((\lambda I - A)^{-1}\) and call it resolvent. The set \( \sigma(A) := \mathbb{C} \setminus \rho(A) \) is called the spectrum of \( A \). It is easy to see that \( \lambda I - A \) is closed. Hence, \( R(\lambda, A) \) is closed with domain \( X \) and thus bounded thanks to the closed graph theorem (see Remark 1.10). It is known that \( \rho(A) \) is open (and so \( \sigma(A) \) is closed). Moreover, if \( T \in \mathcal{B}(X) \), then \( \sigma(T) \) is even compact and always nonempty, and the spectral radius is given by
\[
r(T) := \max\{ |\lambda| ; \lambda \in \sigma(A) \} = \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}} = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}.
\]

There are closed operators \( A \) with \( \sigma(A) = \emptyset \) or \( \sigma(A) = \mathbb{C} \) (see below). We have the resolvent equation
\[
R(\mu, A) - R(\lambda, A) = (\lambda - \mu) R(\lambda, A) R(\mu, A) = (\lambda - \mu) R(\mu, A) R(\lambda, A).
\]
Furthermore, the resolvent function
\[
R(\cdot, A) : \rho(A) \to \mathcal{B}(X) ; \lambda \mapsto R(\lambda, A)
\]
is infinitely often differentiable (even analytic) with
\[
R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1}
\]
for all \( \lambda \in \rho(A) \) and \( n \in \mathbb{N}_0 \).

We continue to discuss the basic properties of \( C_0 \)-semigroups.

**Remark 1.11** (Rescaling)
Let \( A \) generate the \( C_0 \)-semigroup \( T(\cdot) , \lambda \in \mathbb{C} \), and \( a > 0 \). Then \( (S(t))_{t \geq 0} := \left( e^{\lambda t} T(at) \right)_{t \geq 0} \) is also a \( C_0 \)-semigroup with generator \( B = \lambda I + a A \) and \( \mathcal{D}(B) = \mathcal{D}(A) \).

**Proof.** For \( t, s \geq 0 \) we have \( S(t + s) = e^{\lambda t} T(at)e^{\lambda s} T(as) = S(t)S(s) \), and also \( S(0) = I \). The strong continuity of \( S(\cdot) \) is clear. Now, let \( B \) be the generator of \( S(\cdot) \). Because of
\[
\frac{1}{t}(S(t)x - x) = ae^{\lambda t} \frac{1}{at} (T(at)x - x) + \frac{1}{t}(e^{\lambda t} - 1)x,
\]
we have \( x \in \mathcal{D}(B) \) if and only if \( x \in \mathcal{D}(A) \), and in this case it holds that \( Bx = aAx + \lambda x \). □

**Lemma 1.12**

Let \( A \) generate the \( C_0 \)-semigroup \( T(\cdot) \), \( \lambda \in \mathbb{C} \), \( t > 0 \) and \( x \in X \). Then \( \int_0^t e^{-\lambda s}T(s)x \, ds \) belongs to \( \mathcal{D}(A) \) and

\[
(1.5) \quad e^{-\lambda t}T(t)x - x = (A - \lambda I) \int_0^t e^{-\lambda s}T(s)x \, ds;
\]

Furthermore, for \( x \in \mathcal{D}(A) \) we have

\[
(1.6) \quad e^{-\lambda t}T(t)x - x = \int_0^t e^{-\lambda s}T(s)(A - \lambda I)x \, ds.
\]

**Proof.** We only have to prove the case \( \lambda = 0 \) since the general case then follows by applying Remark 1.11. For \( h > 0 \) we compute

\[
\frac{1}{h}(T(h) - I) \int_0^t T(s)x \, ds = \frac{1}{h} \left( \int_0^t T(s + h)x \, ds - \int_0^t T(s)x \, ds \right)
\]

\[
= \frac{1}{h} \left( \int_0^{t+h} T(s)x \, ds - \int_0^t T(s)x \, ds \right)
\]

(1.7)

the last term tends to \( T(t)x - x \) as \( h \to 0 \) due to the continuity of the orbits and (1.2). But this precisely means that \( \int_0^t T(s)x \, ds \) is an element of \( \mathcal{D}(A) \) and (1.5) holds. If in addition \( x \in \mathcal{D}(A) \), we have \( T(\cdot)x \in \mathcal{C}^1([a, b], X) \) with \( \frac{d}{ds}T(\cdot)x = T(\cdot)Ax \) by Proposition 1.7 so that (1.3) implies (1.6). □

**Proposition 1.13**

Let \( A \) generate a \( C_0 \)-semigroup \( T(\cdot) \). Then \( A \) is closed and densely defined. Moreover \( T(\cdot) \) is the only \( C_0 \)-semigroup generated by \( A \). If \( \lambda \in \rho(A) \), then \( R(\lambda, A)T(t) = T(t)R(\lambda, A) \) holds for all \( t \geq 0 \).

**Proof.** 1) To show closedness of \( A \), we take any sequence \( (x_n)_n \) from \( \mathcal{D}(A) \) with limit \( x \in X \) such that \( \lim_{n \to \infty} Ax_n = y \) for some \( y \in X \). For \( t > 0 \) the equation (1.6) yields

\[
\frac{1}{t}(T(t)x_n - x_n) = \frac{1}{t} \int_0^t T(s)Ax_n \, ds
\]
for all \( n \in \mathbb{N} \) which leads to
\[
\frac{1}{t} (T(t)x - x) = \frac{1}{t} \int_0^t T(s)y \, ds
\]
by letting \( n \to \infty \). Because of (1.2) we see that the term on the right hand side converges to \( y \) as \( t \) tends to 0. Therefore we have
\[
\lim_{t \to 0} \frac{1}{t} (T(t)x - x) = y.
\]
Consequently, \( x \in \mathcal{D}(A) \) and \( Ax = y \); i.e., \( A \) is closed.

2) For \( x \in X \) and \( n \in \mathbb{N} \) we define
\[
x_n := \frac{1}{n} \int_0^1 T(s)x \, ds \in \mathcal{D}(A).
\]
As formula (1.2) gives us \( \lim_{n \to \infty} x_n = x \), we arrive at \( \mathcal{D}(A) = X \).

3): Let \( A \) generate another \( C_0 \)-semigroup \( S(\cdot) \). Then \( S(\cdot)x \) solves (1.1) for any \( x \in \mathcal{D}(A) \).

But the uniqueness of such a solution (see Proposition 1.7) forces \( T(\cdot)x = S(\cdot)x \) for all \( t \geq 0 \) and all \( x \in \mathcal{D}(A) \). Thus the bounded operators \( T(t) \) and \( S(t) \) coincide on a dense subset of \( X \) for every \( t \geq 0 \) which leads to \( T(\cdot) = S(\cdot) \) as desired.

4): Let \( \lambda \in \rho(A), t \geq 0 \) and \( x \in X \). Set \( y = R(\lambda, A)x \in \mathcal{D}(A) \). Proposition 1.7 yields that
\[
T(t)(\lambda y - Ay) = (\lambda I - A)T(t)y.
\]
Applying \( R(\lambda, A) \), we conclude that \( R(\lambda, A)T(t)x = T(t)R(\lambda, A)x \) as asserted. \( \square \)

Proposition 1.14

Let \( A \) generate the \( C_0 \)-semigroup \( T(\cdot) \) and \( \lambda \in \mathbb{C} \). Then the following assertions hold.

(a) If the improper integral
\[
R(\lambda)x := \int_0^\infty e^{-\lambda s}T(s)x \, ds := \lim_{t \to \infty} \int_0^t e^{-\lambda s}T(s)x \, ds
\]
exists in \( X \) for all \( x \in X \), then \( \lambda \in \rho(A) \) and \( R(\lambda) = R(\lambda, A) \).

(b) The integral in (a) exists even absolutely for all \( x \in X \) if \( \text{Re}\lambda > \omega_0(T) \) and so the spectral bound (of \( A \))
\[
s(A) := \sup\{\text{Re}\lambda; \lambda \in \sigma(A)\}
\]
is less than or equal to \( \omega_0(T) \).

(c) For all \( n \in \mathbb{N} \), all \( \lambda \in \mathbb{C} \) with \( \text{Re}\lambda > \omega_0(T) \) and all \( \omega \in (\omega_0(T), \text{Re}\lambda) \) we have
\[
\| R(\lambda, A)^n \| \leq \frac{M}{(\text{Re}\lambda - \omega)^n},
\]
where we take any \( M = M(\omega) \geq 1 \) with \( \| T(t) \| \leq Me^{\omega t} \) for all \( t \geq 0 \) (see Lemma 1.3).
Proof. (a): Let $h > 0$ and $x \in X$. Recall from Remark 1.11 that $T_{\lambda}(s) = e^{-\lambda s}T(s)$ is generated by $A - \lambda I$ with domain $\mathcal{D}(A)$. Then (1.7) yields
\[
\frac{1}{h}(T_{\lambda}(h) - I)R(\lambda)x = \lim_{t \to \infty} \frac{1}{h}(T_{\lambda}(h) - I) \int_{0}^{t} T_{\lambda}(s)x \, ds
\]
\[
= \lim_{t \to \infty} \frac{1}{h} \int_{t}^{t+h} T_{\lambda}(s)x \, ds - \frac{1}{h} \int_{0}^{h} T_{\lambda}(s)x \, ds
\]
\[
= -\frac{1}{h} \int_{0}^{h} T_{\lambda}(s)x \, ds,
\]
using the convergence of $-\frac{1}{h} \int_{0}^{\infty} T_{\lambda}(s)x \, ds$. By (1.2) we can let $h \to 0$ and obtain that $R(\lambda)x \in \mathcal{D}(A - \lambda I) = \mathcal{D}(A)$ and $(\lambda I - A)R(\lambda)x = x$. If $x \in \mathcal{D}(A)$, we deduce from $T(s)Ax = AT(s)x$ for $s \geq 0$ (see Proposition 1.7) and from Remark 1.10 (d) that
\[
R(\lambda)(\lambda I - A)x = \lim_{t \to \infty} \int_{0}^{t} e^{-\lambda s}T(s)(\lambda I - A)x \, ds
\]
\[
= \lim_{t \to \infty} (\lambda I - A) \int_{0}^{t} e^{-\lambda s}T(s)x \, ds
\]
using that $A$ is closed as shown in the previous proposition. The closedness of $A$ also implies that
\[
R(\lambda)(\lambda I - A)x = (\lambda I - A) \lim_{t \to \infty} \int_{0}^{t} e^{-\lambda s}T(s)x \, ds = (\lambda I - A)R(\lambda)x.
\]
Hence, part (a) is shown.

(b): Observe that $\|e^{-\lambda s}T(s)x\| \leq Me^{(\omega - \text{Re}\lambda)s}$ for each $x \in X$, $s \geq 0$, $\omega \in (\omega_0(T), \text{Re}\lambda)$ and some $M \geq 1$. For $0 < a < b$ we can thus estimate
\[
\left\| \int_{a}^{b} T_{\lambda}(s)x \, ds - \int_{0}^{a} T_{\lambda}(s)x \, ds \right\| \leq \int_{a}^{b} \|T_{\lambda}(s)x\| \, ds \leq M \int_{a}^{b} e^{(\omega - \text{Re}\lambda)s} \, ds \|x\| \to 0
\]
as $a, b \to \infty$. Consequently, $\int_{0}^{t} T_{\lambda}(s)x \, ds$ converges (absolutely) in $X$ as $t \to \infty$ for all $x \in X$. 

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(c): Let \( n \in \mathbb{N} \) and \( x \in X \). Note that it holds
\[
\left( \frac{d}{d\lambda} \right)^{-n} \int_{0}^{t} e^{-\lambda s} T(s)x \, ds = \int_{0}^{t} (-1)^{n-1} s^{n-1} e^{-\lambda s} T(s)x \, ds
\]
for all \( n \in \mathbb{N} \) and \( x \in X \). For the integrals the limits \( t \to \infty \) exist uniformly for \( \Re \lambda \geq \omega + \epsilon > \omega > \omega_0(T) \) and any \( \epsilon > 0 \). So (1.4), part (b) and Remark 1.10 (f) imply
\[
\begin{aligned}
R(\lambda, A)^n x &= \frac{(-1)^{n-1}}{(n-1)!} \left( \frac{d}{d\lambda} \right)^{n-1} \lim_{t \to \infty} \int_{0}^{t} T_\lambda(s)x \, ds \\
&= \lim_{t \to \infty} \frac{1}{(n-1)!} \int_{0}^{t} s^{n-1} T_\lambda(s)x \, ds = \frac{1}{(n-1)!} \int_{0}^{\infty} s^{n-1} e^{-\lambda s} T(s)x \, ds.
\end{aligned}
\]
As in (b), we conclude that
\[
\| R(\lambda, A)^n x \| \leq \frac{M}{(n-1)!} \| x \| \int_{0}^{\infty} s^{n-1} e^{(\omega - \Re \lambda)s} \, ds = \frac{M}{(\Re \lambda - \omega)^n} \| x \|
\]
for all \( \Re \lambda > \omega > \omega_0(T), x \in X \) and \( n \in \mathbb{N} \) since \( \epsilon \) is arbitrary. \( \square \)

**Example 1.15**

We investigate the generators of translation semigroups.

(a) Let \( T(t) f = f(\cdot + t) \) be the translation group on \( X := C_0(\mathbb{R}) \) (see Example 1.6). We want to compute its generator \( A \) and the spectrum \( \sigma(A) \). For \( f \in \mathcal{D}(A) \) and \( s \in \mathbb{R} \), there exist the pointwise limits
\[
Af(s) = \lim_{t \to 0} \frac{1}{t} (T(t) f(s) - f(s)) = \lim_{t \to 0} \frac{1}{t} (f(s + t) - f(s)) = f'(s)
\]
so that \( f \) is differentiable with \( f' = Af \in C_0(\mathbb{R}) \), i.e.,
\[
\mathcal{D}(A) \subseteq C^1_0(\mathbb{R}) := \{ f \in C^1(\mathbb{R}); \ f, f' \in X \}.
\]
Conversely, let \( f \in C^1_0(\mathbb{R}) \). For \( s \in \mathbb{R} \), we obtain
\[
\left| \frac{1}{t} (T(t) f(s) - f(s)) - f'(s) \right| = \left| \frac{1}{t} (f(s + t) - f(s)) - f'(s) \right| \leq \frac{1}{t} \int_{0}^{t} |f'(s + \tau) - f'(s)| \, d\tau
\]
thus arriving at
\[
\left| \frac{1}{t} (T(t) f(s) - f(s)) - f'(s) \right| \leq \sup_{0 \leq |\tau| \leq |t|} |f'(s + \tau) - f'(s)| \to 0,
\]
as \( t \to 0 \) uniformly in \( s \in \mathbb{R} \), since \( f' \in C_0(\mathbb{R}) \) is uniformly continuous. As a result, \( f \in \mathcal{D}(A) \) and thus \( A = \frac{d}{ds} \) with \( \mathcal{D}(A) = C_0^1(\mathbb{R}) \). By Proposition 1.14 we have \( s(A) \leq \omega_0(A) = 0 \), because
\[ T(t) = 1. \] Later, we shall see that \(-A\) generates the contraction \(C_0\)-semigroup \((S(t))_{t \geq 0} = (T(-t))_{t \geq 0}\). Hence, \(s(-A) \leq 0\). Due to \(-\lambda - A = -(\lambda - (-A))\), we have \(\sigma(-A) = -\sigma(A)\) as well as \(-R(\lambda, -A) = R(\lambda, A)\). So we have shown that \(\sigma(A) \subseteq i\mathbb{R}\). To verify the converse inclusion, let \(\text{Re}(\lambda) > 0\), \(f \in X\) and \(s \in \mathbb{R}\). Since all of the following limits exist with respect to the supremum norm in \(s\), Proposition 1.14 yields
\[
(R(\lambda, A)f)(s) = \left( \lim_{b \to \infty} \int_0^b e^{-\lambda t} T(t) f \, dt \right)(s) = \lim_{b \to \infty} \int_0^b e^{-\lambda t} (T(t)f)(s) \, dt
\]
\[ = \lim_{b \to \infty} \int_0^b e^{-\lambda s} f(t + s) \, dt = \lim_{b \to \infty} \int_s^{b+s} e^{\lambda(s-\tau)} f(\tau) \, d\tau = \int_s^{\infty} e^{\lambda(s-\tau)} f(\tau) \, d\tau,
\]
where we substituted \(\tau = s + t\). Consider now \(\lambda := \alpha + i\beta\) with \(\alpha > 0\) and \(\beta \in \mathbb{R}\). Take any \(f \in X\) with \(\|f\|_\infty = 1\). Then
\[
\|R(\lambda, A)\| \geq \|R(\lambda, A)f\|_\infty \geq |(R(\lambda, A)f)(0)| = \left| \int_0^{\infty} e^{-\alpha \tau} e^{-i\beta \tau} f(\tau) \, d\tau \right|.
\]
For \(n \in \mathbb{N}\) take any \(\varphi_n \in C_c(\mathbb{R})\) with \(0 \leq \varphi_n \leq 1\) and \(\varphi_n = 1\) on \([0, n]\). Inserting \(f_n(\tau) := e^{i\beta \tau} \varphi_n(\tau)\) in the above estimate, it follows
\[
\|R(\lambda, A)\| \geq \left| \int_0^{\infty} e^{-\alpha \tau} \varphi_n(\tau) \, d\tau \right| = \int_0^{\infty} e^{-\alpha \tau} \varphi_n(\tau) \, d\tau \geq \int_0^{n} e^{-\alpha \tau} \, d\tau = \frac{1}{\alpha} (1 - e^{-\alpha n}).
\]
Letting \(n \to \infty\), we arrive at \(\|R(\lambda, A)\| = \frac{1}{\text{Re}(\lambda)}\) (use Proposition 1.14 (c) with \(M = 1\), \(\omega = 0\) and \(n = 1\) there). If it were true that \(i\beta \in \rho(A)\) holds for some \(\beta \in \mathbb{R}\), then we would get
\[
\infty \leftarrow \frac{1}{\alpha} = \|R(\alpha + i\beta, A)\| \to \|R(i\beta, A)\|,
\]
as \(\alpha \to 0\), which is impossible. We thus obtain \(\sigma(A) = i\mathbb{R}\).

(b) We now consider the nilpotent left translation semigroup (see Example 1.6). For \(f \in X = C_0([0,1])\) and \(t \geq 0\) let
\[
(T(t)f)(s) = \begin{cases} 
  f(s + t), & \text{if } s \in [0, 1) \text{ and } s + t < 1, \\
  0, & \text{if } s \in [0, 1) \text{ and } s + t \geq 1.
\end{cases}
\]
Furthermore, let \(A\) be its generator. As in part (a), one shows
\[
\mathcal{D}(A) \subseteq C^1_0([0,1]) = \{ f \in C^1([0,1]); \ f, f' \in X \}
\]
as well as \(Af = f'\). Note that in this situation one can only consider \(t \to 0^+\). Therefore one has to use the following statement, which can be found, e.g., in \([P]\) (see Corollary 2.1.2 there). If \(g : [a, b] \to \mathbb{R}\) is continuous and everywhere differentiable from the right, such that the right hand side derivative \((g^{'})^+\) is continuous, then one already has \(g \in C^1([a, b])\).
For \( f \in C^1_0([0,1]), \ s \in [0,1) \) and \( t > 0 \), we set

\[
\tilde{f}(s) := \begin{cases} 
  f(s), & \text{if } s \in [0,1), \\
  0, & \text{if } s \geq 1.
\end{cases}
\]

Observe that \( \tilde{f} \in C^1_0(\mathbb{R}_+) \) and \( \tilde{f}|_{[0,1)} = f' \) hold. As in part (a), it follows

\[
\frac{1}{t}(T(t)f(s) - f(s)) = \begin{cases} 
  \frac{1}{t}(f(s + t) - f(s)), & \text{if } 0 \leq s < 1 - t, \\
  -\frac{1}{t}f(s), & \text{if } 1 - t \leq s < 1,
\end{cases}
\]

as \( t \to 0 \) uniformly in \( s \in [0,1) \), since \( \tilde{f} \) is uniformly continuous. Hence, \( \mathcal{D}(A) = C^1_0([0,1]) \) and \( Af = f' \). Here we have \( \omega_0(A) = -\infty \) so that \( \sigma(A) = \emptyset \) and \( \rho(A) = \mathbb{C} \).

(c) The operator \( Af = f' \) with \( \mathcal{D}(A) = C^1([0,1]) \) on \( X = C([0,1]) \) has the spectrum \( \sigma(A) = \mathbb{C} \). In fact, the function \( e_\lambda(t) := e^{\lambda t} \) (for \( 0 \leq t \leq 1 \)) belongs to \( \mathcal{D}(A) \) with \( Ae_\lambda = \lambda e_\lambda \) for all \( \lambda \in \mathbb{C} \). Thus \( \lambda \in \sigma(A) \). As a consequence, \( A \) is not a generator in view of Proposition 1.14.

(d) Let \( X = C_0(\mathbb{R}_-) := \{ f \in C((\infty,0]); \lim_{s \to -\infty} f(s) = 0 \} \) and \( A = \frac{d}{ds} \) with \( \mathcal{D}(A) = \{ f \in C^1(\mathbb{R}_-); f, f' \in C_0(\mathbb{R}_-) \} \). Then \( A \) is not a generator. Indeed, for all \( \lambda \) with \( \text{Re}(\lambda) > 0 \) we have \( e_\lambda \in \mathcal{D}(A) \) and \( A e_\lambda = \lambda e_\lambda \) so that \( \lambda \in \sigma(A) \), violating \( s(A) < \infty \) in Proposition 1.14.

(e) On \( X = C([0,1]) \) the operator \( Af = f' \) with \( \mathcal{D}(A) = \{ f \in C^1([0,1]); f(1) = 0 \} \) is not a generator, since \( \mathcal{D}(A) = \{ f \in X; f(1) = 0 \} \neq X \) (see Proposition 1.13).

### 1.2. Characterization of generators

In this section we want to show that the necessary conditions of a generator proved in Proposition 1.13 and 1.14 are even sufficient. Our main tool to achieve this aim is the so-called Yosida approximation: For \( \lambda \in \rho(A) \) we define

\[
A_\lambda := \lambda A R(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I \in \mathcal{B}(X). \tag{1.8}
\]

We note that

\[
AR(\lambda, A) = \lambda R(\lambda, A) - I \quad \text{and} \quad AR(\lambda, A)x = R(\lambda, A)Ax \tag{1.9}
\]

holds for all \( x \in \mathcal{D}(A) \).

#### Lemma 1.16

Let \( A \) be closed and \( M, \omega \geq 0 \) such that \( [\omega, \infty) \subseteq \rho(A) \) and \( \| R(\lambda, A) \| \leq \frac{M}{\lambda} \) for all \( \lambda \geq \omega \). We then have \( \lambda R(\lambda, A)x \to x \) as \( \lambda \to \infty \) for all \( x \in \mathcal{D}(A) \) and \( \lambda A R(\lambda, A)y \to Ay \) as \( \lambda \to \infty \) for all \( y \in \mathcal{D}(A) \).

**Proof.** Let \( x \in \mathcal{D}(A) \). Equation (1.9) and the assumption yield that

\[
\| \lambda R(\lambda, A)x - x \| = \| R(\lambda, A)Ax \| \leq \frac{M}{\lambda} \| Ax \| \to 0
\]

as \( \lambda \to \infty \).
as \( \lambda \to \infty \). Since \( \lambda R(\lambda, A) \) is uniformly bounded, the first assertion follows. The second one is an immediate consequence of the first assertion, taking \( x = Ay \) and using (1.9).

For linear operators \( A, B \) on \( X \) we write \( A \subseteq B \) if \( \text{gr}(A) \subseteq \text{gr}(B) \), i.e., if \( \mathcal{D}(A) \subseteq \mathcal{D}(B) \) and \( Ax = Bx \) for all \( x \in \mathcal{D}(A) \). In this case we call \( B \) an extension of \( A \).

**Lemma 1.17**

Let \( A \) and \( B \) be linear operators with \( A \subseteq B \) such that \( A \) is surjective and \( B \) is injective. Then \( A = B \) holds. In particular, we have \( A = B \) whenever \( A \subseteq B \) and \( \rho(A) \cap \rho(B) \neq \emptyset \) are satisfied.

**Proof.** We have to show that \( \mathcal{D}(B) \subseteq \mathcal{D}(A) \) holds. Let \( x \in \mathcal{D}(B) \). By the assumption, there is a \( y \in \mathcal{D}(A) \) with \( Bx = Ay = By \), from where we conclude \( x = y \in \mathcal{D}(A) \), using the injectivity of \( B \).

The addendum follows by considering \( \lambda I - A \) and \( \lambda I - B \) for some \( \lambda \in \rho(A) \cap \rho(B) \). Obviously, \( \lambda I - A \subseteq \lambda I - B \) for \( A \subseteq B \). The statement just shown thus gives \( \lambda I - A = \lambda I - B \), from where one easily deduces \( A = B \).

A *contraction semigroup* is a \( C_0 \)-semigroup \( T(t) \) with \( \|T(t)\| \leq 1 \) for all \( t \geq 0 \).

**Theorem 1.18** (Hille, Yosida 1948)

A linear operator \( A \) generates a contraction semigroup \( T(t) \) if and only if \( A \) is closed, densely defined, \( (0, \infty) \subseteq \rho(A) \) and

\[
\| R(\lambda, A) \| \leq \frac{1}{\lambda}
\]

holds for all \( \lambda > 0 \). In this case \( \mathbb{C}_+ := \{ z \in \mathbb{C}; \ \text{Re}(z) > 0 \} \) belongs to \( \rho(A) \) and we have \( \| R(\lambda, A)^n \| \leq \frac{1}{(\text{Re}(\lambda))^n} \) for all \( n \in \mathbb{N} \) and \( \lambda \in \mathbb{C}_+ \).

**Proof.** The necessity of the conditions and also the addendum follow from Propositions 1.13 and 1.14 (with \( M = 1 \) and \( \omega = 0 \)). In order to prove sufficiency, put \( A_n := nAR(n, A) = n^2 R(n, A) - nI \) for \( n \in \mathbb{N} \). Lemma 1.16 and the assumption imply that \( \lim_{n \to \infty} A_n y = Ay \) for all \( y \in \mathcal{D}(A) \). Let \( t \geq 0 \). Due to the exercises and the assumptions, we have

\[
\| e^{tA_n} \| = \| e^{-tn} e^{t^2 R(n, A) t} \| \leq e^{-tn} \sum_{j=0}^{\infty} \left( \frac{n^2 t \| R(n, A) \|}{j!} \right)^j \\
\leq e^{-tn} e^{n^2 \| R(n, A) \| t} \leq e^{-tn} e^{nt} = 1.
\]

(1.10)

Next, take \( n, m \in \mathbb{N}, t_0 > 0, y \in \mathcal{D}(A) \) and \( t \in [0, t_0] \). It holds \( A_n A_m = A_m A_n \) and hence

\[
A_n e^{tA_m} = A_n \sum_{j=0}^{\infty} \frac{t^j}{j!} A_m^j = \sum_{j=0}^{\infty} \frac{t^j}{j!} A_m^j A_n = e^{tA_m} A_n.
\]

Using (1.3), we then compute

\[
e^{tA_n} y - e^{tA_m} y = \int_0^t \frac{d}{ds} e^{(t-s)A_m} e^{sA_n} y \, ds = \int_0^t e^{(t-s)A_m} (A_n - A_m) e^{sA_n} y \, ds
\]

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Example 1.19
(a) Let $\mathcal{D}(A)$ be the generator of $T(t)$. Since $\mathcal{D}(A)$ is dense, (1.10) holds and $\tau > 0$ is arbitrary, we obtain contractions $T(t) \in \mathcal{B}(X)$ given by $T(t)x := \lim_{n \to \infty} e^{tA_n}x$ for all $t \geq 0$ and $x \in X$. Clearly, $T(0) = I$ and

$$T(t+s)x = \lim_{n \to \infty} e^{(t+s)A_n}x = \lim_{n \to \infty} e^{tA_n}e^{sA_n}x = T(t)T(s)x$$

for all $t, s \geq 0$ (use the remark after Lemma 1.8). Letting $m \to \infty$ in (1.11), we further deduce that

$$\|e^{tA_n}y - T(t)y\| \leq t_0\|A_ny - Ay\|$$

for all $t \in [0, t_0]$. As a result, $e^{tA_n}y$ converges to $T(t)y$ uniformly for $t \in [0, t_0]$ so that $T(\cdot)y$ is continuous for all $y \in \mathcal{D}(A)$. Because of the density of $\mathcal{D}(A)$, $T(\cdot)$ is a contraction $C_0$-semigroup.

Let $B$ be the generator of $T(\cdot)$. Observe that $(0, \infty) \subseteq \rho(A) \cap \rho(B)$ due to Proposition 1.14 and the assumptions. In view of Lemma 1.17 it thus remains to show $A \subseteq B$. For $t > 0$ and $y \in \mathcal{D}(A)$, as above we conclude

$$\frac{1}{t}(T(t)y - y) = \lim_{n \to \infty} \frac{1}{t}(e^{tA_n}y - y) = \lim_{n \to \infty} \frac{1}{t} \int_0^t e^{sA_n}A_nyds = \frac{1}{t} \int_0^t e^{sA}Ayds.$$  

Letting $t \to 0$, it follows that $y \in \mathcal{D}(B)$ and $By = Ay$, i.e., $A \subseteq B$. 

1.2. CHARACTERIZATION OF GENERATORS
As a result, \( R(\lambda)f(s) \to 0 \) as \( s \to -\infty \) so that \( \lambda \in \rho(A) \) and \( R(\lambda) = R(\lambda, A) \). We further have
\[
\| R(\lambda, A)f \|_\infty \leq \sup_{s \leq 0} \int_{-\infty}^{s} e^{-\lambda(s-\tau)} \|f\|_\infty \, d\tau = \|f\|_\infty \int_{0}^{\infty} e^{-\lambda r} \, dr = \frac{\|f\|_\infty}{\lambda}
\]
for all \( f \in X \) and all \( \lambda > 0 \). Theorem 1.18 thus implies that \( A \) generates a contraction semigroup \( T(t) \). Let now \( f \in D(A) \) and set \( u(t, s) = (T(t)f)(s) \) for \( t \geq 0 \) and \( s \leq 0 \). Proposition 1.7 yields that \( u \) solves the problem
\[
\begin{aligned}
\partial_t u(t, s) &= -\partial_s u(t, s), \quad t \geq 0, s \leq 0, \\
u(t, s) &\to 0 \text{ as } s \to -\infty, \quad t \geq 0, \\
u(0, s) &= f(s), \quad s \leq 0.
\end{aligned}
\]
We observe that \( u(t, s) = f(s - t) \) satisfies these equations, which leads to the guess that \( T(t) \) is the right translation. In fact, as in Examples 1.6 and 1.15 one can show that the right translation \( S(t)f = f(\cdot - t) \) \((t \geq 0)\) gives a \( C_0 \)-semigroup on \( X \) generated by \( A \).

Finally, if \( \Re(\lambda) < 0 \), then \( e^{-\lambda} \in D(A) \) satisfies \( Ae^{-\lambda} = -(e^{-\lambda})' = \lambda e^{-\lambda} \) so that \( \lambda \in \sigma(A) \) follows. Since \( s(A) \leq 0 \), we obtain \( \sigma(A) = \mathbb{C}_- := \{z \in \mathbb{C}; \Re(z) \leq 0\} \) due to the closedness of \( \sigma(A) \).

(b) Let \( X = C_0(\mathbb{R}) \times C_0(\mathbb{R}) \) with \( \| (f, g) \| := \max\{\|f\|_\infty, \|g\|_\infty\} \), let \( m(s) := is \) and
\[
A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} mu + mv \\ mv \end{pmatrix} = \begin{pmatrix} m \\ 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
\]
with \( \mathcal{D}(A) = \{(u, v) \in X; (mu, mv) \in X\} \). Since \( C_c(\mathbb{R}) \times C_c(\mathbb{R}) \subseteq \mathcal{D}(A) \), the domain \( \mathcal{D}(A) \) is dense in \( X \). One can check that \( A \) is closed and that for \( \Re(\lambda) > 0 \) one has \( \lambda \in \rho(A) \) with
\[
R(\lambda, A) = \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda - m} & \frac{m}{(\lambda - m)^2} \\ 0 & \frac{1}{\lambda - m} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \frac{f}{\lambda - m} + \frac{mg}{(\lambda - m)^2} \\ \frac{g}{\lambda - m} \end{pmatrix}.
\]
For \( \lambda > 0 \) and \( \|(f, g)\| \leq 1 \) we can therefore estimate
\[
\| R(\lambda, A) \begin{pmatrix} f \\ g \end{pmatrix} \| \leq \max\left\{ \left\| \frac{f}{\lambda - m} \right\|_\infty + \left\| \frac{mg}{(\lambda - m)^2} \right\|_\infty, \left\| \frac{g}{\lambda - m} \right\|_\infty \right\}
\]
\[
\leq \sup_{s \in \mathbb{R}} \left( \frac{1}{|\lambda - is|} + \frac{|s|}{|\lambda - is|^2} \right) \leq \frac{1}{\lambda} + \sup_{s \in \mathbb{R}} \frac{|s|}{\lambda^2 + s^2}
\]
\[
= \frac{3}{2} \lambda.
\]
On the other hand, for \( a > 0 \) and \( n \in \mathbb{N} \) we choose \( g_n \in C_0(\mathbb{R}) \) such that \( g_n(n) = 1 \) and \( \|g_n\|_\infty = 1 \). It then follows
\[
\| R(a + in, A) \| \geq \left\| R(a + in, A) \begin{pmatrix} 0 \\ g_n \end{pmatrix} \right\| \geq \left\| \frac{m}{(a + in - m)^2} g_n \right\|_\infty \geq \left| \frac{in}{(a + in - in)^2} g_n(n) \right| = \frac{n}{a^2}
\]
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where the right hand side tends to $\infty$ as $n \to \infty$. As a result, $R(\lambda, A)$ is unbounded on every imaginary line $\text{Re}(\lambda) = a$, violating the estimate in Proposition 1.14 (c). Thus $A$ does not generate a $C_0$-semigroup though it almost satisfies the assertions of Theorem 1.18.

There are operators satisfying even $\|R(\lambda, A)\| \leq \frac{c}{\text{Re}(\lambda)}$ for all $\lambda \in \mathbb{C}_+$ that do not generate a $C_0$-semigroup (see Example 2 in §12.4 of [H-P]).

(c) Let $X = C_0(\mathbb{R})$ be endowed with the norm
\[ \|f\| := \max\{\sup_{s \geq 0} |f(s)|, \sup_{s < 0} M|f(s)|\} \]
for some $M > 1$, which is equivalent to the supremum norm. Hence the translations $T(t)f = f(\cdot + t)$ give a $C_0$-semigroup on $(X, \| \cdot \|)$, too. Take any $t > 0$. Choose an $f \in C_0(\mathbb{R})$ such that $\|f\|_{\infty} = 1$ and $\text{supp} f \subseteq (0, t)$. Then $\|f\| = 1$, $\text{supp} T(t)f \subseteq (-t, 0)$, and so $\|T(t)\| \geq \|T(t)f\| = \sup_{-t \leq s \leq 0} M|f(s + t)| = M$. Since $\|T(t)\| \leq M$ holds, we have $\|T(t)\| = M$ for all $t > 0$. This means that there are bounded $C_0$-semigroups such that for each $\omega \in \mathbb{R}$ there is a $t > 0$ with $\|e^{\omega t}T(t)\| = e^{\omega t}M > 1$.

**Theorem 1.20** (Feller, Miyadera, Phillips 1952)
A linear operator $A$ generates a $C_0$-semigroup $T(\cdot)$ satisfying $\|T(t)\| \leq M e^{\omega t}$ for all $t \geq 0$ and some constants $M \geq 1$ and $\omega \in \mathbb{R}$ if and only if $A$ is closed, densely defined, $(\omega, \infty) \subseteq \rho(A)$ and
\[ \|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n} \]
holds for all $\lambda \in (\omega, \infty)$ and all $n \in \mathbb{N}$. If this is the case, we have $\{\lambda \in \mathbb{C}; \text{Re}(\lambda) > \omega\} \subseteq \rho(A)$ as well as $\|R(\lambda, A)^n\| \leq \frac{M}{(\text{Re}(\lambda) - \omega)^n}$ for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > \omega$ and all $n \in \mathbb{N}$.

**Proof.** As in Theorem 1.18 only sufficiency is left to be shown.

By considering the operator $B := A - \omega I$ instead of $A$, we may and will assume $\omega = 0$. (Indeed, if $B$ generates a bounded $C_0$-semigroup $S(\cdot)$, the operator $A$ generates the $C_0$-semigroup $(e^{\omega t}S(t))_{t \geq 0}$ due to Remark 1.11.) Using our assumptions, we define
\[ \|x\| := \sup_{\mu > 0, n \in \mathbb{N}_0} \|\mu^n R(\mu, A)^n x\| \leq M \|x\| \]
for all $x \in X$. Since $\|x\| \geq \|x\|$, this gives an equivalent norm on $X$. If we can prove that $\lambda R(\lambda, A)$ is contractive for $\|\cdot\|$ and all $\lambda > 0$, then Theorem 1.18 shows that $A$ generates a $C_0$-semigroup $T(\cdot)$ that is contractive for $\|\cdot\|$ and $\|\cdot\|$. This means $\|T(t)\| \leq M$ as needed, because of
\[ \|T(t)x\| \leq \|T(t)x\| \leq \|x\| \leq M \|x\| \]
for all $x \in X$.

In order to show the desired contractivity, we set
\[ \|x\|_\mu := \sup_{n \in \mathbb{N}_0} \|\mu^n R(\mu, A)^n x\| \]
for $x \in X$ and $\mu > 0$. As above, $\| \cdot \|_\mu$ is an equivalent norm on $X$. We further have
\[
\| \mu R(\mu, A)x \|_\mu = \sup_{n \in \mathbb{N}_0} \| \mu^{n+1} R(\mu, A)^{n+1} x \| \leq \| x \|_\mu.
\]
Let $0 < \lambda \leq \mu$. The resolvent equation yields
\[
y := R(\lambda, A)x = R(\mu, A)x + (\mu - \lambda) R(\mu, A) R(\lambda, A)x = \frac{1}{\mu} \cdot \mu R(\mu, A)(x + (\mu - \lambda)y).
\]
Consequently, we obtain
\[
\| y \|_\mu \leq \frac{1}{\mu} \| x + (\mu - \lambda)y \|_\mu \leq \frac{1}{\mu} \| x \|_\mu + \left( 1 - \frac{\lambda}{\mu} \right) \| y \|_\mu.
\]
We thus deduce $\lambda \| y \|_\mu \leq \| x \|_\mu$ which means $\| \lambda R(\lambda, A) \|_\mu \leq 1$ for all $0 < \lambda \leq \mu$. This estimate then implies
\[
\| \lambda^n R(\lambda, A)^n x \| \leq \| \lambda^n R(\lambda, A)^n x \|_\mu \leq \| x \|_\mu
\]
for all $x \in X$, $0 < \lambda \leq \mu$ and all $n \in \mathbb{N}$. Taking the supremum over $n \in \mathbb{N}_0$, it follows $\| x \|_\lambda \leq \| x \|_\mu$ for all $x \in X$ and all $0 < \lambda \leq \mu$. Using this monotonicity, we can now conclude
\[
\| \lambda R(\lambda, A)x \| = \sup_{\mu > 0} \| \lambda R(\lambda, A)x \|_\mu = \sup_{\mu \geq \lambda} \| \lambda R(\lambda, A)x \|_\mu \leq \sup_{\mu \geq \lambda} \| x \|_\mu = \| x \|
\]
for all $x \in X$ and $\lambda > 0$. \qed

Theorem 1.20 is also called the Hille-Yosida theorem.

**Remark 1.21**
If $T(\cdot)$ is a $C_0$-semigroup with $\| T(t) \| \leq M e^{\omega t}$, then
\[
\|||x||| = \sup_{s > 0} e^{-\omega s} \| T(s)x \|, \quad x \in X,
\]
defines an equivalent norm on $X$ for which $e^{-\omega t} T(t)$ is contractive for every $t > 0$. However, this renorming can destroy additional properties, for instance the Hilbert space structure (see Remark 1.2.19 in [G]).

**Lemma 1.22**
Let $T(\cdot)$ be a $C_0$-semigroup and $t_0 > 0$ such that $T(t_0)$ is invertible. Then $T(\cdot)$ can be extended to a $C_0$-group $(T(t))_{t \in \mathbb{R}}$.

**Proof.** Take $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\| T(t) \| \leq M e^{\omega t}$ holds for all $t \geq 0$ and set $c := \| T(t_0)^{-1} \|$. Let $0 \leq t \leq t_0$. We then have
\[
T(t_0) = T(t_0 - t)T(t) = T(t)T(t_0 - t),
\]
or equivalently
\[
I = T(t_0)^{-1}T(t_0 - t)T(t) = T(t)T(t_0 - t)T(t_0)^{-1}
\]
so that $T(t)$ has the inverse $T(t_0)^{-1}T(t_0 - t)$ with norm less than or equal to $M e^{\omega |t_0|}$. Furthermore, let $t = nt_0 + \tau$ for some $n \in \mathbb{N}$ and $\tau \in [0, t_0)$. In this case $T(t) = T(\tau)T(t_0)^n$
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has the inverse $T(t_0)^{-n}T(\tau)^{-1}$. We now define $T(t) := T(-t)^{-1}$ for $t \leq 0$. This definition gives a group, since for $t, s \geq 0$ we can calculate

$$T(-t)T(-s) = T(t)^{-1}T(s)^{-1} = (T(s)T(t))^{-1} = T(s + t)^{-1} = T(-s - t),$$

$$T(-t)T(s) = (T(s)T(t - s))^{-1}T(s) = T(t - s)^{-1}T(s)^{-1}T(s).$$

and similarly for $T(s)T(-t)$. Let $t \in [0, t_0]$ and $x \in X$. We then obtain

$$\|T(-t)x - x\| = \|T(-t)(x - T(t)x)\| \leq M_1\|x - T(t)x\| \to 0$$

as $t \to 0$. So $(T(t))_{t \in \mathbb{R}}$ is a $C_0$-group by Lemma 1.5.

**Corollary 1.23**

Let $A$ be a linear operator, $M \geq 1$ and $\omega \geq 0$. The following assertions are equivalent.

(a) The operator $A$ generates a $C_0$-group $(T(t))_{t \in \mathbb{R}}$ with $\|T(t)\| \leq M e^{\omega|t|}$ for all $t \in \mathbb{R}$.

(b) The operator $A$ generates a $C_0$-semigroup $(T_+(t))_{t \geq 0}$ and $-A$ with $\mathcal{D}(-A) := \mathcal{D}(A)$ generates a $C_0$-semigroup $(T_-(t))_{t \geq 0}$ with $\|T_+(t)\| \leq M e^{\omega t}$ for all $t \geq 0$.

(c) The operator $A$ is closed and densely defined and for all $\lambda \in \mathbb{R}$ with $|\lambda| > \omega$ we have $\lambda \in \rho(A)$ and $\|(|\lambda| - \omega)^n R(\lambda, A)\| \leq M$ for all $n \in \mathbb{N}$.

If any (and thus all) of these conditions is (are) fulfilled, one has $T_+(t) = T(t)$ and $T_-(t) = T(-t)$ for all $t \geq 0$.

**Proof.** “(a)⇒(b)”: Setting $T_+(t) := T(t)$ and $T_-(t) := T(-t)$ for each $t \geq 0$, one obtains $C_0$-semigroups with generators $A_\pm$. Since there exists $\frac{d}{dt}T(0)x = Ax$ for all $x \in \mathcal{D}(A)$, we have $A \subseteq \pm A_\pm$. Lemma 1.17 then yields $A = A_+$ because of $s(A), s(A_+) < \infty$. Let $x \in \mathcal{D}(A_-)$ and $t > 0$. We then obtain

$$\frac{1}{-t}(T(-t)x - x) = \frac{1}{-t}(T_-(t)x - x) \to -A_-x$$

and

$$\frac{1}{t}(T(t)x - x) = -T(t)\frac{1}{t}(T_-(t)x - x) \to -A_-x$$

as $t \to 0$ so that $x \in \mathcal{D}(A)$ and hence $A = -A_-.$

“(b)⇒(c)”:

For $\lambda > \omega$, the assertion follows from Propositions 1.13 and 1.14. For $\lambda < -\omega$, we recall that $\sigma(-A) = -\sigma(A)$ with $R(-\lambda, -A) = -R(\lambda, A)$. So we also obtain the asserted estimate for $\lambda < -\omega$.

“(c)⇒(a)”:

Theorem 1.20 implies that $A$ generates a $C_0$-semigroup $(T_+(t))_{t \geq 0}$ and $-A$ generates a $C_0$-semigroup $(T_-(t))_{t \geq 0}$ (arguing for $-A$ as in the previous step). Let $x \in \mathcal{D}(A) = \mathcal{D}(-A)$ and $t \geq s \geq 0$. Proposition 1.7 and its proof yield

$$\frac{d}{ds} T_+(s)T_-(s)x = T_+(s)AT_-(s)x + T_+(s)(-A)T_-(s)x = 0,$$
and hence $T_+(t)T_-(t)x = x$. Analogously, one obtains $T_-(t)T_+(t)x = x$. By approximation, it follows that $I = T_+(t)T_-(t) = T_-(t)T_+(t)$. According to Lemma 1.22 the $C_0$-semigroup $T_+(\cdot)$ can be extended to a $C_0$-group with generator $B$. We have $B \subseteq A$ by definition and so $A = B$ thanks to Lemma 1.17. □

1.3. Dissipative operators

Even in the contraction case, the Hille-Yosida theorem poses the difficult task to check a resolvent estimate for all $\lambda > 0$. In this section we prove the Lumer-Phillips Theorem 1.32 which allows to reduce this task to a more simple property of $A$, its dissipativity, and a certain range condition. To this aim, we have to discuss several concepts.

The duality set $J(x)$ of $x \in X$ is defined by

$$J(x) := \{ x^* \in X^*; \langle x, x^* \rangle = \| x \|^2 = \| x^* \|^2 \},$$

where $\langle x, x^* \rangle = x^*(x)$ for all $x \in X$ and $x^* \in X^*$. The Hahn-Banach theorem ensures that $J(x) \neq \emptyset$. We shall often use that $\| x \| = \| x^* \|$ if $x^* \in J(x)$. In the standard function spaces elements in the duality set can explicitly be computed.

**Example 1.24**

(a) Let $X$ be a Hilbert space. By the Riesz isomorphism theorem for each $y^* \in X^*$ there is a unique $y \in X$ such that $\langle x, y^* \rangle = (x|y)$ holds for all $x \in X$, where $(\cdot|\cdot)$ denotes the inner product on $X$. We further have $\| y \| = \| y^* \|$. As a result, $y^* \in J(x)$ implies $\| x \| = \| y \|$ and $(x|y) = \| x \| \cdot \| y \|$. The last equality holds if and only if $x$ and $y$ are linearly dependent (due to the characterization of equality in the Cauchy-Schwarz inequality) which gives together with the first equality the existence of some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $x = \alpha y$. Inserting this in the equations above leads to the fact that $y^* \in J(x)$ implies $y = x$. The reverse implication is obvious. Consequently, $J(x) = \{ (\cdot|y) \}$.

(b) Let $E = L^p(B)$ for a $p \in [1, \infty)$ and a Borel set $B \subseteq \mathbb{R}^d$. Furthermore, let $f \in E \setminus \{0\}$. We identify $E^*$ with $L^{p'}(B)$ (via the usual duality pairing), where $p' := \frac{p}{p-1}$ for $p > 1$ and $1' := \infty$. We set

$$g := \| f \|_p^{2-p} \int |f|^{p-2} =: \| f \|_{p^*}^{2-p} f^*$$

where $\frac{0}{0} := 0$. For $p = 1$, we have $\| f^* \|_{\infty} = 1$. For $p > 1$, it holds

$$\| f^* \|_{p'} = \left( \int_B |f|^{(p-1)\frac{p'}{p}} \right)^{\frac{p-1}{p}} = \| f \|_{p}^{p-1}.$$

In both cases it follows that $\| g \|_{p'} = \| f \|_{p}$. Moreover

$$\langle f, g \rangle = \| f \|_p^{2-p} \int_B |f|^{p-2} \cdot f^* \cdot f \, dx = \| f \|_p^{2-p} \cdot \| f \|_{p}^p = \| f \|_p^2$$

so that $g \in J(f)$. It can be shown that $J(f) = \{ g \}$ if $p \in (1, \infty)$, see Example II.3.26 of [E-N] or Aufgabe VII.5.37 combined with Aufgabe I.4.13 in [W].
1.3. DISSIPATIVE OPERATORS

Let \( \emptyset \neq U \subseteq \mathbb{R}^d \) be open and let \( E := C_0(U) = \{ f \in C(U); f(x) \to 0 \text{ as } x \to \partial U \text{ and as } |x| \to \infty \text{ for unbounded } U \}. \) For \( f \in E \) there is an \( x_0 \in U \) with \( |f(x_0)| = \|f\|_\infty \). Set \( \varphi(g) := \overline{f}(x_0)g(x_0) \) for \( g \in E \). Then \( \varphi \in E^* \), \( \|\varphi\| = |f(x_0)| = \|f\|_\infty \) and \( \varphi(f) = |f(x_0)|^2 = \|f\|^2_\infty \). Hence, \( \varphi \in J(f) \). The same construction works on \( E = C(K) \) for compact \( K \subseteq \mathbb{R}^d \).

**Definition 1.25**

A linear operator \( A \) is called dissipative if for each \( x \in \mathcal{D}(A) \) there is a \( x^* \in J(x) \) such that \( \text{Re} \langle Ax, x^* \rangle \leq 0 \). The operator \( A \) is called accretive if \( -A \) is dissipative.

**Proposition 1.26**

A linear operator \( A \) is dissipative if and only if \( \|\lambda x - Ax\| \geq \lambda \|x\| \) holds for all \( \lambda > 0 \) and all \( x \in \mathcal{D}(A) \). If \( A \) generates a contraction semigroup \( T(\cdot) \), then we have \( \text{Re} \langle Ax, x^* \rangle \leq 0 \) for all \( x \in \mathcal{D}(A) \) and all \( x^* \in J(x) \).

**Proof.**

1) Let \( A \) generate the contraction semigroup \( T(\cdot), x \in \mathcal{D}(A) \) and \( x^* \in J(x) \). The contractivity implies

\[
\text{Re} \langle Ax, x^* \rangle = \lim_{t \to 0} \text{Re} \left\langle \frac{1}{t}(T(t)x - x), x^* \right\rangle = \lim_{t \to 0} \frac{1}{t} (\text{Re} \langle T(t)x, x^* \rangle - \|x\|^2)
\]

\[
\leq \limsup_{t \to 0} \frac{1}{t} (\|x\| \cdot \|x^*\| - \|x\|^2) = 0.
\]

2) Let \( A \) be dissipative and \( x \in \mathcal{D}(A) \). There is an \( x^* \in J(x) \) such that \( \text{Re} \langle Ax, x^* \rangle \leq 0 \). So we obtain for any \( \lambda > 0 \) that

\[
\lambda \|x\|^2 \leq \text{Re}(\lambda(x, x^*)) - \text{Re} \langle Ax, x^* \rangle \leq |\langle \lambda x - Ax, x^* \rangle| \leq \|\lambda x - Ax\| \cdot \|x^*\|.
\]

Hence, \( \lambda \|x\| \leq \|\lambda x - Ax\| \) since \( \|x\| = \|x^*\| \).

3) Assume that \( \|\lambda x - Ax\| \geq \lambda \|x\| \) holds for all \( \lambda > 0 \) and \( x \in \mathcal{D}(A) \). For the rest of the proof we make the additional assumption that \( X \) is separable.\(^2\) Assume without loss of generality that \( \|x\| = 1 \). Take \( y_\lambda^* \in J(\lambda x - Ax) \). Because of \( \|y_\lambda^*\| = \|\lambda x - Ax\| \geq \lambda \|x\| = \lambda > 0 \) we have \( y_\lambda^* \neq 0 \). Next, we set \( x_\lambda^* := \frac{1}{\|y_\lambda^*\|} y_\lambda^* \). We then have \( \|x_\lambda^*\| = 1 \) as well as

\[
\lambda \leq \|\lambda x - Ax\| = \frac{1}{\|y_\lambda^*\|} \|\lambda x - Ax, y_\lambda^*\| = \text{Re} \langle \lambda x - Ax, x_\lambda^* \rangle = \lambda \text{Re} \langle x, x_\lambda^* \rangle - \text{Re} \langle Ax, x_\lambda^* \rangle
\]

\[
\leq \min\{\lambda - \text{Re} \langle Ax, x_\lambda^* \rangle, \lambda \text{Re} \langle x, x_\lambda^* \rangle + \|Ax\|\}.
\]

As a result, \( \text{Re} \langle Ax, x_\lambda^* \rangle \leq 0 \) and \( 1 - \frac{1}{\lambda} \|Ax\| \leq \text{Re} \langle x, x_\lambda^* \rangle \). Since \( \|x_\lambda^*\| \leq 1 \), the Banach-Alaoglu theorem (see, e.g., Theorem 4.33 in [FA]) gives a sequence \( (\lambda_n)_{n=1}^\infty \) in \((0, \infty)\) converging to \( \infty \) and an \( x^* \in X^* \) with \( \|x^*\| \leq 1 \) such that \( \lim_{n \to \infty} \langle y, x_{\lambda_n}^* \rangle = \langle y, x^* \rangle \) holds for all \( y \in X \). Thus we obtain in particular \( \text{Re} \langle Ax, x^* \rangle \leq 0 \) and \( 1 \leq \text{Re} \langle x, x^* \rangle \), implying

\[
1 \leq \text{Re} \langle x, x^* \rangle \leq \|x\| \cdot \|x^*\| = \|x^*\| \leq 1.
\]

\(^2\) The general case can be treated essentially in the same way, see, e.g., Proposition II.3.23 in [E-N].
Hence, \( \|x\| = 1 = \text{Re}\langle x, x^* \rangle = |\langle x, x^* \rangle| = \|x^*\| \) and so \( x^* \in J(x) \). Consequently, \( A \) is dissipative.

We discuss the dissipativity of the first derivative.

**Example 1.27**

(a) Let \( X = \mathcal{C}_0(\mathbb{R}) \) and \( b, c \in \mathcal{C}_0(\mathbb{R}) \) be real-valued. Define \( Au = bu' + cu \) with \( \mathfrak{D}(A) = \mathcal{C}_0^1(\mathbb{R}) \). Take \( u \in \mathfrak{D}(A) \) and let \( \varphi \in J(u) \) given by \( \varphi(v) = \bar{u}(s_0)v(s_0) \), where \( |u(s_0)| = \|u\|_\infty \) (see Example 1.24). We then have

\[
0 = \text{Re}\langle Au - \|c\|_\infty u, \varphi \rangle = b(s_0) \text{Re}(u'(s_0)\bar{u}(s_0)) + (c(s_0) - \|c\|_\infty) \text{Re}(u(s_0)\bar{u}(s_0))
\]

so that \( A - \|c\|_\infty J \) is dissipative.

(b) Let \( X = \mathcal{C}([0,1]) \) and \( b, c \in X \) be real-valued with \( b(0) \geq 0 \). Define \( A_j = bu' + cu \) with \( \mathfrak{D}(A_j) = \{u \in \mathcal{C}^1([0,1]) ; u'(j) = 0 \} \) with \( j \in \{0,1\} \). For \( u \in \mathfrak{D}(A_j) \) consider the functional \( \varphi(v) = \bar{u}(s_0)v(s_0) \) on \( X \), where \( |u(s_0)| = \|u\|_\infty \) for some \( s_0 \in [0,1] \). As in (a), one sees that \( \varphi \in J(u) \) and

\[
r = \text{Re}\langle A_j u - \|c\|_\infty u, \varphi \rangle \leq b(s_0) \text{Re}(u'(s_0)\bar{u}(s_0)) = b(s_0)h'(s_0)
\]

for the function \( h(s) = \text{Re}(\bar{u}(s_0)u(s)) \) attaining its maximum at \( s_0 \). If \( s_0 = 0 \), then it again follows that \( r \leq 0 \). If \( s_0 = 1 \), then \( h'(0) \leq 0 \); i.e., \( r \leq b(0)h'(0) \leq 0 \). However, if \( s_0 = 1 \), then \( h'(1) \geq 0 \). Here we have \( r \leq 0 \) for \( u \in \mathfrak{D}(A_1) \) so that \( A_1 - \|c\|_\infty J \) is dissipative. Let \( \omega \in \mathbb{R} \). If \( b(1) > 0 \), we can find a real-valued function \( u \in \mathfrak{D}(A_0) \) with maximum \( u(1) = 1 \) and a large derivative \( u'(1) \) such that

\[
\text{Re}\langle A_0 u - \omega u, \varphi \rangle = b(1)u'(1) + c(1) - \omega > 0
\]

so that \( A_0 - \omega J \) is not dissipative for any \( \omega \in \mathbb{R} \).

(c) Let \( X = L^2(\mathbb{R}) \) and \( A = \frac{d}{ds} \) with \( \mathfrak{D}(A) = \mathcal{C}_0^1(\mathbb{R}) \). For \( u \in \mathfrak{D}(A) \) we have \( \bar{u} \in J(u) \). Integration by parts yields

\[
2 \text{Re}\langle Au, \bar{u} \rangle = \langle Au, \bar{u} \rangle + \overline{\langle Au, \bar{u} \rangle} = \int_\mathbb{R} u'\bar{u} \, ds + \int_\mathbb{R} \bar{u}'u \, ds = 0.
\]

Hence, \( A \) is dissipative (but surely not closed).

(d) Let \( X = L^2([0,1]), A_j = \frac{d}{ds} \) and \( \mathfrak{D}(A_j) = \{u \in \mathcal{C}^1([0,1]); u(j) = 0 \} \) for \( j \in \{0,1\} \). For \( u \in \mathfrak{D}(A_j) \) we take again \( \bar{u} \in J(u) \) and obtain

\[
2 \text{Re}\langle Au, \bar{u} \rangle = \int_0^1 u'\bar{u} \, ds + \int_0^1 \bar{u}'u \, ds = u\bar{u}|^1_0 = (|u(1)|^2 - |u(0)|^2).
\]

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Hence, $A_1$ is dissipative. Moreover, $A_0 + \omega I$ is not dissipative for any $\omega \in \mathbb{R}$, since we can find $u \in \mathcal{D}(A_0)$ (depending on $\omega$) such that
\[
\text{Re} \langle A_0 u + \omega u, \pi \rangle = \frac{1}{2} |u(1)|^2 + \omega \|u\|_2 > 0.
\]

The examples (c) and (d) can be extended to $L^p$ with $p \in [1, \infty)$. In these examples, we have encountered dissipative, but nonclosed operators. To treat these (rather natural) operators, we need another concept.

**Intermezzo 2: Closable operators**

A linear operator $A$ is called *closable* if it possesses a closed extension.

**Lemma 1.28**

For a linear operator $A$, the following assertions are equivalent.

(a) The operator $A$ is closable.

(b) If $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{D}(A)$ tending to 0 such that $(Ax_n)_{n=1}^{\infty}$ converges to some $y \in X$, then $y = 0$ must hold.

(c) There is a smallest closed linear extension of $A$, called the closure $\overline{A}$ of $A$, which is given by defining $\mathcal{D}(\overline{A})$ as the set
\[
\{ x \in X; \exists (x_n)_{n=1}^{\infty} \in \mathcal{D}(A)^N, y \in X : \lim_{n \to \infty} x_n = x, \lim_{n \to \infty} Ax_n = y \}
\]

and setting $\overline{A}x := y$ for $x \in \mathcal{D}(\overline{A})$, and the vector $y$ from the definition of $\mathcal{D}(\overline{A})$.

If this is the case, then $\overline{\text{gr}(A)} = \text{gr}(\overline{A})$ and $\mathcal{D}(A)$ is dense in $[\mathcal{D}(\overline{A})]$.

**Proof.** “(a)⇒(b)”: Let $B$ be a closed extension of $A$. If $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ tending to 0 with $(Ax_n)_{n=1}^{\infty} = (Bx_n)_{n=1}^{\infty}$ converging to some $y \in X$, then the closedness of $B$ implies that $y = B(0) = 0$.

“(b)⇒(c)”: Let $(x_n)$ and $(z_n)$ be sequences in $\mathcal{D}(A)$ tending to $x \in X$ such that $(Ax_n)_n$ converges to $y$ and $(Az_n)_n$ converges to $w$ in $X$. Then $(x_n - z_n)_n$ is a null sequence in $\mathcal{D}(A)$ with $A(x_n - z_n) = Ax_n - Az_n \to y - w$ as $n \to \infty$. We conclude $y = w$ using the assumption (b).

Therefore $\overline{A}$ is well defined mapping. One easily verifies that $\overline{A}$ is linear with $\text{gr}(A) = \text{gr}(\overline{A})$, which shows the first part of the addendum. Hence, $\overline{A}$ is closed due to Remark 1.10 and $\overline{A}$ extends $A$. If $B$ is another closed extension of $A$, we have $\text{gr}(A) \subseteq \text{gr}(B)$ and thus $\text{gr}(\overline{A}) = \overline{\text{gr}(A)} \subseteq \text{gr}(B)$ because of the closedness of $B$. In particular, $B$ extends $\overline{A}$.

“(c)⇒(a)”: This implication is obvious.

The final assertion is an immediate consequence of $\overline{\text{gr}(A)} = \text{gr}(\overline{A})$ and the definition of the graph norm.

**Example 1.29**

(a) Let $X = L^1([0, 1])$ and $Af = f(0)1$ with $\mathcal{D}(A) = C([0, 1])$. This operator is not closable. In fact, the functions $f_n(s) = \max\{1 - ns, 0\}$ satisfy $\|f_n\|_1 = 1/2n \to 0$ as $n \to \infty$, but $Af_n = 1$
for all $n \in \mathbb{N}$, contradicting Lemma 1.28 (b).

(b) Let $X = \mathcal{C}([0, 1])$ and $A_0 u = u'$ with $\mathcal{D}(A_0) = \mathcal{C}_c^1(0, 1) := \mathcal{C}_c^1((0, 1))$. Since $A = \frac{d}{ds}$ with $\mathcal{D}(A) = \mathcal{C}_c^1(0, 1) := \mathcal{C}_c^1((0, 1))$ is closed with $A_0 \subseteq A$, we see that $A_0$ is closable and $\overline{A_0} \subseteq A$. Let $f \in \mathcal{C}_c^1(0, 1)$. Take $\varphi_n \in \mathcal{C}_c^1(0, 1)$ such that $\varphi = 1$ on $[1/n, 1-1/n]$, $0 \leq \varphi_n \leq 1$ and $\|\varphi_n\|_{\infty} \leq cn$ for some $c > 0$ and all $n \in \mathbb{N}$ with $n \geq 2$. For instance, one could take

$$\varphi_n(s) := \begin{cases} 
0, & \text{if } 0 < s < \frac{1}{4n}, \\
8n^2 \left(s - \frac{1}{4n}\right)^2, & \text{if } \frac{1}{4n} \leq s \leq \frac{1}{2n}, \\
1 - 8n^2 \left(\frac{3}{4n} - s\right)^2, & \text{if } \frac{1}{2n} \leq s \leq \frac{3}{4n}, \\
1, & \text{if } \frac{3}{4n} \leq s \leq \frac{1}{2}, \\
\varphi_n(1-s), & \text{if } \frac{1}{2} < s < 1,
\end{cases}$$

where $c = 4$. Then $f_n := \varphi_n f$ belongs to $\mathcal{D}(A_0)$ and we have

$$\|f_n - f\|_{\infty} = \sup_{0 \leq s \leq 1} |(\varphi_n(s) - 1)f(s)| \leq \sup_{0 \leq s \leq \frac{1}{4n}} |f(s)| \to 0$$

as well as

$$\|\varphi_n f' - f'\|_{\infty} \leq \sup_{0 \leq s \leq \frac{1}{4n}} |\varphi_n(s) f(s)| \to 0$$

as $n \to \infty$, since $f, f' \in \mathcal{C}_0(0, 1)$. We further obtain

$$\|\varphi_n f\|_{\infty} \leq \sup_{0 \leq s \leq \frac{1}{4n}} |\varphi_n(s) f(s)| + \sup_{1 - \frac{1}{4n} \leq s \leq 1} |\varphi_n(s) f(s)|$$

$$\leq \sup_{0 \leq s \leq \frac{1}{4n}} \int_0^s |f'(\tau)| d\tau + \sup_{1 - \frac{1}{4n} \leq s \leq 1} \int_s^1 |f'(\tau)| d\tau$$

$$\leq cn \int_0^{\frac{1}{2n}} |f'(\tau)| d\tau + cn \int_{\frac{1}{2n}}^1 |f'(\tau)| d\tau \to 0$$

as $n \to \infty$, again because of $f, f' \in \mathcal{C}_0(0, 1)$. Hence, $A_0(\varphi_n f) = \varphi_n f + \varphi_n f'$ converges to $Af = f'$. Consequently, $A \subseteq \overline{A_0}$ and thus $\overline{A_0} = A$.

There are further closed extensions of $A_0$. By an exercise, $A_1 = \frac{d}{ds}$ with $\mathcal{D}(A_1) = \{u \in \mathcal{C}^1([0, 1]); u'(1) = 0\}$ generates a $\mathcal{C}_0$-semigroup on $X$ and $\sigma(A_1) = \{0\}$. Moreover, $A_3 = \frac{d}{ds}$ with $\mathcal{D}(A_3) = \mathcal{C}^1([0, 1])$ has the spectrum $\sigma(A_3) = \mathbb{C}$ by Example 1.15. Clearly, $A_0 \subseteq \overline{A_0} = A \subseteq A_1 \subseteq A_2$. Lemma 1.17 thus implies $\rho(A) \cap \rho(A_1) = \emptyset$. Since $Au \neq 1$ for all $u \in \mathcal{C}_0^1(0, 1)$, $A$ is not surjective and so $\sigma(A) = \mathbb{C}$. In particular, $A$ is not a generator. On the other hand, also $A_2 = \frac{d}{ds}$ with $\mathcal{D}(A_2) = \{u \in \mathcal{C}^1([0, 1]); u(0) = 0\}$ is not a generator on $X$ (because $\overline{\mathcal{D}(A_2)} \neq X$), and we have $A \not\subseteq A_2 \not\subseteq A_3$, but $A_1$ and $A_2$ are not comparable.
Lemma 1.3. Employing Lemma 1.12, we can now estimate

\[ \text{such that} \]

\[ \|D\| \leq \frac{1}{2n}. \]

Such a \( D \) exists since for all \( y \in \mathcal{D}(A) \) the function

\[ [0, \infty) \to [\mathcal{D}(A)]; t \mapsto T(t)y \]

is continuous since \( T(t)y \in \mathcal{D}(A) \) with \( AT(t)y = T(t)Ay \), see Proposition 1.7. Observe furthermore that the above Riemann integral coincides with the Riemann integral of the function

\[ [0, \infty) \to (X, \| \cdot \|); t \mapsto T(t)y \]

due to the fact that \( \|y\| \leq \|y\|_A \) holds for all \( y \in \mathcal{D}(A) \) and thus the respective Riemann sums have the same limit. According to our assumption we can find a sequence \( (y_m)_{m=1}^{\infty} \) in \( \mathcal{D} \) converging to \( x \) in \( X \). We define \( x_m := \frac{1}{\tau} \int_0^\tau T(t)y_m \, dt \) for \( m \in \mathbb{N} \). Because of \( T(t)y_m \in \mathcal{D} \) for all \( t \geq 0 \) we deduce that \( x_m \) belongs to the closure \( \overline{\mathcal{D}}^A \) of \( \mathcal{D} \) in \( [\mathcal{D}(A)] \). Let \( M \) and \( \omega \) be as in Lemma 1.3. Employing Lemma 1.12, we can now estimate

\[ \|x - x_m\|_A \leq \left\| \frac{1}{\tau} \int_0^\tau T(t)x \, dt \right\|_A + \frac{1}{\tau} \left\| \int_0^\tau T(t)(x - y_m) \, dt \right\| + \frac{1}{\tau} \left\| A \int_0^\tau T(t)(x - y_m) \, dt \right\| \]

\[ \leq \frac{1}{2n} + \frac{1}{\tau} \int_0^\tau M e^{\omega t} \|x - y_m\| \, dt + \frac{1}{\tau} \|T(\tau) - I\| (x - y_m) \]

arriving at

\[ \|x - x_m\|_A \leq \frac{1}{2n} + \left( Me^{\omega} + \frac{1}{\tau} (1 + Me^{\omega}) \right) \|x - y_m\| \]

It follows that \( \limsup_{m \to \infty} \|x - x_m\|_A \leq \frac{1}{2n} \). As a consequence, we can find for every \( n \in \mathbb{N} \) an \( z_n \in \overline{\mathcal{D}}^A \) such that \( \|x - z_n\|_A < \frac{1}{n} \); i.e., the sequence \( (z_n)_n \) in \( \overline{\mathcal{D}}^A \) converges to \( x \) in \([\mathcal{D}(A)]\). □
Proposition 1.31
Let $A$ be dissipative. Then the following assertions hold.

(a) For all $\lambda > 0$ the operator $\lambda I - A$ is injective and for $y \in R(\lambda I - A) := (\lambda I - A)(\mathcal{D}(A))$ we have $\|(\lambda I - A)^{-1}y\| \leq \frac{1}{\lambda}\|y\|$. 

(b) Let $\lambda_0 I - A$ be surjective for some $\lambda_0 > 0$. Then $A$ is closed, $(0, \infty) \subseteq \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}$$

for all $\lambda > 0$.

(c) If $\mathcal{D}(A)$ is dense in $X$, then $A$ is closable and $\overline{A}$ is also dissipative.

Proof. Assertion (a) immediately follows from Proposition 1.26.

(b): If the assumptions in (b) hold, then part (a) implies that $\lambda_0 - A$ has an inverse with a norm less than or equal to $\frac{1}{\lambda_0}$; in particular, $A$ is closed. For any $\lambda \in (0, 2\lambda_0)$, we have $|\lambda - \lambda_0| < \lambda_0 \leq \|R(\lambda_0, A)\|^{-1}$. Therefore the series $R(\lambda) := \sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}$ converges in $\mathcal{B}(X, [\mathcal{D}(A)])$ for all such $\lambda$. One can check that $\lambda \in \rho(A)$ and $R(\lambda) = R(\lambda, A)$ hold (see, e.g., Theorem 1.13 in [ST]). This argument shows that we have $(0, 2\lambda_0) \subseteq \rho(A)$, whenever $\lambda \in \rho(A)$ is fulfilled. Applying this to some $\mu \in (\lambda, 2\lambda)$ for a $\lambda \in \rho(A)$ yields $2\lambda \in \rho(A)$. Thus we even have $(0, 2\lambda) \subseteq \rho(A)$, whenever $\lambda \in \rho(A)$ holds. As a consequence, one inductively obtains $(0, 2^n\lambda_0) \subseteq \rho(A)$ for all $n \in \mathbb{N}$ arriving at $(0, \infty) \subseteq \rho(A)$. The asserted estimate for $R(\lambda, A)$ is an immediate consequence of part (a).

(c): Assume $\overline{\mathcal{D}(A)} = X$. Let $(x_n)_n$ be a sequence in $\mathcal{D}(A)$ with limit 0 such that $(Ax_n)_n$ converges in $X$ to some $y \in X$. Due to the density assumption, there is a sequence $(y_n)_n$ in $\mathcal{D}(A)$ with limit $y$. Take any $\lambda > 0$ and $n, m \in \mathbb{N}$. Proposition 1.26 then implies

$$\|\lambda(\lambda I - A)x_n + (\lambda I - A)y_m\| \leq \|(\lambda I - A)(\lambda x_n + y_m)\| \geq \lambda\|\lambda x_n + y_m\|$$

Letting $n \to \infty$, we deduce $\| - \lambda y + (\lambda I - A)y_m\| \geq \lambda\|y_m\|$ or, equivalently,

$$\| - y + \left(I - \frac{1}{\lambda}A\right)y_m\| \geq \|y_m\|.$$ 

As $\lambda \to \infty$, it follows that $\| - y + y_m\| \geq \|y_m\|$. Taking the limit $m \to \infty$, we conclude $y = 0$. Due to Lemma 1.28, the operator $A$ is closable and for $x \in \mathcal{D}(\overline{A})$ there are $z_n \in \mathcal{D}(A)$ such that

$$\lim_{n \to \infty} z_n = x \text{ and } \lim_{n \to \infty} Az_n = \overline{A}x.$$ 

From Proposition 1.26 we can now infer the estimate

$$\|\lambda x - \overline{A}x\| = \lim_{n \to \infty}\|\lambda z_n - Az_n\| \geq \lim_{n \to \infty}\|z_n\| = \|x\|,$$

and thus the dissipativity of $\overline{A}$. \hfill \square

Theorem 1.32 (Lumer-Phillips, 1961)
Let $A$ be linear and densely defined. Then the following assertions hold.

(a) If $A$ is dissipative and $\lambda_0 I - A$ has dense range for some $\lambda_0 > 0$, then $\overline{A} \text{ generates a contraction semigroup.}$

(b) If $A$ is dissipative and $\lambda_0 I - A$ is surjective for some $\lambda_0 > 0$, then $A$ generates a contraction semigroup.

(c) If $A$ generates a contraction semigroup, then $A$ is dissipative and $\mathbb{C}_+ \subseteq \rho(A)$. 

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Intermezzo 3: Weak derivatives and Sobolev spaces

We refer to, e.g., Chapter 3 of [ST] for the following definitions and results. They are needed for many of the following examples.

Let $\emptyset \neq U \subseteq \mathbb{R}^d$ be open, $k \in \mathbb{N}$ and $p \in [1, \infty]$. A function $u \in L^p(U)$ has a \textit{weak derivative in} $L^p(U)$ with respect to the \textit{jth coordinate} for some $j \in \{1, \ldots, k\}$ if there is a $v \in L^p(U)$ such that

$$
\langle x - y, \lambda x^* \rangle = \langle x, A^* x^* \rangle$$

for all $x, y \in \mathcal{D}(A)$ and all $x^* \in \mathcal{D}(A^*)$. A corollary of the Hahn-Banach theorem then implies the density of $R(\lambda I - A)$. \hfill \Box

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For a linear operator $A$ with dense domain, we define its \textit{adjoint} $A^*$ by

$$
\mathcal{D}(A^*) = \{ x^* \in X^*; \exists y^* \in X^* \forall x \in \mathcal{D}(A) : \langle Ax, x^* \rangle = \langle x, y^* \rangle \}
$$

This means that $\langle Ax, x^* \rangle = \langle x, A^* x^* \rangle$ for all $x \in \mathcal{D}(A)$ and all $x^* \in \mathcal{D}(A^*)$. Recall from Remark 1.23 in [ST] that $A^*$ is a well defined closed linear operator.

Corollary 1.33

Let $A$ be dissipative and densely defined. If $\lambda I - A^*$ is injective for some $\lambda > 0$, then $A$ generates a contraction semigroup. (Observe that $\lambda I - A^*$ is injective for all $\lambda > 0$ if $A^*$ is dissipative according to Proposition 1.31.)

Proof. Due to Theorem 1.32, it suffices to show that $\lambda I - A$ has dense range. Let $x^* \in X^*$ belong to $R(\lambda I - A)^\perp$; i.e.,

$$
\langle (\lambda I - A)x, x^* \rangle = 0 \quad \text{for all } x \in \mathcal{D}(A).
$$

This fact is equivalent to

$$
\langle x, \lambda x^* \rangle = \langle x, A^* x^* \rangle \quad \text{for all } x \in \mathcal{D}(A),
$$

so that $\lambda x^* = A^* x^*$. Now the injectivity assumption yields $x^* = 0$. A corollary of the Hahn-Banach theorem then implies the density of $R(\lambda I - A)$. \hfill \Box
that
\[ \int_U u \partial_j \varphi \, dx = - \int_U v \varphi \, dx \]
holds for all \( \varphi \in C_c^\infty(U) \). The function \( v \) is (up to a null function) uniquely determined and we set in this case \( \partial_j u := v \). The set \( W_p^k(U) \) of all \( L^p(U) \)-functions \( u \) possessing weak derivatives in \( L^p(U) \) with respect to all coordinates is called a Sobolev space. The linear space \( W_p^k(U) \) becomes a Banach space when endowed with the norm
\[
\| u \|_{1,p} := \begin{cases} 
\left( \| u \|_p^p + \sum_{j=1}^d \| \partial_j u \|_p^p \right)^{1/p}, & \text{if } p < \infty, \\
\max_{j=1,\ldots,d} \{ \| u \|_\infty, \| \partial_j u \|_\infty \}, & \text{if } p = \infty.
\end{cases}
\]
(Here we identify, as usual, functions which are equal almost everywhere.) This norm is equivalent to the norm given by
\[
\| u \|_p + \sum_{j=1}^d \| \partial_j u \|_p.
\]

Higher order weak derivatives and Sobolev spaces \( W_p^k(U) \) are defined analogously. We summarize some important properties.

(a) If \( u \in C^k(U) \) and \( u \) and all its derivatives up to order \( k \) belong to \( L^p(U) \), then \( u \in W_p^k(U) \) and the classical and the weak derivatives coincide.

(b) Let \( p < \infty \). A function \( u \in L^p(U) \) belongs to \( W_p^k(U) \) if and only if there are \( u_n \in C^k(U) \cap W_p^k(U) \) such that \( u_n \to u \) in \( L^p(U) \) and all derivatives of \( u_n \) up to order \( k \) converge in \( L^p(U) \). We then have \( \partial_j u = \lim_{n \to \infty} \partial_j u_n \) in \( L^p(U) \) and analogously for higher derivatives up to order \( k \).

(c) \( C_c^\infty(\mathbb{R}^d) \) is dense in \( W_p^k(\mathbb{R}^d) \) for \( p < \infty \).

(d) A function \( u \in L^p((a,b)) \) (where \( -\infty \leq a < b \leq \infty \)) belongs to \( W_p^k((a,b)) \) if and only if \( u \) is continuous and there is a function \( v \in L^p((a,b)) \) such that
\[
(1.13) \quad u(t) = u(s) + \int_s^t v(\tau) \, d\tau \quad \text{for all } t, s \in (a,b).
\]

It then holds \( u' := \partial_1 u = v \) and \( u \) has a continuous extension to \( a \) (or \( b < \infty \)). For instance, let \( u(t) = |t| \) and \( v(t) = 1 \) for \( t > 0 \) and \( v(t) = -1 \) for \( t < 0 \). Then (1.13) holds, and thus \( u \in W_p^1(-1,1) \) with \( u' = v \).

(e) If \( u \in W_p^k(U) \) and \( v \in W_p^k(U) \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \), then \( uv \in W_p^k(U) \) and \( \partial_j (uv) = u \partial_j v + v \partial_j u \). Analogous results hold for higher derivatives.

We illustrate the above concepts by a simple example concerning generation properties of \( \frac{d}{ds} \) in \( L^2(\mathbb{R}) \).
Example 1.34

Let \( X = L^2(\mathbb{R}) \) and \( A = \frac{d}{dx} \) with \( \mathcal{D}(A) = C^1_c(\mathbb{R}) \). The operator \( A \) is densely defined and dissipative, thus closable, by Example 1.27. We have \( u \in W^1_2(\mathbb{R}) \) if and only if there are \( u_n \in C^1_c(\mathbb{R}) \) such that \( u_n \to u \) and \( u'_n = Au_n \to v \) in \( L^2(\mathbb{R}) \) for some \( v \in X \). Hence, the closure of \( A \) is given by \( \overline{A} = \frac{d}{dx} \) with \( \mathcal{D}(\frac{d}{dx}) = W^1_2(\mathbb{R}) \). Due to Proposition 1.31, \( \overline{A} \) is dissipative. Next, we want to compute \( \overline{A}^* \).

For \( u, v \in C^1_c(\mathbb{R}) \) we have

\[
\langle Au, v \rangle = \int u'v \, ds = -\int uv' \, ds.
\]

By approximation, the second equation holds for all \( u, v \in W^1_2(\mathbb{R}) = \mathcal{D}(\overline{A}) \). Thus, \( \langle \overline{A}u, v \rangle = \langle u, -v' \rangle \) for all \( u, v \in W^1_2(\mathbb{R}) \) so that \( (-\frac{d}{dx}, W^1_2(\mathbb{R})) \subseteq \overline{A}^* \). Conversely, let \( v \in \mathcal{D}(\overline{A}^*) \). Then \( v \) and \( \overline{A}^*v \) belong to \( L^2(\mathbb{R}) \) and

\[
\int u\overline{A}^*v \, ds = \langle u, \overline{A}^*v \rangle = \langle Au, v \rangle = \int u'v \, ds
\]

for all \( u \in C^\infty(\mathbb{R}) \subseteq \mathcal{D}(A) \) which means that \( v \in W^1_2(\mathbb{R}) \) and \( \overline{A}^*v = -v' = -A\overline{v} \). As a result, \( \overline{A}^* = -\overline{A} = (-\frac{d}{dx}, W^1_2(\mathbb{R})) \). As for \( \overline{A} \), we see that \( \overline{A}^* \) is dissipative. So Corollary 1.33 shows that \( \overline{A} \) generates a contraction semigroup \( T(\cdot) \). On the other hand, the translation group \( S(t)f = f(\cdot + t) \) on \( X \) (see Example 1.6) has a generator \( B \). For \( f \in \mathcal{D}(A) \) the functions \( w(t) = \frac{1}{t}(S(t)f - f) \) converge uniformly to \( f' \). Moreover, \( \text{supp} \, w(t) \subseteq \text{supp} \, f + [-1, 0] \) for \( 0 \leq t \leq 1 \). As a result, \( w(t) \) tends to \( f' \) in \( X \). This means \( A \subseteq B \) and so \( \overline{A} \subseteq B \). Lemma 1.17 now yields \( \overline{A} = B \) and hence \( T(\cdot) = S(\cdot) \).

We conclude this section by a discussion of isometric groups.

**Corollary 1.35**

For a linear operator \( A \), the following assertions are equivalent.

(a) The operator \( A \) generates an isometric \( C_0 \)-group \( T(\cdot) \), i.e., \( \|T(t)x\| = \|x\| \) for all \( x \in \mathcal{D}(A) \) and for all \( t \in \mathbb{R} \).

(b) The operator \( A \) is closed and densely defined, \( A \) and \( -A \) are dissipative and \( \lambda I - A \) and \( \lambda I + A \) are surjective for some \( \lambda > 0 \).

(c) The operator \( A \) is closed and densely defined, \( \mathbb{R} \setminus \{0\} \subseteq \rho(A) \) and \( \|R(\lambda, A)\| \leq \frac{1}{|\lambda|} \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \).

**Proof.** The Lumer-Phillips theorem 1.32 says that (b) holds if and only if \( A \) and \( -A \) generate contraction semigroups. So Corollary 1.23 implies the equivalence of assertion (b) and (c) and that (b) holds if and only if \( A \) generates a contractive \( C_0 \)-group \( T(\cdot) \). It remains to show that a contractive \( C_0 \)-group \( T(\cdot) \) is already isometric. Indeed, we have for all \( x \in X \) and
all \( t \in \mathbb{R} \)
\[
\|T(t)x\| \leq \|x\| = \|T(-t)T(t)x\| \leq \|T(-t)\| \cdot \|T(t)x\| \leq \|T(t)x\|
\]
so that \( T(t) \) is isometric. \( \square \)

Let \( X \) be a Hilbert space and \( A \) be a linear operator on \( X \) with dense domain. We define the 
Hilbert space adjoint \( A’ \) as in (1.12) replacing the duality pairing \( \langle x, x^* \rangle \) by the inner product \( (x|y) \). A linear operator \( A \) on \( X \) is called symmetric if \( A \subseteq A’ \); i.e.,
\[
\forall x, y \in \mathcal{D}(A) : (Ax|y) = (x|Ay),
\]
and it is called selfadjoint if \( A = A’ \), i.e., \( A \) is symmetric and
\[
\mathcal{D}(A) = \{ y \in X; \exists z \in X \forall x \in \mathcal{D}(A) : (Ax|y) = (x|z) \}
\]
\[
= \{ y \in X; \mathcal{D}(A) \ni x \mapsto (Ax|y) \text{ is continuous} \},
\]
where the last equality holds due to the Riesz representation theorem on Hilbert spaces. Since \( A’ \) is closed (which can be seen as in Remark 1.23 in [ST]), any (densely defined) symmetric operator is closable with \( \overline{A} \subseteq A’ \) and any selfadjoint operator is closed. Now, let \( A \) be symmetric. For \( u, v \in \mathcal{D}(\overline{A}) \), there are sequences \( (u_n)_n \) and \( (v_n)_n \) in \( \mathcal{D}(A) \) with limits \( u \), respectively \( v \), such that \( \lim_{n \to \infty} Au_n = \overline{A} u \), respectively \( \lim_{n \to \infty} Av_n = \overline{A} v \). We now obtain
\[
(\overline{A}u|v) = \lim_{n \to \infty} (Au_n|v_n) = \lim_{n \to \infty} (u_n|Av_n) = (u|\overline{A}v)
\]
so that \( \overline{A} \) is also symmetric. However, there are symmetric, closed operators that are not selfadjoint. (For instance, \( A = i \mathbb{I}_{\mathbb{D}} \) with \( \mathcal{D}(A) = \{ u \in W^2_1(0, 1); u(0) = u(1) \} \), where \( \mathcal{D}(A’) = W^2_2(0, 1) \). Recall from Theorem 4.18 of [ST] that a symmetric closed operator is selfadjoint if and only if \( \sigma(A) \subseteq \mathbb{R} \), and that this holds if \( \rho(A) \cap \mathbb{R} \neq \emptyset \). A linear operator \( A \) is called skewadjoint if \( A = -A’ \) which is equivalent to the selfadjointness of \( iA \). Finally, \( T \in \mathcal{B}(X) \) is unitary if it has the inverse \( T^{-1} = T’ \).

The next result belongs to the mathematical foundations of quantum mechanics.

Theorem 1.36 (Stone, 1930)
Let \( X \) be a Hilbert space and \( A \) be a linear operator on \( X \) with a dense domain. Then \( A \)
generates a \( C_0 \)-group of unitary operators if and only if \( A \) is skewadjoint.

Proof. 1): Let \( A’ = -A \). For \( x \in \mathcal{D}(A) \), we have \( J(x) = \{ \varphi_x \} \) where \( \varphi_x := (|x) \) (see Example 1.27). Hence, \( \langle Ax, \varphi_x \rangle = (Ax|x) = -(x|Ax) = -(\overline{Ax}|x) \), so that \( \text{Re}(Ax, \varphi_x) = 0 \). Therefore \( A \) and \( A’ = -A \) are dissipative as well as \( A” = (-A’)’ = A \). Corollary 1.33 thus yields that \( A \) and \( A’ \) generate contraction semigroups. Therefore \( A \) generates a \( C_0 \)-group \( (T(t))_{t \in \mathbb{R}} \) of invertible isometrical operators due to Corollary 1.35, implying that each \( T(t) \) is unitary (see, e.g., Proposition 2.24 in [FA]).

2): Let \( A \) generate a unitary \( C_0 \)-group \( (T(t))_{t \in \mathbb{R}} \). Since \( T(t)' = T(t)^{-1} = T(-t) \) for \( t \geq 0 \), the family \( (T(t)’\rangle_{t \geq 0} \) is a contraction semigroup having the generator \( -A \) by Corollary 1.23.
For \( x, y \in \mathcal{D}(A) \) we thus obtain
\[
(Ax|y) = \lim_{t \to 0} \left( \frac{1}{t} (T(t)x - x)\right) y = \lim_{t \to 0} \left( x \frac{1}{t} (T(t)'y - y) \right) = (x - Ay).
\]
This means that \(-A \subseteq A'\). We further know from Corollary 1.23 that \( \sigma(A) \subseteq i\mathbb{R} \). Since \( \sigma(A') = \{ \lambda ; \lambda \in \sigma(A) \} \subseteq i\mathbb{R} \) (see, e.g., (4.1) in [ST]), Lemma 1.17 yields \(-A = A'\). \(\square\)

1.4. Examples with the Laplacian

Example 1.37 (Laplacian in the supremum norm) Let \( \emptyset \neq U \subseteq \mathbb{R}^d \) be open and \( E = C_0(U) \). We define \( A_0 u := \Delta u := \sum_{j=1}^d \partial_j \partial_j u \) for \( u \in \mathcal{D}(A_0) = \{ u \in C^2(U); u, \Delta u \in E \} \) Clearly, \( \mathcal{D}(A_0) \) is dense in \( E \) (in fact, \( C_c^\infty(U) \) is dense in \( E \)). For \( u \in \mathcal{D}(A_0) \), we have the functional \( \varphi \in J(U) \) given by \( \varphi(v) = \hat{u}(x_0)v(x_0) \), where \( x_0 \in U \) satisfies \( |u(x_0)| = \|u\|_\infty \). Setting \( h = \Re(u(x_0)u) \), it follows
\[
\Re(A_0 u, \varphi) = \Re(\Delta u(x_0)u(x_0)) = \Delta h(x_0).
\]
Since \( u \in C_0(U) \), we see as in Example 1.27 (a) that \( h(x_0) \) is a maximum of \( h \). By Analysis 2, \( D^2 h(x_0) \) is negative semidefinite so that \( \Delta h(x_0) = \text{tr}(D^2 h(x_0)) \leq 0 \) and \( A_0 \) is dissipative. Proposition 1.31 now gives the dissipative closure \( A \) of \( A_0 \). In what follows, we discuss this operator.

(a) Let \( U = \mathbb{R}^d \). Let \( f \) belong to the Schwartz space
\[
\mathcal{S}_d = \{ u \in C^\infty(\mathbb{R}^d); \forall m \in \mathbb{N}_0 \forall \alpha \in \mathbb{N}_0^d : \sup_{x \in \mathbb{R}^d} |x|^{2m} |\partial^\alpha u(x)| < \infty \}.
\]
Here \( \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} \). The Fourier Transform
\[
\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \text{e}^{-i\xi \cdot x} f(x) \, dx
\]
is bijective from \( \mathcal{S}_d \) to \( \mathcal{S}_d \), where \( \xi \cdot x \) denotes the usual inner product \( \sum_{j=1}^d \xi_j x_j \). It holds \( \mathcal{F}^{-1}f(y) = f(-y) \) (see, e.g., Chapter 5 of [FA]). Moreover,
\[
\partial^\alpha \hat{f} = (-i)^{|\alpha|} \mathcal{F}(x^\alpha f) \quad \text{and} \quad \mathcal{F}(\partial^\alpha f) = i^{|\alpha|} \xi^\alpha \hat{f},
\]
where \( |\alpha| = \sum_{j=1}^d \alpha_j \), \( x^\alpha = \prod_{j=1}^d x_j^{\alpha_j} \) and \( x^\alpha f \) is the function \( x \mapsto x^\alpha f(x) \). Hence, \( \mathcal{F}\Delta u = -|\xi|^2 \hat{u} \) and \( -\Delta \mathcal{F}^{-1}v = \mathcal{F}^{-1}|\xi|^2 v \) for \( u, v \in \mathcal{S}_d \) (where \( |\xi| \) denotes the Euclidean norm). Note that \( \frac{1}{1 + |\xi|^2} \hat{f} \in \mathcal{S}_d \) and set \( u = \mathcal{F}^{-1} \left( \frac{1}{1 + |\xi|^2} \hat{f} \right) \in \mathcal{S}_d \subseteq \mathcal{D}(A_0) \). As a result,
\[
uu - \Delta u = \mathcal{F}^{-1} \left( \frac{1}{1 + |\xi|^2} \hat{f} \right) = f
\]
We have thus shown that \( I - A_0 \) has dense range and so \( A = \overline{A_0} \) generates a contraction semigroup on \( C_0(\mathbb{R}^d) \) by the Hille-Yosida theorem. Actually, we have proved that \( A \) is also the closure of \( (\Delta, \mathcal{S}_d) \). Hence, \( u \in \mathcal{D}(A) \) if and only if there are \( u_n \in \mathcal{S}_d \) such that \( u_n \to u \) and \( \Delta u_n \to Au \) as \( n \to \infty \). We want to determine \( \mathcal{D}(A) \) at least in special cases.
Let \( d = 1 \) and thus \( E = C_0(\mathbb{R}) \). Then \( u_n \to u \) and \( u_n'' \to Au \) as \( n \to \infty \) if \( u \in \mathcal{D}(A) \). We further need to control the first derivative. To achieve this aim let \( v \in C^2([a, b]), \delta \in (0, b - a) \) and \( a \leq r < s \leq b \) with \( \delta \leq s - r \leq 2\delta \). There is a \( \sigma \in (r, s) \) such that

\[
v(s) = v(r) + v'(r)(s - r) + v''(\sigma) \frac{(s - r)^2}{2},
\]
or equivalently

\[
(1.14) \quad v'(r) = \frac{v(s) - v(r)}{s - r} - v''(\sigma) \frac{s - r}{2}.
\]
The last equation yields

\[
|v'(r)| \leq \frac{2}{\delta} \|v\|_{\infty} + \delta \|v''\|_{\infty}
\]
which implies

\[
(1.15) \quad \|v'\|_{\infty} \leq \frac{2}{\delta} \|v\|_{\infty} + \delta \|v''\|_{\infty}
\]
Applying this with \( v = u_n - u_m \), it follows that \( u'_n \) converges in \( C_0(\mathbb{R}) \) and by Analysis 1/2 the limit equals \( u' \). As a result, \( u \in C^2(\mathbb{R}) \) and \( Au = u'' \in E \). Therefore we have

\[
\mathcal{D}(A) \subseteq C_0^2(\mathbb{R}) = \{u \in C^2(\mathbb{R}) ; u, u', u'' \in E\} \subseteq \mathcal{D}(A_0).
\]
As a result, \( \mathcal{D}(A) = C_0^2(\mathbb{R}) = \mathcal{D}(A_0) \). The corresponding result in \( \mathbb{R}^d \) with \( d \geq 2 \) is however wrong! Here one obtains

\[
\mathcal{D}(A) = \{u \in C_0(\mathbb{R}^d) ; u \in W_p^2(B(0, r)) \text{ for all } p \in (1, \infty), r > 0, \text{ and } \Delta u \in C_0(\mathbb{R}^d)\}
\]
(see, e.g., Corollary 3.19 in [L]). See also Example 4.12 in [ST] for an example indicating that \( \mathcal{D}(A) \neq \mathcal{D}(A_0) \) if \( d \geq 2 \).

(b) Let \( U = (0, 1) \) and \( E = C_0(0, 1) \). We define \( Au = u'' \) and \( \mathcal{D}(A) = \{u \in C^2(0, 1) ; u, u'' \in E\} \).

Let \( \lambda \in \mathbb{C} \setminus \mathbb{R}_- \) and let \( \mu = \sqrt{\lambda} \). Take \( f \in E \). We then have

\[
u \in \mathcal{D}(A), \lambda u - u'' = f \iff u \in C^2([0, 1]), \lambda u - u'' = f \text{ and } u(0) = u(1) = 0
\]

\[
\iff u(s) = \frac{1}{2\mu} \int_0^1 e^{-\mu|s-\tau|} f(\tau) \, d\tau + a e^{\mu s} + b e^{-\mu s}
\]
and \( u(0) = u(1) = 0 \) for some \( a, b \in \mathbb{C} \).

(The above integral term can be guessed using the formula for \( u \) in (a); cf. Beispiel 3.14 in [FT] and Chapter 5 in [FA].) As a result, \( \lambda \in \rho(A) \) and

\[
(1.16) \quad R(\lambda, A)f(s) = a(f, \mu) e^{\mu s} + b(f, \mu) e^{-\mu s} + \frac{1}{2\mu} \int_0^1 e^{-\mu|s-\tau|} f(\tau) \, d\tau
\]

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\[
\begin{pmatrix}
    a(f, \mu) \\
    b(f, \mu)
\end{pmatrix} = \frac{1}{2\mu(e^{-\mu} - e^{-\mu})}
\begin{pmatrix}
    e^{-\mu} \int_0^1 (e^{\mu \tau} - e^{-\mu \tau}) f(\tau) \, d\tau \\
    \int_0^1 (e^{\mu \tau} - e^{-\mu \tau}) f(\tau) \, d\tau
\end{pmatrix}
\]

(see Example 5.9 in [ST]). In particular, \( A \) generates a contraction semigroup.

(c) Let \( U \subseteq \mathbb{R}^d \) be bounded with a boundary of class \( C^2 \). Then \( A = \overline{A_0} \) also generates a contraction semigroup on \( C_0(U) \) and its domain is given as in (a), see Corollary 3.1.21 in [L] (where also other boundary conditions are treated). The needed density of \( (I - \Delta)\mathcal{D}(A_0) \) in \( C_0(U) \) also follows from Theorem 4.3 in [G-T].

Example 1.38 (Laplacian on \( L^2(\mathbb{R}^d) \))

Let \( E = L^2(\mathbb{R}^d) \) and \( A_0 u = \Delta u \) with the dense domain \( \mathcal{D}(A_0) = \mathcal{S}_d \). For \( u, v \in \mathcal{D}(A_0) \), Green’s formula gives

\[
(A_0 u | v) = \int_{\mathbb{R}^d} \Delta u \, \overline{v} \, dx = \lim_{r \to \infty} \int_{B(0,r)} \Delta u \, \overline{v} \, dx
\]

\[
= \lim_{r \to \infty} \left( \int_{B(0,r)} u \Delta \overline{v} \, dx + \int_{\partial B(0,r)} \overline{v} (\nabla u) \cdot d\sigma - \int_{\partial B(0,r)} u \nu \cdot \nabla \overline{v} \, d\sigma \right)
\]

\[
= \int_{\mathbb{R}^d} u \Delta \overline{v} \, dx = (u | A_0 v)
\]

since \( u, v \) and their derivatives converge faster to 0 than \( |x|^{-2m} \) for any \( m \in \mathbb{N} \), as \( |x| \to \infty \). (Here \( \nu \) denotes the outer unit normal.) Hence, \( A_0 \) is symmetric. In the same way, one obtains

\[
(A_0 u | u) = \int_{\mathbb{R}^d} \Delta u \, \overline{u} \, dx = - \int_{\mathbb{R}^d} \nabla u \cdot \nabla \overline{u} \, dx = - \int_{\mathbb{R}^d} |\nabla u|^2 \, dx \leq 0
\]

so that \( A_0 \) is dissipative. By Example 1.37 (a), the range of \( I - A_0 \) contains \( \mathcal{S}_d \) and is thus dense in \( E \). Therefore \( A = \overline{A_0} \) generates a contraction semigroup and \( A \) is symmetric. Since \( s(A) \leq 0 \), \( A \) is even selfadjoint (recall Theorem 4.18 in [ST]). As in, e.g., Example 4.11 of [ST], it can be shown that on \( \mathcal{S}_d \) the graph norm of \( A_0 \) is equivalent to the norm of \( W^2_2(\mathbb{R}^d) \). (The argument given there only uses integration by parts for \( u \in \mathcal{S}_d \).) Since \( \mathcal{S}_d \) is dense both in \( \mathcal{D}(A) \) and in \( W^2_2(\mathbb{R}^d) \), it follows that \( \mathcal{D}(A) = W^2_2(\mathbb{R}^d) \).

Let \( 1 \leq p < \infty \) and \( U \subseteq \mathbb{R}^d \) be bounded with a \( C^2 \) boundary. The trace map

\[
\mathcal{C}(U) \cap W^1_p(U) \to \mathcal{C}(\partial U); \quad u \mapsto u|_{\partial U}
\]

can be extended to a linear continuous map

\[
T : W^1_p(U) \to L^p(\partial U, \, d\sigma).
\]

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The kernel of $T$ is equal to $\hat{W}_p^1(U)$, the closure of $C_0^\infty(U)$ in $W_p^1(U)$, see §5.5 in [E] or Theorem 4.8 of [ST]. We also have the integration by parts formula

$$\int_U \text{div} \, u \, v \, dx = - \int_U u \cdot \nabla v \, dx + \int_{\partial U} (v' - u) v \, d\sigma$$

for $u \in W_p^1(U)^d$ and $v \in W_p^1(U)$, where we write $u_j$ and $v$ instead of $Tu_j$ and $Tv$ (cf. Lemma 4.20 of [ST]). If $U = (a, b)$ is one-dimensional, one easily deduces from (1.13) the special case

$$\int_a^b u' \, dx = - \int_a^b u \, v' \, dx + u(b)v(b) - u(a)v(a)$$

for all $u \in W_p^1(a, b)$ and $v \in W_p^1(a, b)$.

**Example 1.39** (one-dimensional Dirichlet Laplacian on $L^2$)

Let $E = L^2(0, 1)$ and $Au = u''$ with the dense domain $\mathfrak{D}(A) = \{u \in W^2_0(0, 1); u(0) = u(1) = 0\}$. For $u, v \in \mathfrak{D}(A)$, formula (1.18) gives

$$\langle Au | v \rangle = \int_0^1 u'' \tau \, d\tau = - \int_0^1 u' \tau' \, d\tau + \int_0^1 u \tau'' \, d\tau = (u | Av)$$

as well as

$$\langle Au | u \rangle = - \int_0^1 |u'|^2 \, d\tau \leq 0$$

since the boundary terms vanish because of $u, v \in \mathfrak{D}(A)$. So $A$ is symmetric and dissipative. For $f \in E$, $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$ and $\mu = \sqrt{\lambda}$ we introduce as in (1.16) the function

$$R(\mu)f(s) = a(f, \mu) e^{\mu s} + b(f, \mu) e^{-\mu s} + \frac{1}{2\mu} \int_0^s e^{\mu \tau} f(\tau) \, d\tau + \frac{1}{2\mu} \int_s^1 e^{-\mu \tau} f(\tau) \, d\tau$$

for $s \in [0, 1]$. By (1.13), the integrals define functions with the weak derivative $\pm e^{\pm \mu s} f(s)$ which belong to $E$. Thus we can deduce that $R(\mu)f \in W^2_0(0, 1)$ and $(\lambda I - \frac{d^2}{ds^2}) R(\mu)f = f$. Moreover, $R(\mu)f(0) = R(\mu)f(1) = 0$. This solution is unique since all solutions of the homogeneous problem are given by $ce^{\pm \mu s}$. Summing up, $\lambda \in \rho(A)$ and $R(\mu) = R(\mu, A)$. We conclude that $A$ is selfadjoint and generates a contraction semigroup on $E$.

Below we will need Poincaré’s inequality. For any bounded open nonempty set $U \subseteq \mathbb{R}^d$ and any $p \in [1, \infty)$, there is a constant $c = c(U, p) > 0$ such that

$$\|f\|_p \leq c \|\nabla f\|_p$$

for all $f \in \hat{W}_p^1(U)$ (see, e.g., Theorem 5.6.3 in [E] or (3.13) and Remark 3.20 in [ST]).
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**Theorem 1.40** (Lax-Milgram lemma)

Let $Y$ be a Hilbert space and $a : Y^2 \to \mathbb{C}$ be sesquilinear such that

$$|a(x, y)| \leq c\|x\| \cdot \|y\| \quad \text{(boundedness)}$$

and

$$\text{Re } a(x, x) \geq \delta \|x\|^2 \quad \text{(strict accretivity)}$$

hold for all $x, y \in Y$ and some constants $c, \delta > 0$. Then for each $\psi \in Y^*$ there is a unique $z \in Y$ such that $a(u, z) = \psi(u)$ for all $u \in Y$. The map $\psi \mapsto z$ is antilinear and bounded.

**Proof.** The map $\varphi_y := a(\cdot, y)$ belongs to $Y^*$ with $\|\varphi_y\| \leq c\|y\|$ for each $y \in Y$. By Riesz’ theorem there is a unique $S_y \in Y$ such that $\|S_y\| = \|\varphi_y\| \leq c\|y\|$ and $(\cdot | S_y) = \varphi_y$. Moreover,

$$\delta \|x\|^2 \leq \text{Re } a(x, x) = \text{Re}(x | Sx) \leq |(x | Sx)| \leq c\|x\| \cdot \|Sx\|$$

for every $x \in X$ which implies

$$\frac{\delta}{c} \|x\| \leq \|Sx\| \leq c\|x\|.$$ 

As a consequence, $S$ is linear, bounded, injective and has a closed range $\text{R}(S)$ (see, e.g., Corollary 3.19 in [FA]). For $x \in \text{R}(A)^\perp$, we obtain in particular

$$0 = (x | Sx) = \text{Re}(x | Sx) = \text{Re } a(x, x) \geq \delta \|x\|^2$$

so that $x = 0$. As a result, $\text{R}(S) = \overline{\text{R}(S)} = Y$ and $S$ is invertible with $\|S^{-1}\| \leq \frac{\delta}{c}$.

Now, let $\psi \in Y^*$ be arbitrary. Then there is a unique $v \in Y$ such that $\psi = (\cdot | v)$ thanks to Riesz’ theorem. Hence,

$$a(u, S^{-1}v) = (u | SS^{-1}v) = (u | v) = \psi(u)$$

for all $u \in Y$ and the result follows with $z = S^{-1}v = S^{-1}T\psi$, where $T : Y^* \to Y$ denotes the antilinear isomorphism from Riesz’ theorem. \[\square\]

**Example 1.41** (Dirichlet Laplacian on $L^2$)

Let $\emptyset \neq U \subseteq \mathbb{R}^d$ be open and bounded with $C^2$ boundary. Define $A = \Delta$ with the dense domain $\mathcal{D}(A) = W_0^2(U) \cap W_1^2(U)$. For $u, v \in \mathcal{D}(A)$, formula (1.17) and estimate (1.19) yield

$$(Au | v) = \int_U \text{div}(\nabla u)v \, dx = -\int_U \nabla u \cdot \nabla v \, dx + \int_{\partial U} (v \cdot \nabla u) v \, d\sigma$$

and

$$(Au | u) = -\int_U |\nabla u|^2 \, dx \leq -\delta \|u\|_2^2$$

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for a constant $\delta > 0$ since the boundary terms vanish for $u, v \in \mathcal{D}(A)$. Hence, $A$ is symmetric and dissipative. In order to show the range condition in the Lumer-Phillips theorem, we consider the sesquilinear form

$$a(u, v) := \int_U \nabla u \cdot \nabla v \, dx$$

for $u, v \in W^1_2(U) =: Y$. Observe that $a$ is bounded and strictly accretive on $W^1_2(U)$. Let $f \in L^2(U)$. Then $u \mapsto \int_U f u \overline{f} \, dx$ defines an element in $Y^*$. The Lax-Milgram lemma gives a unique $v_f \in Y$ such that $a(u, v_f) = \int_U f u \overline{f} \, dx$ for all $u \in Y$. We define

$$\mathcal{D}(\tilde{A}) = \{ v \in Y; \exists c > 0 \forall u \in Y : |a(u, v)| \leq c \|u\|_2 \}.$$  

(Note that the above $v_f$ belongs to $\mathcal{D}(\tilde{A})$ with $c = \|f\|_2$.) If $v \in \mathcal{D}(\tilde{A})$, the map $a(\cdot, v)$ can be extended to an element in $L^2(U)^*$ and there thus exists a unique $w \in L^2(U)$ such that $a(\cdot, v) = (\cdot | w)$ on $Y$. We then define $Av = w$. Observe that $A v_f = f$ and so $\tilde{A}$ is surjective. Moreover, $-\tilde{A} \subseteq \tilde{A}$ since

$$a(u, v) = \int_U \nabla u \cdot \nabla v \, dx = -\int_U u \Delta \overline{v} \, dx + \int_{\partial U} u \nu \cdot \nabla \overline{v} \, d\sigma = (u | -\Delta v)$$

if $u \in Y$ and $v \in \mathcal{D}(A)$ due to (1.17). For $u, v \in \mathcal{D}(\tilde{A})$ we further compute

$$(u | \tilde{A} v) = a(u, v) = a(v, u) = (v | Au) = (\tilde{A} u | v)$$

as well as

$$(-\tilde{A} u | u) = -(u | \tilde{A} u) = -a(u, u) \leq -\delta \|u\|^2_2 = -\delta (u | u).$$

Hence, $-\tilde{A}$ is symmetric and dissipative. Moreover, $\delta I - \tilde{A}$ is also dissipative and $\delta I - (\delta I - \tilde{A}) = \tilde{A}$ is surjective. So, by the Lumer-Phillips theorem, $\delta I - \tilde{A}$ generates a contraction semigroup which means that also $-\tilde{A}$ generates a contraction semigroup and $s(-\tilde{A}) \leq -\delta$ because of $s(\delta I - \tilde{A}) \leq 0$. Therefore $\tilde{A}$ is selfadjoint.

So far we only know that $(\Delta, \mathcal{D}(A)) \subseteq -\tilde{A}$. However, it can be shown that $(\Delta, \mathcal{D}(A))$ is surjective, and hence $\tilde{A} = (\Delta, \mathcal{D}(A))$ by Lemma 1.17 (see Theorem 8.3, 8.8, 8.12 in [G-T]).

**Example 1.42** (wave equation)

Let $\emptyset \neq U \subseteq \mathbb{R}^d$ be open and bounded with $C^2$ boundary. We want to consider the wave equation

$$\begin{cases}
\partial_t u(t, x) = \Delta u(t, x),  \\
u(t, x) = 0,  \\
u(0, x) = f(x), \quad \partial_t u(0, x) = g(x),
\end{cases}$$

$t \in \mathbb{R}, x \in U,$ $t \in \mathbb{R}, x \in \partial U,$ $x \in U,$

\[38\]
for given $f \in \dot{W}^1_2(U) =: Y$ and $g \in L^2(U)$. We define $\Delta_D u = \Delta u$ with $\mathcal{D}(\Delta_D) = W^2_2(U) \cap Y$ on $L^2(U)$. We further introduce the Hilbert space $E := Y \times L^2(U)$ with the inner product

$$
\left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_E = \int_U (\nabla u_1 \cdot \nabla v_1 + u_2 v_2) \, dx
$$

and the operator

$$
A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix}
$$
on $\mathcal{D}(A) = \mathcal{D}(\Delta_D) \times Y$. For $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{D}(A)$ we compute

$$
\left( A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_E = \left( \begin{pmatrix} u_2 \\ \Delta_D u_1 \end{pmatrix} \middle| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_E = \int_U (\nabla u_2 \cdot \nabla v_1 + \Delta u_1 v_2) \, dx
$$

$$
= - \int_U (u_2 \Delta v_1 + \nabla u_1 \cdot \nabla v_2) \, dx = - \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_{\Delta_D v_1}
$$

using (1.17) and that the traces of $u_2$ and $v_2$ on $\partial U$ vanish. We thus arrive at

$$
\left( A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_E = \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| - A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_E
$$

Hence, $-A \subseteq A'$ and so $iA \subseteq (iA)'$. We define

$$
R = \begin{pmatrix} 0 & \Delta_D^{-1} \\ I & 0 \end{pmatrix} : E \to \mathcal{D}(\Delta_D) \times Y = \mathcal{D}(A),
$$

where $\Delta_D^{-1}$ exists thanks to Example 1.41. It is easy to see that $AR = I$ and $RA \omega = \omega$ for every $\omega \in \mathcal{D}(A)$. Hence, $A$ is invertible so that $0 \in \rho(iA)$ and $iA$ is selfadjoint by Theorem 4.18 in [ST]. As a consequence, $A$ is skewadjoint, and thus generates a unitary $C_0$-group on $E$ due to Stone’s theorem 1.36. The connection to the wave equation will be discussed below in Example 2.4.
CHAPTER 2

Wellposedness and regularity

2.1. Wellposedness

Let \( A \) be a closed operator on \( X \). For any \( x \in \mathcal{D}(A) \), we consider the Cauchy problem or evolution equation

\[
\begin{cases}
u'(t) = Au(t), & t \geq 0, \\ u(0) = x.
\end{cases}
\]

Recall that a (classical) solution of (2.1) (on \( \mathbb{R}_+ \)) is a function \( u \in C^1(\mathbb{R}_+, X) \) such that \( u \) is even \( \mathcal{D}(A) \)-valued and satisfies (2.1) for all \( t \geq 0 \). Observe that then \( Au \in C(\mathbb{R}_+, X) \) and thus \( u \in C(\mathbb{R}_+, [\mathcal{D}(A)]) \).

Definition 2.1

The Cauchy problem (2.1) is called wellposed if \( \mathcal{D}(A) \) is dense in \( X \), if for each \( x \in \mathcal{D}(A) \) there is a unique solution \( u = u(\cdot ; x) \) of (2.1) and if \( u(t; x_n) \) converges uniformly for \( t \) in compact subsets of \( \mathbb{R}_+ \) to \( u(\cdot ; x) \) whenever \( x_n, x \in \mathcal{D}(A) \) and \( x_n \) tends to \( x \) in \( X \) as \( n \to \infty \) ("continuous dependence on initial data ").

Theorem 2.2

A closed operator \( A \) generates a \( C_0 \)-semigroup if and only if (2.1) is wellposed. In this case, the function \( u = T(\cdot)x \) solves (2.1) for any given \( x \in \mathcal{D}(A) \).

Proof. If \( A \) is a generator, then \( T(\cdot)x \) is the unique solution of (2.1) according to Proposition 1.7 and the solution continuously depends on the initial data since \( T(t) \) is locally uniformly bounded.

Conversely, let (2.1) be wellposed. We define the operator \( T(t) \) by \( T(t)x = u(t; x) \) for \( x \in \mathcal{D}(A) \) and \( t \geq 0 \) using uniqueness. For \( x, y \in \mathcal{D}(A) \) and \( \alpha, \beta \in \mathbb{C} \), the function \( \alpha u(\cdot ; x) + \beta u(\cdot ; y) \) solves (2.1) with initial value \( \alpha x + \beta y \) since \( A \) is linear. Hence, \( T(t) \) is linear for every \( t \geq 0 \). For each \( t_0 > 0 \) there is a \( c > 0 \) such that \( \| T(t)x \| \leq c \| x \| \) for all \( x \in \mathcal{D}(A) \) and all \( t \in [0, t_0] \). In fact, if this assertion were wrong, there would exist \( t_0 > 0 \), a sequence \( (x_n)_n \) in \( \mathcal{D}(A) \) and a sequence \( (t_n)_n \) in \( [0, t_0] \) such that \( \| x_n \| = 1 \) and \( \| T(t_n)x_n \| =: c_n \to \infty \) as \( n \to \infty \). Set \( y_n := \frac{1}{c_n} x_n \in \mathcal{D}(A) \) for every \( n \in \mathbb{N} \). The initial values \( y_n \) tend to 0 as \( n \to \infty \), but \( \| u(t_n; y_n) \| = \frac{1}{c_n} \| T(t_n)x_n \| = 1 \) does not converge to 0. This contradicts the wellposedness of (2.1) and thus \( T(\cdot) \) is locally uniformly bounded. So we can extend each single operator \( T(t) \) to a continuous operator on \( \overline{\mathcal{D}(A)} = X \) (also denoted by \( T(t) \)) without enlarging the operator norm. Therefore the extended family \( (T(t))_{t \geq 0} \) is still locally uniformly bounded.
Clearly, $T(0) = I$. Since $t \mapsto T(t)x \in X$ is continuous on $\mathbb{R}_+$ for any $x \in \mathcal{D}(A)$ and $\overline{\mathcal{D}(A)} = X$, we also deduce the strong continuity of $T(\cdot)$ on $X$. Furthermore, let $t, s \geq 0$ and $x \in \mathcal{D}(A)$. Then $u(s; x)$ belongs to $\mathcal{D}(A)$ so that $v(t) = T(t)u(s; x) = u(t; u(s; x))$ for $t \geq 0$ is the unique solution of (2.1) with initial value $u(s; x)$. On the other hand, $u(t+s; x) = T(t+s)x$ for $t \geq 0$ solves this problem, too. As a result, $T(t)T(s)x = T(t+s)x$ which gives the semigroup property by approximation. Let $B$ be the generator of $T(\cdot)$. By definition, we have $A \subseteq B$. Since $\mathcal{D}(A)$ is dense in $X$ and $T(t)\mathcal{D}(A) \subseteq \mathcal{D}(A)$ for all $t \geq 0$, Proposition 1.30 shows that $\mathcal{D}(A)$ is a core of $B$. So for any $x \in \mathcal{D}(B)$, there are $x_n \in \mathcal{D}(A)$ such that $x_n \rightarrow x$ and $Ax_n = Bx_n \rightarrow Bx$ in $X$ as $n \rightarrow \infty$. The closedness of $A$ now implies $x \in \mathcal{D}(A)$ and $A = B$. \hfill \bbox

**Remark 2.3**

One cannot drop the continuous dependance on initial data in Theorem 2.2. Consider, e.g., a closed, densely defined, unbounded operator $A$ on a Banach space $Y$. Set $X = Y \times Y$ and $A = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ with $\mathcal{D}(A) = Y \times \mathcal{D}(B)$. For $(x, y) \in \mathcal{D}(A)$ one has the unique solution $u(t) = \begin{pmatrix} x + tBy \\ y \end{pmatrix}$ of (2.1) with $u(0) = (x, y)$. But for $t > 0$ one cannot continuously extend the map $T(t) : (x, y) \mapsto u(t)$ to a map on $X$, since $T(t) \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} tBy \\ y \end{pmatrix}$. However, it can be shown that (2.1) has a unique solution for a closed operator $A$ and each $x \in \mathcal{D}(A)$ if and only if the operator $A_1$ on $X_1 = [\mathcal{D}(A)]$ given by $A_1x = Ax$ with $\mathcal{D}(A_1) = \{ x \in \mathcal{D}(A) : Ax \in \mathcal{D}(A) \}$ generates a $C_0$-semigroup on $X_1$, see Proposition II.6.6 in [E-N]. Moreover, if $\rho(A) \neq \emptyset$ and (2.1) has a unique solution for each $x \in \mathcal{D}(A)$, then $A$ is a generator (and in particular densely defined), see Theorem II.6.7 in [E-N].

**Example 2.4** (wave equation)

Let $\emptyset \neq U \subseteq \mathbb{R}^d$ be open and bounded with $C^2$ boundary. We study the wave equation

$$
\begin{cases}
\partial_t u(t, x) = \Delta u(t, x), & t \in \mathbb{R}, x \in U, \\
u(t, x) = 0, & t \geq 0, x \in \partial U, \\
u(0, x) = u_0(x), & \partial_t u(0, x) = u_1(x), \quad x \in U
\end{cases}
$$

(2.2)

for given functions $u_0, u_1$. We interpret (2.2) as a second order evolution equation in $X = L^2(U)$ by setting $\Delta_Dv = \Delta v$ with $\mathcal{D}(\Delta_D) = W_2^2(U) \cap \dot{W}_1^1(U)$. For $u_0 \in \mathcal{D}(\Delta_D)$ and $u_1 \in Y := \dot{W}_1^1(U)$ we then consider the problem

$$
\begin{cases}
u''(t) = \Delta_Du(t), & t \geq 0, \\
u(0) = u_0, & u'(0) = u_1.
\end{cases}
$$

(2.3)

Here we look for solutions $u \in C^2(\mathbb{R}_+, X) \cap C^1(\mathbb{R}_+, Y) \cap C(\mathbb{R}_+, W_2^2(U))$. In particular, the boundary condition in (2.2) is understood “in the sense of trace”, i.e., as $u(t) \in \dot{W}_1^1(U)$. To obtain a first order evolution equation, we set $E = Y \times X$ and define
2.1. WELLPOSEDNESS

\[ A = \begin{pmatrix} 0 & I \\ \Delta D & 0 \end{pmatrix} \text{ with } \mathcal{D}(A) = \mathcal{D}(\Delta D) \times Y. \]

We observe that (2.3) has a solution \( u \) if and only if the problem

\[
\begin{align*}
\frac{du}{dt} &= \begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} = \begin{pmatrix} w_2(t) \\ \Delta D w_1(t) \end{pmatrix} = Aw(t), \quad t \geq 0, \\
\end{align*}
\]

(2.4)

\[ w(0) = w_0 := \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{D}(A), \]

has a \( \mathcal{D}(A) \)-valued solution \( w \in C^1(\mathbb{R}_+, E) \). In this case, we have \( w(t) = (u(t), u'(t)) \). Indeed, let \( w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \) solve (2.4). Set \( u = w_1 \). Then \( w_1' = w_2 \in C^1(\mathbb{R}_+, X) \) and \( w_1' = [Aw]_1 \in C(\mathbb{R}_+, Y) \).

Observe that \( \|\varphi\|_{\Delta D} \leq \tilde{c}\|\varphi\|_{2,2} \) for all \( \varphi \in \mathcal{D}(\Delta D) = W^2_2(U) \cap Y \) and some \( \tilde{c} > 0 \). Since \([\mathcal{D}(\Delta D)]\) is a Banach space by Example 1.41, the open mapping theorem then shows that

\[ \|\varphi\|_{2,2} \leq C (\|\Delta D \varphi\|_2 + \|\Delta D^{-1} \Delta D \varphi\|_2) \leq c\|\varphi\|_{2,2} \]

for some constants \( C, c > 0 \) and all \( \varphi \in \mathcal{D}(\Delta D) \). Since \( \Delta D w_1 = [Aw]_2 \in C(\mathbb{R}_+, X) \), it follows that \( u = w_1 \in C(\mathbb{R}_+, W^2_2(U)) \). Finally, (2.4) implies that

\[ u'' = u_1'' = w_2'' = \Delta D w_1 = \Delta Du \]

and so \( u \) solves (2.3). Conversely, if \( u \) solves (2.3), then \( w = (u, u') \in C^1(\mathbb{R}_+, E) \), \( w(t) \in \mathcal{D}(\Delta D) \times Y \) for all \( t \geq 0 \) and

\[ w' = (u', u'') = (u', \Delta Du) = Aw \]

as asserted. By Example 1.42, \( A \) generates a unitary \( C_0 \)-group. Consequently, (2.3) is uniquely solvable (even for \( t \in \mathbb{R} \)).

Let \( J \in \{[0, b], (0, b], \mathbb{R}_+, (0, \infty); \ b > 0 \} \), \( x \in X \) and \( f \in C(J, X) \) such that

\[ \int_0^\delta \|f(s)\| \, ds < \infty \quad \text{for some } \delta \in J \setminus \{0\}. \]

Let \( A \) generate the \( C_0 \)-semigroup \( T(\cdot) \). We study the inhomogeneous Cauchy problem or inhomogeneous evolution equation

\[
\begin{align*}
\frac{du}{dt} &= Au(t) + f(t), \quad t \in J, \\
\end{align*}
\]

(2.5)

\[ u(0) = x. \]

**Definition 2.5**

A function \( u : J \to X \) is a (classical) solution of (2.5) on \( J \) if \( u \) belongs to \( C^1(J, X) \cap C(\bar{J}, X) \), \( u(t) \in \mathcal{D}(A) \) for all \( t \in J \) and (2.5) holds.
Proposition 2.6
Let $A$ generate the $C_0$-semigroup $T(\cdot)$, $x \in X$, and $f \in \mathcal{C}(J, X)$ with $\int_0^\delta \| f(s) \| \, ds < \infty$ for some $\delta \in J \setminus \{0\}$. If $u$ solves (2.5) on $J$, then $u$ is given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s) \, ds, \quad t \in J. \tag{2.6}$$

In particular, solutions of (2.5) are unique.

Observe that the function in (2.6) is always defined in our setting. It is called the mild solution of (2.5).

Proof. Let $t \in J \setminus \{0\}$ and set $v(s) = T(t-s)u(s)$ for $0 \leq s \leq t$, where $u$ solves (2.5) on $J$. As in the proof of Proposition 1.7, one shows that $v$ is continuously differentiable with derivative

$$v'(s) = T(t-s)u'(s) - T(t-s)Au(s) = T(t-s)f(s)$$

for all $0 < s \leq t$. Let $\epsilon \in (0, t)$. By integration we deduce

$$\int_0^t T(t-s)f(s) \, ds = v(t) - v(\epsilon) = u(t) - T(t-\epsilon)u(\epsilon).$$

Since $u \in \mathcal{C}(J, X)$ and $\int_0^t \| T(t-s)f(s) \| \, ds \leq M e^{\omega|t|} \int_0^t \| f(s) \| \, ds$ for suitable constants $M \geq 1$ and $\omega \in \mathbb{R}$, we can let $\epsilon \to 0$ and obtain the assertion. \qed

Example 2.7
Let $X = C_0(\mathbb{R})$, $A = \frac{d}{ds}$ with $\mathfrak{D}(A) = C^1(\mathbb{R})$ and $\varphi \in X$ be nowhere differentiable. The operator $A$ generates the $C_0$-group $T(\cdot)$ given by $T(t)g = g(\cdot + t)$. Clearly, $T(t)\varphi \notin \mathfrak{D}(A)$ for all $t \in \mathbb{R}$ and for every $s \in \mathbb{R}$ the map $t \mapsto (T(t)\varphi)(s) = \varphi(s+t)$ is not differentiable. Hence, $t \mapsto T(t)\varphi \in X$ is a continuous, but non-differentiable function. Set $f(s) = T(s)\varphi$ for $s \in \mathbb{R}$. Then $f \in \mathfrak{C}(\mathbb{R}, X)$ and the mild solution of (2.5) with $x = 0$ is given by $u(t) = \int_0^t T(t-s)T(s)\varphi \, ds = tT(t)\varphi$ for $t \geq 0$. Then $u(t) \notin \mathfrak{D}(A)$ and $u$ is not differentiable for $t > 0$, i.e., $u$ is not a classical solution of (2.5).

Lemma 2.8
Let $A$ generate the $C_0$-semigroup $T(\cdot)$, $x \in \mathfrak{D}(A)$, and $f \in \mathcal{C}(J, X)$ with $\int_0^\delta \| f(s) \| \, ds < \infty$ for some $\delta \in J \setminus \{0\}$. Define $v(t) = \int_0^t T(t-s)f(s) \, ds$ for $t \in J$ and $v(0) := 0$ if $0 \notin J$. Then the following assertions are equivalent.

(a) $v \in C^1(J, X)$.
(b) $v(t) \in \mathfrak{D}(A)$ for all $t \in J$ and $Av \in \mathcal{C}(J, X)$.

In this case, (2.6) gives the unique solution of (2.5) on $J$. If (2.5) has a solution on $J$, then assertions (a) and (b) hold.
In the setting of Example 2.4, we consider the inhomogeneous wave equation

\[ \partial_t u(t, x) = \Delta u(t, x) + g(t, x), \quad t \geq 0, x \in U, \]

\[ u(t, x) = 0, \quad t \geq 0, x \in \partial U, \]

\[ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \quad x \in U. \]

Theorem 2.9 (existence result for inhomogeneous evolution equations)

Let \( A \) generate the \( C_0 \)-semigroup \( T(\cdot) \), \( x \in \mathcal{D}(A) \) and \( J \in \{ \lbrack 0, b \rbrack, \mathbb{R}_+; b > 0 \} \). Assume either that \( f \in C^1(J, X) \) or that \( f \in C(J, [\mathcal{D}(A)]) \). Then the mild solution \( u \) given by (2.6) is the unique solution of (2.5) on \( J \).

Proof. Let \( f \in C^1(J, X) \). Since \( v(t) = \int_0^t T(s) f(t - s) \, ds \) for all \( t \in J \), it follows that \( v \in C^1(J, X) \) and hence (a) in Lemma 2.8 is satisfied.

Let \( f \in C(J, [\mathcal{D}(A)]) \). Since \( A \) is closed and commutes with \( T(t - s) \) on \( \mathcal{D}(A) \), we obtain

\[ v(t) \in \mathcal{D}(A) \] and \( Av(t) = \int_0^t T(t - s) Av(s) \, ds \) so that \( Av \in C(J, X) \). In this case, (b) in Lemma 2.8 is fulfilled. The theorem is now an immediate consequence of Lemma 2.8.

Example 2.10

In the setting of Example 2.4, we consider the inhomogeneous wave equation

\[ \frac{\partial^2 u}{\partial t^2}(t, x) + \Delta u(t, x) = f(t, x), \quad t \geq 0, x \in \Omega, \]

\[ u(t, x) = 0, \quad t \geq 0, x \in \partial \Omega, \]

\[ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \quad x \in \Omega. \]
for given \( u_0 \in \mathcal{D}(\Delta_D) = W^2_2(U) \cap \dot{W}^1_2(U) \), \( u_1 \in \dot{W}^1_2(U) \) and \( g \in C(\mathbb{R}_+, L^2(U)) \), we set \( g(t, x) = (g(t))(x) \) for all \( t \geq 0 \) and almost every \( x \in U \). We write these equations as

\[
\begin{cases}
  u''(t) = \Delta_D u(t) + g(t), & t \geq 0, \\
  u(0) = u_0, & u'(0) = u_1.
\end{cases}
\]

(2.8)

As in Example 2.4, the second order problem is equivalent to the first order problem (2.5) on \( E = \dot{W}^1_2(U) \times L^2(U) \) with

\[
A = \left( \begin{array}{cc} 0 & I \\ \Delta_D & 0 \end{array} \right)
\]

on \( (W^2_2(U) \cap \dot{W}^1_2(U)) \times \dot{W}^1_2(U) \) and \( f = (0, g) \).

In view of Theorem 2.9 and Example 2.4, we obtain a unique solution

\[
u \in C^2(\mathbb{R}_+, L^2(U)) \cap C^1(\mathbb{R}_+, \dot{W}^1_2(U)) \cap C(\mathbb{R}_+, \dot{W}^2_2(U))
\]

if either \( g \in C^1(\mathbb{R}_+, L^2(U)) \) (then we have \( f \in C^1(\mathbb{R}_+, E) \)) or \( g \in C(\mathbb{R}_+, \dot{W}^1_2(U)) \) (then we have \( f \in C(\mathbb{R}_+, [\mathcal{D}(A)]) \)).

### 2.2. Analytic semigroups

Let \( \emptyset \neq U \subseteq \mathbb{C} \) be open, \( Y \) be a Banach space, \( f : U \to Y \) be continuous and \( \gamma : J \to \mathbb{C} \) be piecewise continuously differentiable with \( \Gamma := \gamma(J) \subseteq U \), where \( J \) is a closed interval. If \( J = [a, b] \) and \( \gamma(a) = \gamma(b) \), then the curve \( \Gamma \) is called closed. We define the curve integral

\[
\int_{\Gamma} f \, dz := \int_{J} f(\gamma(t))\gamma'(t) \, dt,
\]

assuming that the improper Riemann integral exists if \( J \) is unbounded. As in the proof of Proposition 1.14, one sees that the improper integral exists if the function \( \| f \circ \gamma \| \cdot |\gamma'| \) is integrable on \( J \).

Now, let \( f \) be holomorphic on \( U \), \( U \) be starshaped, \( \Gamma \) be closed and \( z \in U \setminus \Gamma \). We then have Cauchy’s theorem

\[
\int_{\Gamma} f(w) \, dw = 0,
\]

and Cauchy’s formula

\[
f(z)n(\Gamma, z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} \, dw, \quad \text{where} \quad n(\Gamma, z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{w - z} \, dw.
\]

(2.10)

In fact, these equations hold with \( f \) replaced by \( \langle f, x^* \rangle \) for each \( x^* \in X^* \). Hence, we obtain \( \int_{\Gamma} f \, dz, x^* \rangle = 0 \) for every \( x^* \in X^* \), implying (2.9) due to the Hahn-Banach theorem. Formula (2.10) is shown similarly. If \( Y = \mathbb{C} \), identity (2.10) yields

\[
e^{-a} = \frac{1}{2\pi i} \int_{\partial B(a,1)} e^{\lambda(z - a)^{-1}} \, d\lambda \quad \text{for} \ a \in \mathbb{C} \ \text{and} \ z \in \mathbb{C}.
\]

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We want to imitate this formula for unbounded $A$. For $\alpha \in (0, \pi]$, we set $\Sigma_\alpha = \{ \lambda \in \mathbb{C} \setminus \{0\}; |\arg(\lambda)| < \alpha \}$. We remark that $\mathbb{C}_+ = \Sigma_{\frac{\pi}{2}}$, $\mathbb{C} \setminus \mathbb{R}_- = \Sigma_{\pi}$ and $\mathbb{C}_- = \mathbb{C} \setminus \mathbb{C}_+$.

**Definition 2.11**

A closed operator $A$ is called sectorial if for some $\varphi \in (0, \pi)$ we have $\Sigma_\varphi \setminus \{0\} \subseteq \rho(A)$ and
\[
\| R(\lambda, A) \| \leq \frac{K}{|\lambda|}
\]
for all $\lambda \in \Sigma_\varphi \setminus \{0\}$ and some constant $K = K_\varphi > 0$. The supremum $\phi \in (0, \pi]$ of all such $\varphi$ is called the angle of $A$.

An analytic semigroup of angle $\theta \in (0, \pi]$ is a family of operators $\{T(z); z \in \Sigma_\theta \cup \{0\}\}$ such that

1. $T(0) = I$ and $T(w)T(z) = T(w + z)$ for all $z, w \in \Sigma_\theta$.
2. $T: \Sigma_\theta \to \mathcal{B}(X)$ is differentiable.
3. $T(z)x \to x$ in $X$ as $z \to 0$ in $\Sigma_\theta$ for all $x \in X$ and each $\vartheta \in (0, \theta)$.

The generator of $T(\cdot)$ is defined as the generator of the $C_0$-semigroup $(T(t))_{t \geq 0}$.

If $\| T(z) \|$ is bounded for all $z \in \Sigma_\theta$ and each $\vartheta \in (0, \theta)$, the analytic semigroup is called bounded.

Let $A$ be sectorial with angle $\phi > \frac{\pi}{2}$. For any $r > 0$ and $\varphi \in \left(\frac{\pi}{2}, \phi\right)$ we define $\Gamma = \Gamma(r, \varphi) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ by setting
\[
\Gamma_1 = \{ \gamma_1(s) = -s e^{-i\varphi}; -\infty < s \leq -r \},
\Gamma_2 = \{ \gamma_2(s) = r e^{i\alpha}; -\varphi \leq \alpha \leq \varphi \},
\Gamma_3 = \{ \gamma_3(s) = s e^{i\varphi}; r \leq s < \infty \}.
\]

Let $t > 0$. For $\lambda = \gamma_1(s) \in \Gamma_1$ we estimate
\[
\| e^{\lambda t} \| \leq e^{-st} |e^{-i\varphi}| - e^{-i\varphi} |s e^{-i\varphi}| = K \frac{e^{s t \cos \varphi}}{|s|}
\]
where $s \leq -r$ and $K = K_\varphi$. The same inequality holds for $\lambda \in \Gamma_3$ with $s \geq r$. For $\lambda = \gamma_2(\alpha) \in \Gamma_2$, we further obtain
\[
\| e^{\lambda t} \| \leq e^{tr} |e^{i\alpha}| \frac{K}{r e^{i\alpha}} = K e^{tr}.
\]
As a result, the improper Riemann integral
\[
e^{tA} := \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} R(\lambda, A) \, d\lambda
\]
exists in $\mathcal{B}(X)$ for $t > 0$, and we have
\[
\| e^{tA} \| \leq \frac{1}{2\pi} \left| \int_{\Gamma} \| e^{\lambda t} R(\lambda, A) \| \, d\lambda \right| \leq \frac{1}{2\pi} \left( 2K \int_{-\varphi}^{\varphi} \frac{e^{st \cos \varphi}}{s} \, ds + K \int_{-\varphi}^{\varphi} e^{r \cos \varphi} \, d\alpha \right).
\]
The substitution \( \sigma = -st\cos \varphi \) in the first integral of the right hand side thus yields

\[
\| e^{tA} \| \leq \frac{K}{\pi} \int_{rt\cos \varphi}^{\infty} \frac{e^{-\sigma}}{\sigma} (-t\cos \varphi) \frac{d\sigma}{-t\cos \varphi} + \frac{K\varphi}{\pi} e^{rt}.
\]

If we choose \( r = \frac{1}{t} \), we find a constant \( M_0 = M_0(\varphi, K) > 0 \) such that

\[
\| e^{tA} \| = \left\| \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} R(\lambda, A) d\lambda \right\| \leq M_0, \quad \text{for all } t > 0.
\]

For later use, we note that we obtain similarly

\[
\| e^{tA} \| = \left\| \int_{S_R} e^{\lambda t} R(\lambda, A) d\lambda \right\| \leq M_0.
\]

We can make the choice \( r = \frac{1}{t} \) since the above integrals do not depend on \( r > 0 \) and \( \varphi \in \left( \frac{\pi}{2}, \pi \right) \). In fact, take any \( r' > 0 \) and \( \varphi' \in \left( \frac{\pi}{2}, \varphi \right) \). Choose \( R > \max\{r, r'\} \) and set \( \Gamma_R = \Gamma(r, \varphi) \cap \overline{B(0, R)} \). Denote by \( C_R^+ \) and \( C_R^- \) the circle arcs connecting the endpoints \( \Gamma_R \) with those of \( \Gamma' \) inside \( \{ \lambda \in \mathbb{C}; \text{Im} \lambda > 0 \} \) and \( \{ \lambda \in \mathbb{C}; \text{Im} \lambda < 0 \} \), respectively. Then \( S_R = \Gamma_R \cup C_R^+ \cup (-\Gamma^-_R) \cup (-C^-_R) \) is a closed curve in the starshaped domain \( \Sigma_\phi \) so that

\[
\int_{S_R} e^{\lambda t} R(\lambda, A) d\lambda = 0
\]
due to Cauchy’s theorem (2.9). For \( \theta = \min\{\varphi, \varphi'\} \in \left( \frac{\pi}{2}, \varphi \right) \) it follows as in (2.12) and (2.14) that

\[
\left\| \int_{C_R^+} e^{\lambda t} R(\lambda, A) d\lambda \right\| \leq \int_{\varphi}^{\varphi'} \int_{\Gamma_R} e^{\lambda t} R e^{i\alpha} d\lambda \frac{K}{|R e^{i\alpha}| |iR e^{i\alpha}|} d\alpha \leq K|\varphi' - \varphi| e^{tR\cos \theta} \rightarrow 0
\]
as \( R \rightarrow \infty \), because of \( \cos \theta < 0 \). These facts imply together that

\[
\int_{\Gamma(r, \varphi)} e^{\lambda t} R(\lambda, A) d\lambda = \lim_{R \rightarrow \infty} \int_{\Gamma_R} e^{\lambda t} R(\lambda, A) d\lambda = \lim_{R \rightarrow \infty} \int_{\Gamma_R} e^{\lambda t} R(\lambda, A) d\lambda = \int_{\Gamma'(r', \varphi')} e^{\lambda t} R(\lambda, A) d\lambda
\]
as required. The integral in (2.16) is treated in the same way.

For \( n \in \mathbb{N} \) we inductively define

\[
\mathcal{D}(A^n) := \{ x \in \mathcal{D}(A^{n-1}); A^{n-1}x \in \mathcal{D}(A) \} \text{ and } A^n x := A(A^{n-1}x)
\]

**Theorem 2.12** (Hille, 1948)

For a closed linear operator \( A \) on \( X \), the following assertions are equivalent.

(a) The operator \( A \) is densely defined and sectorial of angle \( \phi \in \left( \frac{\pi}{2}, \pi \right) \).

(b) The operator \( A \) is densely defined, \( C_+ \subseteq \rho(A) \) and there is a constant \( C > 0 \) such that
Employing again Proposition 1.14, we estimate
\[ ||R(\lambda, A)|| \leq \frac{C}{\text{Re} \lambda} \quad \text{and} \quad ||R(\lambda, A)|| \leq \frac{C}{\text{Im} \lambda} \]
for all \( \lambda \in \mathbb{C}_+ \).

(c) For some \( \vartheta \in (0, \frac{\pi}{2}) \), the operators \( e^{\pm i\vartheta} A \) generate bounded \( C_0 \)-semigroups.

(d) The operator \( A \) generates a bounded \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) such that \( T(t)X \subseteq \mathcal{D}(A) \)
and
\[ ||AT(t)|| \leq \frac{M_1}{t} \]
for all \( t > 0 \) and a constant \( M_1 > 0 \). Moreover, \( T(\cdot) \) belongs to \( C^1((0, \infty), \mathcal{B}(X)) \) with \( \frac{d}{dt}T(t) = AT(t) \) for all \( t > 0 \).

(e) The operator \( A \) generates a bounded analytic semigroup with angle \( \theta \in (0, \frac{\pi}{2}] \).

If this is the case, \( T(t) \) is given by (2.13), for each \( n \in \mathbb{N} \) we have \( T(t)X \subseteq \mathcal{D}(A^n) \) and \( ||A^nT(t)|| \leq \frac{M_n}{t^n} \) for all \( t > 0 \) and \( T(\cdot) \in C^n((0, \infty), \mathcal{B}(X)) \) with \( \frac{d^n}{dt^n}T(t) = A^nT(t) \) for all \( t > 0 \) and some constants \( M_n > 0 \) only depending on \( n, K \) and \( \theta \) (or on \( C \)).

**Remark 2.13**

(a) One can develop the theory of analytic semigroups without using \( C_0 \)-semigroups and for possibly nondense \( \mathcal{D}(A) \). Then \( T(t)x \to x \) as \( t \to 0^+ \) only holds for \( x \in \mathcal{D}(A) \). See [L].

(b) By rescaling, one obtains a version of Theorem 2.12 where \( A - \omega I \) is sectorial for some \( \omega \in \mathbb{R} \) and \( (e^{-\omega z} T(z))_{z \in \Sigma_{\vartheta}} \) is a bounded analytic semigroup, cf. §2.1 in [L].

(c) One can check that one can take \( \theta = \frac{\vartheta}{2} \). In addition, \( \theta = \sup \{ \vartheta \in (0, \frac{\pi}{2}) ; (c) \text{ holds} \} \). Moreover, \( T(z) \) is given as in (2.13) with \( t > 0 \) replaced by \( z \in \Sigma_{\vartheta} \) and with \( \varphi \in (|\arg z| + \frac{\pi}{2}, \phi) \) in the definition of \( \Gamma \), see Proposition II.4.3 in [E-N] and the proof given below.

(d) There are generators of \( C_0 \)-semigroups which are not sectorial of angle \( \phi > \frac{\pi}{2} \). For instance, \( A = \frac{d}{dz} \) with \( \mathcal{D}(A) = W_2^1(\mathbb{R}) \) on \( X = L^2(\mathbb{R}) \) generates the shift \( T(t)f = f(\cdot + t) \) that does not map \( X \) into \( \mathcal{D}(A) \). In this case \( A \) is sectorial with angle \( \frac{\pi}{2} \), see Example 5.10 in [ST].

**Proof.** We prove the following chain of implications: \( (e) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a) \Rightarrow (d) \Rightarrow (e) \).

\( "(e)\Rightarrow(c)" \): For all \( \vartheta \in (0, \theta) \), the operators \( T(e^{\pm i\vartheta} t) \) with \( t \geq 0 \) yield two bounded \( C_0 \)-semigroups. As in Remark 1.11 one sees that they are generated by \( e^{\pm i\vartheta} A \).

\( "(c)\Rightarrow(b)" \): Observe that \( \rho(A) = e^{\pm i\vartheta} \rho(e^{\pm i\vartheta} A) \supseteq e^{\pm i\vartheta} \mathbb{C}_+ \) due to assertion (c) and Proposition 1.14. As a result, \( \rho(A) \supseteq \Sigma_{\frac{\pi}{2} + \vartheta} \supseteq \mathbb{C}_+ \). Let \( e^{i\vartheta} = a + ib \) for \( a, b > 0 \) and take \( r, s > 0 \). Employing again Proposition 1.14, we estimate
\[
||R(r + is, A)|| = \left| ||e^{-i\vartheta} R(e^{-i\vartheta}(r + is), e^{-i\vartheta} A)|| \right| \leq \frac{M}{\text{Re}((a - ib)(r + is))} = \frac{M}{ar + bs} \leq \min\{(M/a)r^{-1}, (M/b)s^{-1}\},
\]
where the semigroup generated by $e^{-i\theta} A$ is bounded by $M$. The case $s < 0$ can be treated using $e^{i\theta} A$.

“(b)⇒(a)”: Proposition 1.13 of [ST] and part (b) imply that

$$d(\lambda, \sigma(A)) \geq \frac{1}{\|R(\lambda, A)\|} \geq \frac{|\text{Im } \lambda|}{C}$$

for all $\lambda \in \mathbb{C}_+$. Considering $\lambda = \epsilon + is$ for $\epsilon > 0$ and $s \in \mathbb{R} \setminus \{0\}$ as well as $\mu \in \mathbb{C}$ with $\text{Im } \mu = s$, we see that all $\mu$ with $\text{Re } \mu \leq 0$ and $|\text{Re } \mu| < \frac{|\text{Im } \mu|}{C}$ belong to $\rho(A)$. Setting $\phi = \frac{\pi}{2} + \arctan \frac{1}{\epsilon}$, we deduce that $\Sigma_\phi \subseteq \rho(A)$. Moreover, by continuity, assertion (b) implies $\|R(is, A)\| \leq \frac{C}{|s|}$ for all $s \in \mathbb{R} \setminus \{0\}$. Take any $q \in (0, 1)$. Furthermore, take $\lambda = r + is$ with $r, s \in \mathbb{R}$ and $|r| \leq q|\epsilon|$. Writing

$$R(\lambda, A) = (rI + isI - A)^{-1} = (isI - A)^{-1}(r(isI - A)^{-1} + I)^{-1},$$

we can estimate

$$\|R(\lambda, A)\| \leq \frac{C}{|s|} \cdot \frac{1}{1 - \|rR(is, A)\|} \leq \frac{C}{|s|} \cdot \frac{1}{1 - \frac{C}{|s|}} \leq \frac{C}{\cos \left(\frac{\phi - \frac{\pi}{2}}{2}\right)} \cdot \frac{1}{1 - \frac{1}{q}},$$

using that $-1 \in \rho(S)$ with $\|S\| < 1$ if $\|S\| < 1$ where $S = rR(is, A)$ (see, e.g., Theorem 3.12 in [FA]) and that $|s| = |\lambda| \cos(\arg(\lambda) - \frac{\pi}{2}) \geq |\lambda| \cos(\phi - \frac{\pi}{2})$. Finally, for $\lambda = r + is$ with $r \geq \frac{2}{q}|s|$, we have

$$|\text{arg } \lambda| = \arctan \frac{|s|}{r} \leq \arctan \frac{C}{q} =: \theta_q \in \left(0, \frac{\pi}{2}\right),$$

and thus assertion (b) yields

$$\|R(\lambda, A)\| \leq \frac{C}{r} = \frac{C}{|\lambda| \cos(\arg(\lambda))} \leq \frac{C}{\cos \theta_q} \cdot \frac{1}{|\lambda|}.$$

Summarizing, we see that $A$ is sectorial.

“(a)⇒(d)”: In this main step we define $T(t) := e^{tA}$ for $t > 0$ by (2.13) and set $T(0) = I$. In, e.g., Theorem 5.12 of [ST] it is shown that $(T(t))_{t \geq 0}$ is bounded $C_0$-semigroup satisfying assertion (d), except for the generator property of $A$. In order to establish this last fact, consider $t > 0$ and $x \in \mathcal{D}(A)$. Theorem 5.12 (b) in [ST] implies that

$$e^{tA} x - e^{sA} x = \int_t^s \frac{d}{ds} e^{sA} x \, ds = \int_t^s e^{sA} A x \, ds.$$

We can now let $\epsilon \to 0$. The strong continuity of $s \mapsto e^{sA}$ then yields that

$$\frac{1}{t} \left( e^{tA} x - x \right) = \frac{1}{t} \int_0^t e^{sA} A x \, ds \to A x \text{ as } t \to 0.$$

Thus the generator $B$ of $(e^{tA})_{t \geq 0}$ extends $A$. Since $s(A), s(B) \leq 0$, Lemma 1.17 now shows that $A = B$. 

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Let \( t \in \mathbb{R} \). Inductively, we obtain that \( AT(t) = \frac{1}{t} AT(t - \delta) \) for \( \delta > 0 \) such that \( \delta \to 0 \). Since \( \frac{d}{dt} T(t) = AT(t) = AT(t - \delta) AT(\delta) \), we obtain that \( T(t)X \subseteq \mathcal{D}(A^2) \) and that there exists \( \frac{d^2}{dt^2} T(t) = AT(t - \delta) AT(\delta) = A^2 T(t) \).

Inductively, we obtain that \( T(t)X \subseteq \mathcal{D}(A^n) \) and that \( \frac{d^n}{dt^n} T(t) = A^n T(t) \) exists for all \( n \in \mathbb{N} \). Assertion (d) further yields

\[
\|A^n T(t)\| = \left\| \left( AT \left( \frac{t}{n} \right) \right)^n \right\| \leq \left( \frac{M_1^n}{t^n} \right)^n =: \frac{M_n}{t^n}.
\]

Observe that \( e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!} \geq \frac{n^n}{n!} \). For any \( q \in (0,1) \), we take \( z \in \mathbb{C}_+ \) with \( |\arg z| = \frac{|\text{Im} z|}{\text{Re} z} \leq \frac{q}{\pi M_1} \). Letting \( t = \text{Re} z \), we define

\[
T(z) = \sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} A^n T(t)
\]

using that this series converges absolutely in \( B(X) \) because of

\[
(2.17) \quad \|T(z)\| \leq \sum_{n=0}^{\infty} \frac{|z-t|^n}{n!} \cdot \frac{M_1^n n^n}{t^n} \leq \sum_{n=0}^{\infty} \left( \frac{qt}{e M_1} \right)^n \frac{M_1^n e^n}{t^n} = \frac{1}{1-q}.
\]

Thus we have extended \( T(\cdot) \) to a bounded differentiable map \( T : \Sigma_\vartheta \to B(X) \) for \( \vartheta = \arctan \frac{q}{\pi M_1} \) and any \( q \in (0,1) \). Let \( x \in X \) and \( x^* \in X^* \). For fixed \( t > 0 \), we note that the holomorphic functions \( \langle T(z)T(t)x, x^* \rangle \) and \( \langle T(z + t)x, x^* \rangle \) coincide for \( z \in (0, \infty) \). Consequently, they are the same for all \( z \in \Sigma_\vartheta \) thanks to the identity theorem of complex analysis. The Hahn-Banach theorem now yields that \( T(z)T(t) = T(z + t) \) for all \( z \in \Sigma_\vartheta \). In the same way one can replace here \( t \) by any \( w \in \Sigma_\vartheta \). It remains to check the strong continuity as \( z \to 0 \) for \( z \in \Sigma_\vartheta \). Let \( x \in X \) and \( \varepsilon > 0 \). We fix \( h > 0 \) such that \( \|T(h)x - x\| < \varepsilon \). Using (2.17) and the continuity of \( T(\cdot) \) on \( \Sigma_\vartheta \), we estimate

\[
\|T(z)x - x\| \leq \|T(z)\| \cdot \|x - T(h)x\| + \|T(z + h)x - T(h)x\| + \|T(h)x - x\| \\
\leq \frac{1}{1-q} \cdot \varepsilon + \|T(z + h) - T(h)\| \cdot \|x\| + \varepsilon,
\]

which leads to \( \limsup_{z \to 0} \|T(z)x - x\| \leq \left( 1 + \frac{1}{1-q} \right) \varepsilon \). As a result, \( T(z)x \to x \) as \( z \to 0 \) in \( \Sigma_\vartheta \). \( \square \)

**Corollary 2.14**

Let \( A \) be closed, densely defined and dissipative. Assume that there is a \( \lambda_0 > 0 \) such that \( \lambda_0 I - A \) is surjective and a \( \vartheta \in (0, \frac{\pi}{2}) \) such that \( e^{\pm i \theta} A \) are dissipative. Then \( A \) generates a bounded analytic \( C_0 \)-semigroup \( T(\cdot) \) of angle \( \theta \in \left[ \vartheta, \frac{\pi}{2} \right] \) with \( \|T(z)\| \leq 1 \) for \( \|\arg(z)\| \leq \vartheta \).
Proof. The Lumer-Phillips theorem 1.32 implies that $C_+ \subseteq \rho(A)$. The operators $I - e^{\pm i\theta}A = e^{\pm i\theta}(e^{\mp i\theta}I - A)$ are thus surjective and so $e^{\pm i\theta}A$ generate contraction semigroups again by the Lumer-Phillips theorem. Hence, $A$ generates a bounded analytic $C_0$-semigroup of angle $\theta \geq \vartheta$ due to Theorem 2.12 and Remark 2.13. Let $|\arg z| \leq \vartheta$. For $x \in X$ and $x^* \in X^*$ the function $f(z) = \langle T(z)x, x^* \rangle$ is bounded on $\overline{\Sigma_{\vartheta}}$ and on $\partial\overline{\Sigma_{\vartheta}}$ it is bounded by $\|x\| \cdot \|x^*\|$. The Phragmén-Lindelöf theorem (see Corollary VI.4.2 in [C]) then shows that $|f(z)| \leq \|x\| \cdot \|x^*\|$ for all $z \in \overline{\Sigma_{\vartheta}}$. We then deduce that $\|T(z)\| \leq 1$ for all $z \in \overline{\Sigma_{\vartheta}}$ from a corollary to the Hahn-Banach theorem.

**Corollary 2.15**

Let $X$ be a Hilbert space and $A$ be densely defined and selfadjoint with $(Ax|x) \leq 0$ for all $x \in \mathcal{D}(A)$ (in this case one writes $A = A' \leq 0$ and says that $A$ is negative). Then $A$ generates a contractive analytic $C_0$-semigroup of angle $\pi/2$.

**Proof.** By assumption $A$ and $A'$ are dissipative. So Corollary 1.33 shows that $I - A$ is invertible. For $x \in \mathcal{D}(A)$ and $\vartheta \in (0, \pi/2)$, we further obtain

$$\Re (e^{\pm i\theta}Ax|x) = \cos \vartheta \ (Ax|x) \leq 0,$$

so that the operators $e^{\pm i\theta}A$ are also dissipative. Corollary 2.14 now implies the assertion since $\vartheta < \pi/2$ is arbitrary.

**Example 2.16**

On $X = L^2(\mathbb{R}^d)$ let $A = \Delta$ with $\mathcal{D}(A) = W^2_{\mathcal{C}}(\mathbb{R}^d)$. Let $\emptyset \neq U \subseteq \mathbb{R}^d$ be bounded and open with $C^2$ boundary. Define $B = \Delta$ with $\mathcal{D}(B) = W^2_{\mathcal{C}}(U) \cap W^1_{\mathcal{C}}(U)$. Due to Examples 1.38 and 1.41, the operators $A$ and $B$ are selfadjoint and negative. Therefore $A$ and $B$ generate contractive analytic $C_0$-semigroups with angle $\pi/2$.

**Example 2.17**

Let $X = C_0(0,1)$ and $Au = u''$ with $\mathcal{D}(A) = \{ u \in C^2(0,1); \ u, u'' \in X \}$. From Example 1.37 we know that $\mathcal{C} \setminus \mathbb{R}_- \subseteq \rho(A)$ and

$$R(\lambda, A)f(s) = a(f, \mu) e^{\mu s} + b(f, \mu) e^{-\mu s} + \frac{1}{2\mu} \int_0^1 e^{-|s-\tau|} f(\tau) \ d\tau$$

with

$$\left( \begin{array}{c} a(f, \mu) \\ b(f, \mu) \end{array} \right) = \frac{1}{2\mu(e^{-\mu} - e^\mu)} \left( \begin{array}{c} e^{-\mu} \int_0^1 (e^{\mu \tau} - e^{-\mu \tau}) f(\tau) \ d\tau \\ \int_0^1 (e^{\mu \tau} - e^{-\mu \tau}) f(\tau) \ d\tau \end{array} \right)$$

for $\lambda \notin \mathbb{R}$, $\mu = \sqrt{\lambda}$, and $f \in X$. Fix $\phi \in \left( \frac{\pi}{2}, \pi \right)$. Take $\lambda \in \Sigma\phi$. Hence, $\mu \in \Sigma\phi$ satisfies $\Re \mu = |\mu| \cos \arg \mu \geq |\mu| \cos \frac{\phi}{2}$. We estimate

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\[ \| R(\lambda, A)f \|_\infty \leq |a(f, \mu)| e^{Re \mu} + |b(f, \mu)| + \frac{\|f\|_\infty}{2|\mu|} \int_{\mathbb{R}} e^{-Re \mu |r|} \, dr \]

\[
\leq \frac{\|f\|_\infty}{2|\mu| (|e^{\mu}| - |e^{-\mu}|)} \left( \int_0^1 (e^{Re \mu \tau} + e^{-Re \mu \tau}) \, d\tau + \int_0^1 (e^{Re \mu} e^{-Re \mu \tau} + e^{-Re \mu} e^{Re \mu \tau}) \, d\tau \right) + \frac{\|f\|_\infty}{|\mu| \text{Re} \mu} \]

\[
= \frac{\|f\|_\infty (e^{Re \mu} - 1 + 1 - e^{-Re \mu} + e^{Re \mu}(1 - e^{-Re \mu}) + e^{-Re \mu}(e^{Re \mu} - 1))}{2|\mu| \text{Re} \mu (e^{Re \mu} - e^{-Re \mu})} + \frac{\|f\|_\infty}{|\mu| \text{Re} \mu} \]

\[
\leq \frac{\|f\|_\infty}{|\mu|^2 \cos \frac{\phi}{2}} + \frac{\|f\|_\infty}{|\mu|^2 \cos \frac{\phi}{2}} = \frac{2 (\cos \frac{\phi}{2})^{-1}}{|\lambda|} \|f\|_\infty.
\]

As a result, \( A \) is sectorial of angle \( \pi \).

**Example 2.18**

For \( p \in (1, \infty) \), we set \( E = L^p(\mathbb{R}^d) \) and \( A_0 = \Delta \) with \( \mathcal{D}(A_0) = \mathcal{S}_d \). For \( u \in \mathcal{D}(A_0) \setminus \{0\} \), we define \( u^* = \overline{u}|u|^{p-2} \). Recall that \( ||u||_{p^*}^{2-p}u^* \in J(u) \) by Example 1.24.

First let \( p \geq 2 \). We then have \( u^* \in C^1(\mathbb{R}^d) \) and, using \( u^* = \overline{u}|u|^{p-2} \), we compute

\[
\partial_k u^* = |u|^{p-2} \partial_k \overline{u} + \frac{p-2}{2} (u \overline{u})^{\frac{p}{2}-2} (\overline{u} \partial_k u + u \overline{\partial_k u}) = (\overline{u} \partial_k \overline{u} + (p-2) \overline{u} \text{Re} (\overline{u} \partial_k u)) |u|^{p-4},
\]

which leads to

\[
\partial_k u \partial_k u^* = \left( |\partial_k u|^2 + (p-2)(\text{Re}(\overline{u} \partial_k u))^2 + i(p-2) \text{Im}(\overline{u} \partial_k u) \text{Re}(\overline{u} \partial_k u) \right) |u|^{p-4}
\]

\[
= \left( (p-1)(\text{Re}(\overline{u} \partial_k u))^2 + (\text{Im}(\overline{u} \partial_k u))^2 + i(p-2) \text{Im}(\overline{u} \partial_k u) \text{Re}(\overline{u} \partial_k u) \right) |u|^{p-4}
\]

for all \( k \in \{1, \ldots, d\} \). Here all summands are bounded by the continuous function \( |\nabla u|^2 |u|^{p-2} \) which decays rapidly as \( |x| \to \infty \) since \( p \geq 2 \). Integrating by parts, we thus obtain

\[
\langle A_0 u, u^* \rangle = -\int_{\mathbb{R}^d} \nabla u \cdot \nabla u^* \, dx
\]

\[
= -\int_{\mathbb{R}^d} |u|^{p-4} (p-1) |\text{Re}(\overline{u} \nabla u)|^2 + |\text{Im}(\overline{u} \nabla u)|^2 + i(p-2) |u|^{p-4} \text{Im}(\overline{u} \nabla u) \text{Re}(\overline{u} \nabla u)) \, dx.
\]
Consequently, $\Re(A_0u, u^*) < 0$ and $A_0$ is dissipative. The inequalities of Hölder and Young further yield
\[
|\Im(A_0u, u^*)| \leq |p - 2| \int_{\mathbb{R}^d} |u|^{\frac{p}{2} - 2} |\Re(\overline{u} \nabla u)| \cdot |\Im(\overline{u} \nabla u)| \cdot |u|^{\frac{p}{2} - 2} \, dx
\leq |p - 2| \left( \sqrt{\frac{p - 1}{p - 1}} \int_{\mathbb{R}^d} |u|^{p - 4} |\Re(\overline{u} \nabla u)|^2 \, dx \right)^{\frac{1}{2}} \left( \frac{1}{\sqrt{p - 1}} \int_{\mathbb{R}^d} |\Im(\overline{u} \nabla u)|^2 |u|^{p - 4} \, dx \right)^{\frac{1}{2}}
\leq \frac{|p - 2|}{2} \int_{\mathbb{R}^d} |u|^{p - 4} |\Re(\overline{u} \nabla u)|^2 \, dx + |p - 2| \frac{1}{2\sqrt{p - 1}} \int_{\mathbb{R}^d} |\Im(\overline{u} \nabla u)|^2 |u|^{p - 4} \, dx
= -\frac{|p - 2|}{2\sqrt{p - 1}} \Re(A_0u, u^*).
\]
Setting $z = -\langle A_0u, u^* \rangle \in \mathbb{C}_+$, we have shown that $|\arg z| = \arctan \frac{|\Im z|}{\Re z} \leq \arctan \frac{|p - 2|}{2\sqrt{p - 1}} = \arctan \frac{p - 2}{2\sqrt{p - 1}} =: \theta_p \in (0, \frac{\pi}{2})$. This means that $-\langle e^{\pm i\theta} A_0u, u^* \rangle \in \mathbb{C}_+$, and thus the operators $e^{\pm i\theta} A_0$ are dissipative for $0 \leq \theta \leq \theta_p$.

Next, let $p \in (1, 2)$. For $\epsilon > 0$ we replace $u^*$ by $u^*_\epsilon = \overline{u} u^{p - 2}_\epsilon$ with $0 < u_\epsilon := \sqrt{\epsilon + |u|^2}$ so that $u^{p - 2}_\epsilon = (\epsilon + u\overline{u})^{\frac{p - 2}{2}} \in C^1(\mathbb{R}^d)$. As above, we calculate
\[
\partial_k u^*_\epsilon = |u_\epsilon|^{p - 4} (\epsilon |\partial_k \overline{u}| + \overline{u} u_\epsilon |\partial_k \overline{u}| + (p - 2) \overline{u} \Re(\overline{u} \partial_k u))
\]
as well as
\[
\partial_k u \partial_k u^*_\epsilon = |u_\epsilon|^{p - 4} \left( \epsilon |\partial_k u|^2 + (p - 1)(\Re(\overline{u} \partial_k u))^2 + (\Im(\overline{u} \partial_k u))^2 + i(p - 2) \Im(\overline{u} \partial_k u) \Re(\overline{u} \partial_k u) \right),
\]
arriving at
\[
-\Re(A_0u, u^*_\epsilon) = \int_{\mathbb{R}^d} |u_\epsilon|^{p - 4} \left( \epsilon |\nabla u|^2 + (p - 1)|\Re(\overline{u} \nabla u)|^2 + |\Im(\overline{u} \nabla u)|^2 \right) \, dx
\]
and
\[
|\Im(A_0u, u^*_\epsilon)| \leq \frac{|p - 2|}{2\sqrt{p - 1}} \int_{\mathbb{R}^d} |u_\epsilon|^{p - 4} \left( (p - 1)|\Re(\overline{u} \nabla u)|^2 + |\Im(\overline{u} \nabla u)|^2 \right) \, dx.
\]
Since $u_\epsilon \geq |u|$, we have $|u_\epsilon|^{p - 2} \leq |u|^{p - 2}$ and thus $|u^*_\epsilon| \leq |u^*| \in L^p(\mathbb{R}^d)$. Moreover, $u_\epsilon$ converges on $\mathbb{R}^d$ to $u$ pointwise as $\epsilon \to 0$. So $u^*_\epsilon$ tends to $u^*$ in $L^p(U)$ by dominated convergence. Using also Fatou’s lemma, we deduce that
\[
-\Re(A_0u, u^*) = \lim_{\epsilon \to 0} \left( -\Re(A_0u, u^*_\epsilon) \right) \geq \liminf_{\epsilon \to 0} \int_{\mathbb{R}^d} \left( (p - 1)|\Re(\overline{u} \nabla u)|^2 + |\Im(\overline{u} \nabla u)|^2 \right) |u_\epsilon|^{p - 4} \, dx
\]
Let $\emptyset \neq U \subseteq \mathbb{R}^d$ be bounded and open with $C^2$ boundary. For $p \in (1, \infty)$ define $A = \Delta$ on $L^p(U)$ with $\mathcal{D}(A) = W^2_p(U) \cap W^1_p(U)$. One sees as above that $A_0$ and $e^{\pm i \theta} A_0$ are dissipative for all $0 \leq \theta \leq \theta_p$ where $A_0$ is the restriction of $A$ to $\mathcal{D}(A_0) = C^2(\overline{U}) \cap C_0(U)$. By Proposition 1.31, the closure $\overline{A_0}$ exists and is dissipative. Hence, $e^{\pm i \theta} \overline{A_0}$ are the closures of $e^{\pm i \theta} A_0$ and dissipative. A modification of the proof of Corollary 3.23 in [ST] shows that $\mathcal{D}(A_0)$ is dense in $\mathcal{D}(A)$ for the Sobolev norm $\| \cdot \|_{2,p}$. Using that $\| \cdot \|_{2,p}$ is stronger than $\| \cdot \|_{A_0}$, we obtain that $A \subseteq \overline{A_0}$. Theorem 9.15 of [G-T] further yields that $I - A$ is surjective. Since $I - \overline{A_0}$ is injective by dissipativity, we derive that $A = \overline{A_0}$. As a result, $A$ generates a contractive analytic $C_0$-semigroup. For related results, we refer to Chapter 3 of [L] (also for $p = \infty$) and to [T79] and [T97] (also for $p = 1$).

**Remark**

Based on the Lax-Milgram theorem, each “closed” and “sectorial” sesquilinear form

$$a : \mathcal{D}(a) \times \mathcal{D}(a) \to \mathbb{C}$$

on a dense subspace $\mathcal{D}(a)$ of $X$ can be “associated” to an operator $A$ satisfying the assumption of Corollary 2.14 (cf. Example 1.41). In particular, $A$ generates a contractive analytic $C_0$-semigroup. See §6.2 of [K], and also [O] for related results on $L^p$-spaces.

Let $x \in X$, $b > 0$, $f \in C([0,b], X)$ and $A - \omega I$ be sectorial of angle $\phi > \frac{\pi}{2}$ for some $\omega \in \mathbb{R}$. We consider the inhomogeneous evolution equation

$$\begin{aligned}
\begin{cases}
u'(t) = Au(t) + f(t), & t \in (0, b], \\
u(0) = x,
\end{cases}
\end{aligned} \tag{2.18}$$
with its mild solution

\[ u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds =: T(t)x + v(t), \quad t \in [0,b], \]

where \( A \) generates the analytic \( C_0 \)-semigroup \( T(\cdot) \).

**Theorem 2.20**

Let \( x \in X, \ b > 0, \ f \in C([0,b],X) \) and \( A - \omega I \) be sectorial of angle \( \phi > \frac{\pi}{2} \) for some \( \omega \in \mathbb{R} \). Then the following assertions hold for the mild solution \( u \) of \( (2.18) \).

(a) We have \( u \in C^\beta([\epsilon,b],X) \) for all \( \beta \in (0,1) \) and \( \epsilon \in (0,b) \). If \( x \in \mathcal{D}(A) \), we can even take \( \epsilon = 0 \).

(b) If \( f \in C^\alpha([0,b],X) \) for some \( \alpha \in (0,1) \), then \( u \) solves \( (2.18) \) on \( [0,b] \). If also \( x \in \mathcal{D}(A) \), then \( u \) solves \( (2.18) \) on \( [0,b] \).

**Remark**

For \( \alpha = 0 \), Theorem 2.20 (b) is wrong due to Example 4.1.7 in [L]. One thus needs a bit of extra regularity of \( f \). Much more detailed information on the regularity of \( u \) can be found in Chapter 4 of [L].

The space \( C^\alpha([a,b],X) \) with \( \alpha \in (0,1) \) is the space of all functions \( u \in C([a,b],X) \) satisfying

\[ [u]_\alpha := \sup_{a \leq s < t \leq b} \frac{\|u(t) - u(s)\|}{(t-s)^\alpha} < \infty. \]

This space is called *Hölder space with exponent \( \alpha \)* and it becomes a Banach space when endowed with the norm

\[ \|u\|_\alpha := \|u\|_\infty + [u]_\alpha. \]

**Proof.** Due to Theorem 2.12 and Remark 2.13, the function \( T(\cdot)x \) solves \( (2.18) \) on \( (0,\infty) \) with \( f = 0 \) if \( x \in X \) and on \( \mathbb{R}_+ \) if \( x \in \mathcal{D}(A) \). Moreover,

\[ T(\cdot)x \in C^1([\epsilon,b],X) \hookrightarrow C^\beta([\epsilon,b],X) \]

for all \( \beta \in (0,1) \) and \( \epsilon > 0 \) and for all \( \epsilon \geq 0 \) if \( x \in \mathcal{D}(A) \). So we only have to consider the function \( v \) from (2.19).

(a): Let \( 0 \leq s < t \leq b \). We write

\[ v(t) - v(s) = \int_s^t T(t-\tau)f(\tau)\,d\tau + \int_0^s (T(t-\tau) - T(s-\tau))f(\tau)\,d\tau =: I_1 + I_2. \]

By Theorem 2.12 and Remark 2.13 there are constants \( c_j = c_j(b) \) with \( j \in \{0,1\} \) such that

\[ \|T(t)\| \leq c_0 \quad \text{and} \quad \|tAT(t)\| \leq c_1 \]

for all \( 0 \leq t \leq b \). It thus holds \( \|I_1\| \leq c_0|t-s||f|_\infty \). Furthermore, we have

\[ \|T(t-\tau) - T(s-\tau)\| \leq c_0|t-s| \frac{c_1}{|s-\tau|}. \]
for all $t > s > \tau \geq 0$ because of

$$T(t - \tau) - T(s - \tau) = (T(t - s) - I)T(s - \tau) = \int_0^{t-s} T(\sigma)AT(s - \tau)\,d\sigma.$$ 

We then estimate

$$\|I_2\| \leq \int_0^s \|T(t - \tau) - T(s - \tau)\|^2\|T(t - \tau) - T(s - \tau)\|^{1-\beta}\|f(\tau)\|\,d\tau$$

$$\leq \int_0^s c_0^2c_1^2|t - s|^\beta|s - \tau|^{1-\beta}2c_0(1-\beta)\|f\|_\infty\|s^{1-\beta}|t - s|^{\beta}. $$

Using that $s \leq b$, we thus find a constant $c = c(\beta, b)$ such that $[v]_\beta \leq c\|f\|_\infty$. Since we know that $v \in C([0, b], X)$ and $\|v\|_\infty \leq bc_0\|f\|_\infty$, assertion (a) is shown.

(b): In view of Lemma 2.8 (with $x = 0$), we have to show that $v \in C([0, b], [\mathcal{D}(A)])$. For $t \in [0, b]$, we write

$$v(t) = \int_0^t T(t - s)(f(s) - f(t))\,ds + \int_0^t T(\tau)f(t)\,d\tau =: v_1(t) + v_2(t),$$

where we have substituted $\tau = t - s$. Lemma 1.12 then says that $v_2(t) \in \mathcal{D}(A)$ and $Av_2 = T(\cdot)f(\cdot) - f(\cdot) \in C([a, b], X)$. For $0 < \epsilon < \epsilon_0 \leq t \leq b$, Theorem 2.12 further implies that

$$v_{1,\epsilon}(t) := \int_0^{t-\epsilon} T(t - s)(f(s) - f(t))\,ds = T(\epsilon)\int_0^{t-\epsilon} T(t - \epsilon - s)(f(s) - f(t))\,ds$$

belongs to $\mathcal{D}(A)$ and that $Av_{1,\epsilon} \in C([\epsilon, b], X)$. Furthermore,

$$Av_{1,\epsilon}(t) = \int_0^{t-\epsilon} AT(t - s)(f(s) - f(t))\,ds.$$

We also estimate

$$\|v_1(t) - v_{1,\epsilon}(t)\| = \left\|\int_0^{t-\epsilon} T(t - s)(f(s) - f(t))\,ds\right\| \leq 2c_0\|f\|_\infty$$

so that $v_{1,\epsilon}(t) \to v_1(t)$ as $\epsilon \to 0$. Next, let $0 < \epsilon < \eta < \epsilon_0 \leq t$. We calculate

$$Av_{1,\epsilon}(t) - Av_{1,\eta}(t) = A\int_{t-\eta}^{t-\epsilon} T(t - s)(f(s) - f(t))\,ds = \int_{t-\eta}^{t-\epsilon} AT(t - s)(f(s) - f(t))\,ds$$
because $A$ is closed and the integrands are continuous on $[t-\eta, t-\epsilon]$. From Theorem 2.12, we now deduce that

$$
\|Av_{1,\epsilon}(t) - Av_{1,\eta}(t)\| \leq c_1 \int_{t-\eta}^{t-\epsilon} (t-s)^{-1}(t-s)^{\alpha} f |t-s|^{-\alpha} ds = \frac{c_1}{\alpha} \int_{t-\eta}^{t-\epsilon} f |t-s|^{-1}\alpha = \frac{c_1}{\alpha} |f|_{\alpha}(\eta^\alpha - \epsilon^\alpha).
$$

Hence, $Av_{1,\epsilon}$ converges in $C([\epsilon_0, b], X)$ as $\epsilon \to 0$. Since $A$ is closed, it follows that $v_1(t) \in \mathcal{D}(A)$ for all $t \in [\epsilon_0, b]$ and $Av_1 \in C([\epsilon_0, b], X)$ for all $\epsilon_0 > 0$. Clearly, $v_1(0) = 0 \in \mathcal{D}(A)$. Moreover,

$$
\|Av_1(t)\| = \lim_{\epsilon \to 0} \|Av_{1,\epsilon}(t)\| \leq \lim_{\epsilon \to 0} \int_{t-\epsilon}^{t-\epsilon} c_1 |t-s|^{-1}[f](t-s)^{\alpha} ds \leq \frac{c_1}{\alpha} |f|_{\alpha} t^\alpha
$$

tends to 0 as $t \to 0$. We conclude that $Av \in C([0, b], X)$ as required. \hfill \Box

**Example 2.21**

Let $\emptyset \neq U \subseteq \mathbb{R}^d$ be bounded and open with $C^2$ boundary, $1 < p < \infty$, $u_0 \in L^p(U)$ and $f \in C^\alpha([0, b], L^p(U))$ for some $\alpha \in (0, 1)$. We then obtain a unique solution $u \in C^1((0, b], L^p(U)) \cap C((0, b], D(\Delta_D)) \cap C([0, b], L^p(U))$

of the problem

$$(2.20) \begin{cases}
\partial_t u(t) = \Delta_D u(t) + f(t), & 0 < t \leq b, \\
u(0) = u_0
\end{cases}$$

where $\Delta_D w = \Delta w$ with $D(\Delta_D) = W^2_p(U) \cap \dot{W}^1_p(U)$ (see Theorem 2.20, Example 2.16 and Example 2.19). Setting $g(t, x) = (g(t))(x)$ for all $0 < t \leq b$ and almost every $x \in U$, we arrive at

$$(2.21) \begin{cases}
\partial_t u(t, x) = \Delta u(t, x) + f(t, x), & t > 0, x \in U, \\
u(t, x) = 0, & t > 0, x \in \partial U, \\
u(0, x) = u_0(x), & x \in U.
\end{cases}$$

In general, here the first and third equality hold almost everywhere and the second one in the sense of trace. However if $p > \frac{d}{2}$, the Sobolev embedding shows that $W^2_p(U) \hookrightarrow C(\overline{U})$ (see, e.g., Corollary 3.22 in [ST]) so that the boundary condition holds pointwise. The solutions become more regular if we improve the regularity of $u_0$, $f$ and $\partial U$ (see Section 5 of [L]).
CHAPTER 3

Perturbation and approximation

3.1. Perturbation

The basic question in this section is: Let \( A \) generate a \( C_0 \)-semigroup and \( B \) be linear. Does “\( A + B \)” generate a \( C_0 \)-semigroup? Here we think of \( A \) as “well known” and \( B \) as an “easier part”. In this setting we have to face two basic problems.

First, how to define “\( A + B \)” if \( D(A) \cap D(B) \) is “small” (e.g., equal to \{0\})? In this section we just consider the case that \( D(A) \subseteq D(B) \). Unless something else is said, we put \( D(A + B) := D(A) \) in this case.

Second, if \( B \) with \( D(B) \supseteq D(A) \) is “too large”, then it can happen that \( A + B \) fails to be a generator. For instance, consider any generator \( A \) whose spectrum is unbounded to the left (e.g. \( \Delta \) on \( L^2(\mathbb{R}^d) \)) and \( B := -(1 + \delta)A \) for any \( \delta > 0 \). Then \( A + B = -\delta A \) and \( s(A + B) = \infty \), thus \( A + B \) is not a generator. The following definition gives an appropriate concept of “smallness”.

Definition 3.1

Let \( A \) and \( B \) be linear operators with \( D(A) \subseteq D(B) \). Then \( B \) is called \( A \)-bounded (or relatively bounded) if there are constants \( a,b \geq 0 \), such that

\[
\|Bx\| \leq a\|Ax\| + b\|x\|
\]

for all \( x \in D(A) \). If this is the case we set \( D(A + B) := D(A) \) (unless something else is specified). The infimum of the possible numbers \( a \geq 0 \) for which (3.1) holds for some \( b = b(a) \geq 0 \) is called \( A \)-bound of \( B \).

We note that the \( A \)-boundedness of \( B \) is equivalent to the statement \( B \in \mathcal{B}(D(A)), X \).

Let \( A \) be closed with \( \lambda \in \rho(A) \). Assume that \( D(A) \subseteq D(B) \), and set \( \gamma := \|BR(\lambda, A)\| < \infty \). For \( x \in D(A) \), write \( y = \lambda x - Ax \in X \). Then

\[
\|Bx\| = ||B(\lambda, A)y|| \leq \gamma \|y\| \leq \gamma \|Ax\| + |\lambda| \gamma \|x\|,
\]

i.e., (3.1) holds with \( a = \gamma \).

Conversely, let \( B \) be \( A \)-bounded and \( z \in X \). Then

\[
\|B(\lambda, A)z\| \leq a\|AR(\lambda, A)z\| + b\|R(\lambda, A)z\| = a\|R(\lambda, A - I)z\| + b\|R(\lambda, A)z\|
\]

so that \( B(\lambda, A) \in \mathcal{B}(X) \).

Lemma 3.2

Let \( A \) and \( B \) be linear with \( D(A) \subseteq D(B) \) and \( \|Bx\| \leq c\|Ax\|^\alpha \|x\|^{1-\alpha} \) for all \( x \in D(A) \) and
for some constants $c \geq 0$ and $\alpha \in (0, 1)$. Then (3.1) holds for all $a > 0$ with some $b = b(a) \geq 0$ so that $B$ has the $A$-bound $0$.

**Proof.** The lemma follows from Young’s inequality

$$ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{p'}$$

for all $a, b \geq 0$, $p \in (1, \infty)$ and $p' = \frac{p}{p-1}$, since we have

$$\|Bx\| \leq \epsilon \|Ax\|^\alpha \cdot c \cdot \frac{1}{\epsilon} \|x\|^{1-\alpha} \leq \alpha \epsilon \frac{1}{\epsilon} \|Ax\| + c \frac{1}{\epsilon^{1-\alpha}} (1 - \alpha) \epsilon^{-\frac{1}{1-\alpha}} \|x\|$$

with $p = \frac{1}{\alpha}$, $p' = \frac{1}{1-\alpha}$, for all $x \in \mathcal{D}(A)$ and $\epsilon > 0$. \hfill \square

The following perturbation result for the resolvent is proved, e.g., in Theorem 1.26 in [ST] using properties of the Neumann series; see, e.g., Proposition 3.12 in [FA].

**Lemma 3.3**

Let $A$ be closed, $\lambda \in \rho(A)$ and $B$ be $A$-bounded with $\|BR(\lambda, A)\| < 1$. Then $A + B$ (with $\mathcal{D}(A + B) := \mathcal{D}(A)$) is closed and $\lambda \in \rho(A + B)$ with

$$R(\lambda, A + B) = R(\lambda, A) \sum_{n=0}^{\infty} (BR(\lambda, A))^n = R(\lambda, A)(I - BR(\lambda, A))^{-1}$$

and with

$$\|R(\lambda, A + B)\| \leq \frac{\|R(\lambda, A)\|}{1 - \|BR(\lambda, A)\|}.$$ 

**Theorem 3.4** (Bounded perturbation)

Let $B$ be a bounded operator and $A$ generate a $C_0$-semigroup $T(\cdot)$ satisfying $\|T(t)\| \leq M e^{\omega t} t$ for all $t \geq 0$ and some constants $M \geq 1$ and $\omega \in \mathbb{R}$. Then the sum $A + B$ with $\mathcal{D}(A + B) = \mathcal{D}(A)$ generates a $C_0$-semigroup $S(\cdot)$ satisfying

$$\|S(t)\| \leq M e^{(\omega + M\|B\|)t}$$

for all $t \geq 0$. Furthermore, it holds

(3.3) \hspace{1cm} S(t)x = T(t)x + \int_0^t T(t - s)BS(s)x \, ds,$$

(3.4) \hspace{1cm} S(t)x = T(t)x + \int_0^t S(t - s)BT(s)x \, ds$$

as well as

(3.5) \hspace{1cm} S(t) = \sum_{n=0}^{\infty} S_n(t), \hspace{0.5cm} \text{where} \hspace{0.5cm} S_0(t) = T(t) \hspace{0.5cm} \text{and} \hspace{0.5cm} S_{n+1}(t)x = \int_0^t T(t - s)BS_n(s)x \, ds,$$
for all $n \in \mathbb{N}, t \geq 0, x \in X$. The so called Dyson-Phillips series in (3.5) converges in $\mathcal{B}(X)$ uniformly on compact subsets of $\mathbb{R}_+$. The operator family $(S(t))_{t \geq 0}$ is the only strongly continuous family of operators satisfying (3.3).

**Proof.** 1. Step: Observe that $A + B$ is densely defined. First let $\omega = 0$ and $M = 1$. Take $\lambda > \|B\|$. Due to Proposition 1.14, we have

$$\|B R(\lambda, A)\| \leq \frac{\|B\|}{\lambda} < 1.$$ 

Lemma 3.3 thus yields $\lambda \in \rho(A + B)$ and

$$\| R(\lambda, A + B)\| \leq \frac{1}{1 - \frac{\|B\|}{\lambda}} = \frac{1}{\lambda - \|B\|}.$$ 

Hence, the Hille-Yosida theorem shows that $A + B - \|B\|I$ generates a contraction semigroup so that $A + B$ generates a $C_0$-semigroup $S(\cdot) \in \mathcal{B}(X)$ satisfying $\|S(t)\| \leq e^{\|B\|t}$ for all $t \geq 0$.

2. Step: In the general case, we recall that $A - \omega I$ generates the $C_0$-semigroup given by $e^{-\omega t} T(t)$ which is contractive for the norm

$$\|x\| = \sup_{s \geq 0} \|e^{-\omega s} T(s)x\|,$$

see Remark 1.21. Since $\|x\| \leq \|x\| \leq M\|x\|$, the operator $B$ satisfies

$$\|Bx\| \leq M\|Bx\| \leq M\|B\| \cdot \|x\| \leq M\|B\| \cdot \|x\|$$

for all $x \in X$ so that $\|B\| \leq M\|B\|$. The first step now implies that $A + B - \omega I$ generates a $C_0$-semigroup $\tilde{S}(\cdot)$ with

$$\|\tilde{S}(t)\| \leq e^{M\|B\|t}$$

for all $t \geq 0$. Hence, $A + B$ generates the $C_0$-semigroup $S(\cdot)$ fulfilling

$$\|S(t)x\| = \|e^{\omega t} \tilde{S}(t)x\| \leq e^{\omega t} \|\tilde{S}(t)\| \leq e^{\omega t} e^{M\|B\|t} \|x\| \leq M e^{(\omega + M\|B\|)t} \|x\|$$

for all $x \in X$ and $t \geq 0$, as asserted.

3. Step: We want to show (3.3)-(3.5) and uniqueness. For any fixed $x \in \mathcal{D}(A)$ the function $u(t) = S(t)x, t \geq 0$, solves

$$\begin{cases} u'(t) = (A + B)u(t) = Au(t) + f(t), & t \geq 0, \\ u(0) = x, \end{cases}$$

where $f(t) := BS(t)x$ is a continuous function for $t \geq 0$. Thus Proposition 2.6 on mild solutions shows that

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)x \, ds$$

which is (3.3) for $x \in \mathcal{D}(A)$. We derive (3.3) for all $x \in X$ by approximation, by means of $\mathcal{D}(A) = X$ and $S(s), T(t-s), S(t), T(t) \in \mathcal{B}(X)$. Equation (3.4) is shown in the same way.
using that \( v(t) = T(t)x \) solves
\[
\begin{cases}
v'(t) = (A + B)v(t) - Bv(t), & t \geq 0, \\
u(0) = x \in \mathcal{D}(A).
\end{cases}
\]
Concerning (3.5), we note that \( S_1(\cdot) \) is strongly continuous and
\[
\|S_1(t)x\| \leq \int_0^t M^2 e^\omega(t-s) \|B\| e^{\omega s} \|x\| \, ds = M^2 t e^{\omega t} \|B\| \cdot \|x\|
\]
for all \( t \geq 0 \) and \( x \in X \). A proof by induction then gives that \( S_n(\cdot) \) is strongly continuous with
\[
\|S_n(t)\| \leq \frac{M^{n+1} \|B\|^n}{n!} t^n e^{\omega t}
\]
for all \( t \geq 0 \) and \( n \in \mathbb{N} \). Thus the series in (3.5) converges in \( \mathcal{B}(X) \) uniformly on compact subsets of \( \mathbb{R}_+ \) to some operator \( R(t) \). Hence \( R(t) \) is strongly continuous and
\[
\int_0^t T(t - s)BR(s)x \, ds = \sum_{n=0}^{\infty} \int_0^t T(t - s)BS_n(s)x \, ds = \sum_{j=1}^{\infty} S_j(t)x = R(t)x - T(t)x
\]
for all \( t \geq 0 \) and all \( x \in X \). The uniqueness assertion for (3.3) will then imply (3.5). In order to show uniqueness, let \( (U(t))_{t \geq 0} \) be strongly continuous and satisfy (3.3) with \( S(\cdot) \) replaced by \( U(\cdot) \). Then
\[
\|S(t)x - U(t)x\| = \left\| \int_0^t T(t - s)B(S(s)x - U(s)x) \, ds \right\| \leq M e^{\omega|t_0|} \|B\| \int_0^t \|S(s)x - U(s)x\| \, ds
\]
holds for all \( x \in X \) and \( t \in [0, t_0] \) where \( t_0 > 0 \) is arbitrary, but fixed. Recall Gronwall’s inequality:

If \( 0 \leq \varphi(t) \leq a + \int_0^t b\varphi(s) \, ds \) holds for all \( t \geq 0 \), then \( \varphi(t) \leq ae^{bt} \) holds for all \( t \geq 0 \), too.

We thus obtain \( \varphi(t) := \|S(t)x - U(t)x\| = 0 \) for all \( t \geq 0 \) and \( x \in X \), i.e., \( U(t) = S(t) \) for all \( t \geq 0 \). 

\[ \square \]

**Example 3.5** (Positivity and perturbation)

Let \( E = C_0(U) \) or \( E = L^p(U) \) for \( 1 \leq p < \infty \) and a nonempty, open set \( U \subseteq \mathbb{R}^d \). Let \( T(\cdot) \) be a positive \( C_0 \)-semigroup on \( E \) with generator \( A \), i.e., \( T(t)f \geq 0 \) for all \( t \geq 0 \) and \( 0 \leq f \in E \), where \( f \geq 0 \) means that \( f(x) \geq 0 \) for (almost) every \( x \in U \). Let \( B \in \mathcal{B}(E) \) be also positive. For \( 0 \leq f \in E \), we thus obtain that \( T(t - s)BT(s)f \) is positive for all \( 0 \leq s \leq t \). Since the set of positive functions is closed in \( E \), we infer that \( S_1(t)f \geq 0 \) and, by induction, that all terms \( S_n(t)f \) in the Dyson-Phillips series (3.5) are positive. So the semigroup \( S(\cdot) \) generated by \( A + B \) is positive and satisfies \( S(t) \geq T(t) \), i.e., \( S(t)f \geq T(t)f \) for all \( 0 \leq f \in E \). Next, let \( Bf = bf \) for a bounded measurable (and in the case \( E = C_0(U) \) continuous) function \( b : U \to \mathbb{R} \). Then
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$B_0 = B + \|b_-\|_\infty I$ is positive, where $b_- = \max\{-b, 0\}$. Hence, $A + B_0$ generates a positive $C_0$-semigroup $S_0(\cdot)$ and so $A + B = A + B_0 - \|b_-\|_\infty I$ generates the positive $C_0$-semigroup $S(\cdot)$ given by $S(t) = e^{-\|b_-\|_\infty t}S_0(t) \geq e^{-\|b_-\|_\infty t}T(t)$.

As a simple example, we take $U = \mathbb{R}$ and $A = \frac{d}{dx}$ with either $\mathcal{D}(A) = C^1_0(\mathbb{R})$ or $\mathcal{D}(A) = W^1_p(\mathbb{R})$. Because $A$ generates the positive translation semigroup on $E$, the operator $Cu = u' + bu$ with $\mathcal{D}(C) = \mathcal{D}(A)$ also generates a positive $C_0$-semigroup if $b$ is real, bounded and measurable (continuous, respectively).

We will continue the investigation of positive semigroups later.

The known perturbation theorems for unbounded perturbations need additional assumptions. We focus on very useful theorems for contraction or analytic semigroups. (See also §III.3 in [E-N] for different results based on the fixed point equation (3.3) for $S(\cdot)$.)

**Theorem 3.6** (Dissipative case)

Let $A$ generate the contraction semigroup $T(\cdot)$ and $B$ be dissipative. Assume that $B$ is $A$-bounded with a constant $a < 1$ in (3.1). Then $A + B$ with $\mathcal{D}(A + B) = \mathcal{D}(A)$ generates a contraction semigroup $S(\cdot)$ which also satisfies (3.3) and (3.4) for all $x \in \mathcal{D}(A)$.

**Proof.** 1) Observe that $A + B$ is densely defined and that we have $\text{Re}(Ax, x^*) \leq 0$ for all $x \in \mathcal{D}(A)$ and $x^* \in J(x)$ due to Proposition 1.26. Since $B$ is dissipative, for each $x \in \mathcal{D}(A)$ there is a $y^* \in J(x)$ such that $\text{Re}(Bx, y^*) \leq 0$. Hence, $A + B$ is dissipative. By the assumption there are constants $a \in [0, 1)$ and $b \geq 0$ such that

$$\|Bx\| \leq a\|Ax\| + b\|x\|$$

for all $x \in \mathcal{D}(A)$. First, assume that $a < \frac{1}{2}$. Take some $\lambda_0 > \frac{b}{1-2a} \geq 0$. Inequality (3.2) and the Hille-Yosida estimate yield

$$\|BR(\lambda_0, A)\| \leq a\lambda_0\|R(\lambda_0, A)\| + a + b\|R(\lambda_0, A)\| \leq a + a + \frac{b}{\lambda_0} < 1.$$

Lemma 3.3 now implies that $A + B$ is closed and $\lambda_0 \in \rho(A + B)$ and thus $A + B$ generates a contraction semigroup by the Lumer-Phillips theorem.

2) If $a \geq \frac{1}{2}$, we take $k \in \mathbb{N}$ with $k > \frac{2a}{1-2a} \leq \frac{1}{2}$. Then $\frac{1}{k}B$ is dissipative and $A$-bounded with a constant $a' = \frac{a}{k} < \frac{1-a}{2} \leq \frac{1}{2}$. By step 1), $A + \frac{1}{k}B$ thus generates a contraction semigroup. By induction, assume that $C_j := A + \frac{j}{k}B$ generates a contraction semigroup for some $j \in \{1, \ldots, k-1\}$. We then obtain

$$\|Bx\| \leq a\|Ax\| + b\|x\| \leq a\|C_jx\| + \frac{aj}{k}\|Bx\| + b\|x\|$$

and so

$$(1 - a)\|Bx\| \leq \left(1 - \frac{aj}{k}\right)\|Bx\| \leq a\|C_jx\| + b\|x\|,$$

which implies 63
The operator matrix for all hold for all last assertion can be shown as in Theorem 3.4. But note that it is not clear that (3.3) and (3.4) contraction semigroup. As a result, for all \( x \in \mathcal{D}(A) \). Since \( \tilde{a} := \frac{a}{k(1-a)} < \frac{1}{2} \), step 1 implies that \( C_j + \frac{1}{k} B = C_{j+1} \) generates a contraction semigroup. As a result, \( A + C_k = A + B \) generates a contraction semigroup. The last assertion can be shown as in Theorem 3.4. But note that it is not clear that (3.3) and (3.4) hold for all \( x \in X \) by approximation since \( B \) may be unbounded.

We refer to Corollaries III.2.8 and III.2.9 in [E-N] for versions of the above theorem with \( a = 1 \) in (3.1).

**Example 3.7** (damped wave equation)
Let \( \emptyset \neq U \subseteq \mathbb{R}^3 \) be bounded and open with \( C^2 \) boundary and \( \Delta_D v = \Delta v \) with \( \mathcal{D}(\Delta_D) = W^2_2(U) \cap W^1_2(U) \). Let \( b \in L^q(U) \) be positive for some \( q \in (3, \infty] \). Hölder’s inequality yields that

\[
\|bv\|_2 \leq \|b\|_q \|v\|_r \leq c\|b\|_q \|v\|_6
\]

for all \( v \in L^r(U) \) with \( r = \left( \frac{1}{2} - \frac{1}{q} \right)^{-1} \in [2, 6) \) and a constant \( c \). We consider the damped wave equation

\[
(3.6) \quad \begin{cases}
  u''(t) = \Delta_D u(t) - bu'(t), & t \geq 0, \\
  u(0) = u_0, \quad \partial_t u(0) = u_1,
\end{cases}
\]

for given \( u_0 \in \mathcal{D}(\Delta_D) \) and \( u_1 \in Y := \tilde{W}^1_2(U) \), where we endow \( Y \) with the norm \( \|\nabla u\|_2 \). We recall from Examples 1.42 and 2.4 that on \( E = Y \times L^2(U) \) with the norm

\[
\| (u, v) \|_E = (\|\nabla u\|_{L^2}^2 + \|v\|_2^2)^{\frac{1}{2}}
\]

the operator matrix

\[
A = \begin{pmatrix}
  0 & 1 \\
  \Delta_D & 0
\end{pmatrix}
\]

with \( \mathcal{D}(A) = \mathcal{D}(\Delta_D) \times Y \) generates the unitary \( C_0 \)-group \( T(\cdot) \). It holds

\[
\begin{pmatrix} v(t) \\ v'(t) \end{pmatrix} = T(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}
\]

for the solution \( v \) of (3.6) with \( b = 0 \). Sobolev’s embedding theorem yields that \( W^1_2(U) \hookrightarrow L^6(U) \) (since \( 1 - \frac{3}{2} = -\frac{3}{6} \)), see, e.g., (3.13) and Remark 3.20 in [ST]. As a result, we can define

\[
B = \begin{pmatrix}
  0 & 0 \\
  0 & -b
\end{pmatrix} : \mathcal{D}(B) \to E
\]

on \( \mathcal{D}(B) := Y \times Y \supseteq \mathcal{D}(A) \). For \( w = (u, v) \in \mathcal{D}(A) \), we further obtain

\[
\|Bw\|_E = \|bv\|_2 \leq \|b\|_q \|v\|_r \leq \|b\|_q \|v\|_2^2 \leq \|v\|_6^{1-\theta} \leq c^{1-\theta} \|b\|_q \|v\|_2^2 \|\nabla v\|_2^{1-\theta} \leq c^{1-\theta} \|b\|_q \|w\|_E^2 \|Aw\|_E^{1-\theta}
\]

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for the Sobolev constant $c$ and the number $\theta = \frac{3}{r} - \frac{1}{2} \in (0,1]$ satisfying $\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{6}$. Here we use a standard consequence of Hölder’s inequality. Lemma 3.2 thus implies that $B$ has the $A$-bound 0. Furthermore, we have

$$(Bw|w)_E = -(bv|v) = -\int b|v|^2 \, dx \leq 0.$$  

Hence, $A + B$ generates a contraction semigroup $S(\cdot)$ on $E$ due to Theorem 3.6. As in Example 2.4, one can see that the solution $u$ of (3.6) is given by $(u(t), u'(t))^T = S(t)(u_0, u_1)^T$.

**Theorem 3.8** (sectorial case)  
Let $A$ be densely defined and closed. Assume there are constants $\omega \geq 0$, $K > 0$ and $\varphi \in \left(\frac{\pi}{2}, \pi\right)$ such that $\omega + \Sigma_\varphi \setminus \{0\} \subseteq \rho(A)$ and

$$\| R(\lambda + \omega, A) \| \leq \frac{K}{|\lambda|}$$

holds for all $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg \lambda| \leq \varphi$. Let $B$ be $A$-bounded with constant $a \in \left[0, \frac{1}{K+1}\right]$ in (3.1). Then there is a number $\varpi \geq 0$ such that $A + B - \varpi I$ generates a bounded analytic $C_0$-semigroup, and hence $A + B$ generates an analytic semigroup.

**Proof.** First assume that $a\omega + b > 0$. Take $q \in (a(K+1), 1)$ and set $r := \frac{K(a\omega+b)}{q-a(K+1)} > 0$. For $\lambda \in \Sigma_\varphi \setminus B(0, r)$ and $x \in X$, we estimate

$$\| B R(\lambda, A - \omega I)x \| \leq a \| A R(\lambda + \omega, A)x \| + b \| R(\lambda + \omega, A)x \|$$

$$\leq a \| (\lambda + \omega) R(\lambda + \omega, A)x \| + a \| x \| + b \frac{K}{|\lambda|} \| x \|$$

$$\leq a \left( \frac{K|\lambda| + K\omega}{|\lambda|} + 1 \right) \| x \| + b \frac{K}{|\lambda|} \| x \|$$

$$\leq \left( a(K+1) + \frac{K(a\omega+b)}{r} \right) \| x \| = q \| x \|.$$  

If $a\omega + b = 0$, the above estimate holds even with $q = a(K+1) < 1$. Lemma 3.3 thus implies that $\lambda \in \rho(A + B - \omega I)$ and

$$\| R(\lambda, A + B - \omega I) \| \leq \frac{\| R(\lambda + \omega, A) \|}{1-q} \leq \frac{K/(1-q)}{|\lambda|}$$

for all $\lambda \in \Sigma_\varphi \setminus B(0, r)$. We can now choose a number $\gamma > 0$ such that $\gamma + \Sigma_\varphi \subseteq \Sigma_\varphi \setminus B(0, r)$. It follows that

$$\| R(\mu, A + B - (\omega + \gamma)I) \| = \| R(\mu + \gamma, A + B - \omega I) \| \leq \frac{K/(1-q)}{|\mu + \gamma|} \leq \frac{K/(1-q)}{|\mu|}$$

for all $\mu \in \mathbb{C} \setminus \{0\}$ with $|\arg \mu| \leq \varphi$. Setting $\varpi = \gamma + \omega$, we arrive at the assertion. \qed

We note that the above proof keeps the angle $\varphi$ but increases the shift $\omega$ considerably.
Example 3.9
Let \( \emptyset \neq U \subseteq \mathbb{R}^d \) be bounded and open with \( C^2 \) boundary and \( A = \Delta \) with \( \mathcal{D}(A) = W^2_p(U) \cap W^1_p(U) \) in \( E = L^p(U) \) for some \( p \in (1, \infty) \). Example 2.16 and 2.19 say that \( A \) is sectorial. For \( b \in L^{\infty}(U)^d \) and \( b_0 \in L^\infty(U) \), we define

\[
Bu = b \cdot \nabla u + b_0 u = \sum_{j=1}^d b_j \partial_j u + b_0 u
\]

for \( u \in W^1_p(U) \). On \( \mathcal{D}(A) \) we have \( \|u\|_A \leq c\|u\|_{2,p} \) and both norms are complete. Hence, they are equivalent thanks to the open mapping theorem. Using also, e.g., Proposition 3.27 in [ST], we then estimate

\[
\|Bu\|_p \leq \|b\|_p \|\Delta u\|_p + \|b_0\| \|u\|_p \leq \|b\|_p \|\Delta u\|_p + \|b_0\| \|u\|_p \leq \|b\|_p \|\Delta u\|_p + \|b_0\| \|u\|_p
\]

for all \( \epsilon > 0 \) and some constants \( c, c_\epsilon > 0 \). Therefore, \( B \) has the \( A \)-bound 0 and Theorem 3.8 shows that \( Cu = \Delta u + b \cdot \nabla u + b_0 u \) with \( \mathcal{D}(C) = \mathcal{D}(A) \) generates an analytic semigroup on \( L^p(U) \) for \( 1 < p < \infty \).

Example 3.10
Let \( E = L^2(\mathbb{R}^3), A = i\Delta \) with \( \mathcal{D}(A) = W^2_2(\mathbb{R}^3), V(x) = -\frac{b}{|x|} \) for \( x \in \mathbb{R}^3 \setminus \{0\} \) and \( Bu = -iVu \) for \( u \in \mathcal{D}(A) \) and some constant \( b \in \mathbb{R} \). By Stone’s theorem 1.36 the Schrödinger equation

\[
\begin{align}
\frac{du}{dt} (t) &= (A + B)u(t) = -i(-\Delta + V)u(t), \quad t \in \mathbb{R}, \\
u(0) &= u_0
\end{align}
\]

is solved by a unitary \( C_0 \)-group if \( (A + B, \mathcal{D}(A)) \) is skewadjoint, i.e., \( H := -\Delta + V \) with \( \mathcal{D}(H) = W^2_2(\mathbb{R}^3) \) is selfadjoint. Recall that \( -\Delta \) is selfadjoint by Example 1.38. Sobolev’s embedding theorem shows that \( W^2_2(\mathbb{R}^3) \hookrightarrow C_0(\mathbb{R}^3) \) since \( 2 - \frac{3}{2} > 0 \) (see, e.g., Theorem 3.16 in [ST]). For \( \epsilon > 0 \), we thus obtain

\[
\|Vu\|_2^2 = \frac{b^2}{2} \int_{B(0,\epsilon)} |u(x)|^2 \frac{dx}{|x|^2} + b^2 \int_{\mathbb{R}^3 \setminus B(0,\epsilon)} |u(x)|^2 \frac{dx}{|x|^2} \leq c \int_{0}^{\epsilon} \|u\|_2^2 r^2 dr + \frac{b^2}{\epsilon^2} \int_{\mathbb{R}^3 \setminus B(0,\epsilon)} |u(x)|^2 dx
\]

which implies

\[
\|Vu\|_2 \leq \sqrt{c\epsilon} \|u\|_2 + \frac{b}{\epsilon} \|u\|_2
\]

for all \( u \in W^2_2(U) \) and a constant \( c > 0 \) arising from polar coordinates and the Sobolev embedding. Hence, the multiplication operator induced by \( V \) has the \( A \)-bound 0 and so \( -\Delta + V \) generates an analytic \( C_0 \)-semigroup, implying \( \rho(-\Delta + V) \cap \mathbb{R} \neq \emptyset \). By, e.g., Theorem 4.18 of [ST], the selfadjointness of \( H \) thus follows from the symmetry of \( H \). In order to check
symmetry, take \( u, v \in C_c^\infty(\mathbb{R}^3) \) and observe that then we have
\[
((-\Delta + V)u|v) = \int_{\mathbb{R}^3} -\Delta u \, v \, dx + \int_{\mathbb{R}^3} Vu \, v \, dx = (u|(-\Delta + V)v)
\]
due to integration by parts. Since \( C_c^\infty(\mathbb{R}^3) \) is dense in \( W^2_2(U) \) and \( \Delta, V \in \mathcal{B}(W^2_2(\mathbb{R}^3), L^2(\mathbb{R}^3)) \), we deduce \((-\Delta + V)u|v) = (u|(-\Delta + V)v)\) for all \( u, v \in W^2_2(\mathbb{R}^3) \) as required.

### 3.2. The Trotter-Kato theorems

The basic question of this section is: If two generators \( A_1 \) and \( A_2 \) with generated \( C_0 \)-semigroups \( T_1(\cdot) \) and \( T_2(\cdot) \), respectively, are “close”, does it follow that \( T_1(\cdot) \) and \( T_2(\cdot) \) are close, too? This is an important question since in applications the system’s parameters enter into the generator and are known only approximately.

The easiest case occurs if \( B := A_2 - A_1 \in \mathcal{B}(X) \). Then formula (3.3) gives
\[
\|T_2(t)x - T_1(t)x\| = \left\| \int_0^t T_1(t-s)BT_2(s)x \, ds \right\| \leq tc(t_0)\|B\| \cdot \|x\|
\]
for all \( x \in X, t \in [0, t_0], t_0 > 0 \) and a constant depending on \( t_0 \).

**Example 3.11**

Let \( \emptyset \neq U \subseteq \mathbb{R}^d \) be open and bounded with \( C^2 \) boundary, \( E = L^2(U), \Delta_D u = \Delta u \) with \( \mathcal{D}(\Delta_D) = W^2_2(U) \cap \dot{W}^2_2(U) \) and \( a, a_n \in L^\infty(U) \) with \( \frac{1}{\delta} \geq a_n(x), a(x) \geq \delta > 0 \) for almost every \( x \in U \), all \( n \in \mathbb{N} \), and a constant \( \delta > 0 \). From Exercise 24 we know that \( A = a\Delta_D \) and \( A_n = a_n\Delta_D \) with domains \( \mathcal{D}(A) = \mathcal{D}(A_n) = \mathcal{D}(\Delta_D) \) generate analytic \( C_0 \)-semigroups \( T(\cdot) \) and \( T_n(\cdot) \) with \( \delta^{-1} \) as a common norm bound. Assume that \( a_n \rightarrow a \) pointwise almost everywhere on \( U \) as \( n \rightarrow \infty \). Then \( A_n u \rightarrow A u \) in \( E \) as \( n \rightarrow \infty \) for all \( u \in \mathcal{D}(\Delta_D) \) by dominated convergence with majorant \( \delta^{-2}\|\Delta u\| \). Can we conclude that \( T_n(\cdot) \) converges somehow to \( T(\cdot) \)?

**Example 3.12**

Let \( X = \ell^2, A((x_k)_k) = (ikx_k)_k \) with \( \mathcal{D}(A) = \{x \in \ell^2; (kx_k)_k \in \ell^2\} \) and \( A_n((x_k)_k) = (ikx_k + \delta_{k,n}kx_k)_k \) with \( \mathcal{D}(A_n) = \mathcal{D}(A) \) for all \( n \in \mathbb{N} \) and the Kronecker delta \( \delta_{k,n} \). As in the Exercises, one sees that \( A \) generates the \( C_0 \)-semigroup given by \( T(t)x = (e^{ikt}x_k)_k \) and \( A_n \) generates the \( C_0 \)-semigroup given by \( T_n(t)x = (e^{ikt}e^{k\delta_{k,n}t}x_k)_k \). We then have
\[
\|A_n x - Ax\|_2 = |inx_n| = |(Ax)_n| \rightarrow 0,
\]
but
\[
\|T_n(t)\| \geq \|T_n(t)e_n\|_2 = |e^{i\int_0^t e^{\delta_{k,n}t}| = e^{\int_0^t \delta_{k,n}t} \rightarrow \infty
\]
as \( n \rightarrow \infty \) for all \( x \in \mathcal{D}(A) \) and all \( t > 0 \). So \( A_n \) converges on the common domain strongly to \( A \). However, \( T_n(t) \) cannot converge strongly, since strong convergence would imply uniform boundedness of \( \{|T_n(t); n \in \mathbb{N}\} \) on compact subsets of \( \mathbb{R}_+ \). As a consequence, we will impose a uniform boundedness condition in the following results.
Theorem 3.13 (Trotter-Kato I, 1958/59)
Let $A_n$ and $A$ generate $C_0$-semigroups $T_n(\cdot)$ and $T(\cdot)$, respectively, which satisfy $\|T_n(t)\|, \|T(t)\| \leq M e^{\omega t}$ for all $t \geq 0$ and $n \in \mathbb{N}$ and some $M \geq 1$ and $\omega \in \mathbb{R}$. Let $\mathcal{D}$ be a core of $\mathcal{D}(A)$. The following relation (a)$\Rightarrow$(b)$\Leftrightarrow$(d)$\Rightarrow$(c) holds among the subsequent assertions, where we always let $n$ tend to infinity.

(a) $\mathcal{D} \subseteq \mathcal{D}(A_n)$ for all $n \in \mathbb{N}$ and $A_n x \rightarrow Ax$ for all $x \in \mathcal{D}$.
(b) For all $x \in \mathcal{D}$ and $n \in \mathbb{N}$ there are $x_n \in \mathcal{D}(A_n)$ such that $x_n \rightarrow x$ and $A_n x_n \rightarrow Ax$.
(c) For some $\lambda \in \omega + \mathbb{C}_+$, we have $R(\lambda, A_n)x \rightarrow R(\lambda, A)x$ for all $x \in X$.
(d) For each $t \geq 0$ we have $T_n(t)x \rightarrow T(t)x$ for all $x \in X$.

If (c) or (d) holds, then (c) holds for all $\lambda \in \omega + \mathbb{C}_+$ and the limit in (d) is uniform on all compact subsets of $\mathbb{R}_+$.

Proof. The implication from (a) to (b) is trivial (take $x_n = x$).

Let (b) hold and take any $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > \omega$. Since $(\lambda I-A)\mathcal{D}$ is dense in $X$ and $\| R(\lambda, A_n) \| \leq \frac{M}{\text{Re}\lambda - \omega}$ for all $n \in \mathbb{N}$ by the assumptions, we only have to show that $R(\lambda, A_n) y \rightarrow R(\lambda, A) y$ for all $y = \lambda x - Ax$ and all $x \in \mathcal{D}$. Let $x \in \mathcal{D}$. Due to (b), there are $x_n \in \mathcal{D}(A_n)$ such that $x_n \rightarrow x$ and $A_n x_n \rightarrow Ax$ as $n \rightarrow \infty$, hence

$$y_n := \lambda x_n - A_n x_n \rightarrow y = \lambda x - Ax$$

as $n$ tends to infinity. Estimating

$$\| R(\lambda, A_n)y - R(\lambda, A)y \| \leq \| R(\lambda, A_n)(y - y_n) \| + \| R(\lambda, A_n)y_n - R(\lambda, A)y \|
\leq \frac{M}{\text{Re}\lambda - \omega} \| y - y_n \| + \| x_n - x \| \rightarrow 0, \quad n \rightarrow \infty,$$

we conclude that (c) holds for all $\lambda \in \omega + \mathbb{C}_+$.

Next, let (c) hold for some $\lambda \in \omega + \mathbb{C}_+$. For $x \in \mathcal{D}$, define $y = \lambda x - Ax$ and $x_n = R(\lambda, A_n)y \in \mathcal{D}(A_n)$. It follows that $x_n \rightarrow x$ and

$$A_n x_n = \lambda R(\lambda, A_n)y - y \rightarrow \lambda R(\lambda, A)y - y = \lambda x - y = Ax$$

as $n \rightarrow \infty$, i.e., (b) holds.

Let (d) hold. Take $x \in X$ and any $\lambda \in \omega + \mathbb{C}_+$. Proposition 1.14 yields

$$\| R(\lambda, A)x - R(\lambda, A_n)x \| \leq \int_0^\infty \| e^{-\lambda t} (T(t)x - T_n(t)x) \| \, dt.$$
and moreover
\[\| (T_n(t) - T(t)) R(\lambda, A)y \| \leq \| T_n(t)(R(\lambda, A)y - R(\lambda, A_n)y) \| + \| R(\lambda, A_n)(T_n(t)y - T(t)y) \| + \| (R(\lambda, A_n) - R(\lambda, A))T(t)y \| =: d_{1,n}(y) + d_{2,n}(y) + d_{3,n}(y).\]

We have \(d_{1,n}(y) \leq M e^{\|\omega\|t_0} \| R(\lambda, A)y - R(\lambda, A_n)y \|.\) Consequently, \(d_{1,n}(y)\) tends to 0 uniformly for \(t \in [0, t_0]\) as \(n \to \infty.\) Since \(\{T(t)y; 0 \leq t \leq t_0\}\) is compact, the same holds for \(d_{3,n}(y)\) (this is a variant of the Banach-Steinhaus theorem). It remains to show this convergence for \(d_{2,n}(y).\)

As above we find an element \(v \in X\) such that \(\| y - R(\lambda, A)v \| \leq \epsilon\) and
\[d_{2,n}(y) \leq \frac{M}{\text{Re} \lambda - \omega} (2M e^{\|\omega\|t_0} \epsilon + d_{1,n}(R(\lambda, A)v) + d_{2,n}(R(\lambda, A)v) + d_{3,n}(R(\lambda, A)v)).\]

Again, one sees that \(d_{1,n}(R(\lambda, A)v)\) and \(d_{3,n}(R(\lambda, A)v)\) converge to 0 as \(n \to \infty\) uniformly for \(t \in [0, t_0].\) It remains to treat \(d_{2,n}(R(\lambda, A)v) = \| R(\lambda, A_n)(T_n(t) - T(t)) R(\lambda, A)v \|.\) For \(0 \leq s \leq t,\) we have
\[\frac{\partial}{\partial s} T_n(t - s) R(\lambda, A_n)T(s) R(\lambda, A)v = -T_n(t - s) A_n R(\lambda, A_n)T(s) R(\lambda, A)v + T_n(t - s) R(\lambda, A_n)T(s) A R(\lambda, A)v = T_n(t - s)(R(\lambda, A) - R(\lambda, A_n))T(s)v,\]
where we used that \(B R(\lambda, B) = \lambda R(\lambda, B) - I\) for \(B = A, A_n.\) Integrating over \(s \in [0, t],\) we deduce
\[d_{2,n}(R(\lambda, A)v) = \left\| T_n(t) R(\lambda, A_n)R(\lambda, A)v - R(\lambda, A_n)T(t) R(\lambda, A)v \right\|
= \left\| -\int_0^t T_n(t - s)(R(\lambda, A) - R(\lambda, A_n))T(s)v ds \right\|
\leq M e^{\|\omega\|t_0} t_0 \sup_{0 \leq s \leq t_0} \| (R(\lambda, A) - R(\lambda, A_n))T(s)v \|.\]

As before, the right side converges to 0 as \(n \to \infty\) uniformly for \(t \in [0, t_0]\) by compactness. Combining these estimates, we derive (d) with local uniform convergence. \( \square \)

**Example 3.14**

In the setting of Example 3.11, we see that the semigroup generated by \(a_n \Delta_D = A_n\) converges strongly on \(L^2(U)\) to the \(C_0\)-semigroup generated by \(A = a \Delta_D.\) Here we have \(\mathcal{D}(\Delta_D) = \mathcal{D} = \mathcal{D}(A) = \mathcal{D}(A_n), \\omega = 0\) and \(M = \delta^{-1}.\)

We now want to drop in Theorem 3.13 the assumption that \(A\) is a generator. For this we need a new tool whose relevance is explained by the subsequent lemma.
Definition 3.15
Let \( \emptyset \neq \Lambda \subseteq \mathbb{C} \). A set \( \{ R(\lambda); \lambda \in \Lambda \} \subseteq \mathcal{B}(X) \) is called pseudoresolvent if
\[
R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu)
\]
holds for all \( \lambda, \mu \in \Lambda \).

Lemma 3.16
Let \( R(\lambda, A_n) \) be resolvents satisfying \( \| R(\lambda, A_n) \| \leq \frac{M}{\operatorname{Re} \lambda - \omega} \) for all \( n \in \mathbb{N} \) and \( \lambda \in \omega + \mathbb{C}_+ =: \Lambda \) and some \( \omega \in \mathbb{R} \) and \( M > 0 \). If \( R(\lambda_0, A_n) \) converges strongly to an operator \( R(\lambda_0) \in \mathcal{B}(X) \) for some \( \lambda_0 \in \Lambda \), then all operators \( R(\lambda, A_n) \), \( \operatorname{Re} \lambda > \omega \), converge strongly as \( n \to \infty \) to a pseudoresolvent \( \{ R(\lambda); \lambda \in \Lambda \} \).

Proof. We only have to show the strong convergence for all \( \lambda \in \omega + \mathbb{C}_+ \) since the resolvent equation for \( R(\lambda, A_n) \) yields (3.8) by strong convergence. Let \( \operatorname{Re} \mu > \omega \). It holds
\[
R(\lambda, A_n) = \sum_{k=0}^{\infty} (\mu - \lambda)^k R(\mu, A_n)^{k+1}
\]
for all \( \lambda \in \omega + \mathbb{C}_+ \) such that \( |\mu - \lambda| \leq \frac{1}{2} \cdot \frac{\operatorname{Re} \mu - \omega}{M} \leq \frac{1}{2} \| R(\mu, A_n) \|^{-1} \). If \( R(\lambda, A_n) \) converges strongly as \( n \to \infty \), then also the partial sums of the above series converge strongly. The norm of the remainder terms \( \sum_{k=N}^{\infty} (\mu - \lambda)^k R(\mu, A_n)^{k+1} \) are bounded by \( c \sum_{k=N}^{\infty} 2^{-k} = c 2^{-N+1} \) which is less than any given \( \epsilon > 0 \) provided that \( N \) is sufficiently large. Thus \( R(\lambda, A_n) \) converges strongly as \( n \to \infty \) for \( \lambda \in B(\mu, \frac{1}{2M}(\operatorname{Re} \mu - \omega)) =: B_\mu \). Hence, the set
\[
\Gamma := \{ \lambda \in \omega + \mathbb{C}_+; R(\lambda, A_n) \text{ converges strongly as } n \to \infty \}
\]
is open (in \( \omega + \mathbb{C}_+ \)), and according to the assumption it is nonempty. Let \( \lambda_j \in \Gamma \) converge to some \( \lambda \in \omega + \mathbb{C}_+ \) and fix an \( \epsilon \in (0, \frac{1}{2M}(\operatorname{Re} \lambda - \omega)) \). For sufficiently large values of \( j \), we then have \( \frac{1}{2M}(\operatorname{Re} \lambda_j - \omega) > \epsilon > |\lambda - \lambda_j| \). This implies for such values of \( j \) that \( \lambda \in B_{\lambda_j} \), which yields \( \lambda \in \Gamma \) as seen above. Consequently, \( \Gamma \neq \emptyset \) is open and closed in \( \omega + \mathbb{C}_+ \). Since \( \omega + \mathbb{C}_+ \) is apparently connected, this is only possible if \( \omega + \mathbb{C}_+ \) and \( \Gamma \) coincide. (Otherwise \( \Gamma \) and \( (\omega + \mathbb{C}_+) \setminus \Gamma \) would be a nontrivial, disjoint, open covering of \( \omega + \mathbb{C}_+ \).)

We note that in Lemma 3.16 the operator \( R(\lambda) \) does not need to be a resolvent. Consider, e.g., \( A_n = -nI \) satisfying \( \| e^{tA_n} \| = e^{-nt} \leq 1 \) for all \( t \geq 0 \). and \( R(\lambda, A_n) = \frac{1}{\lambda + n} I \to 0 = R(\lambda) \) as \( n \to \infty \) for all \( \operatorname{Re} \lambda > 0 \).

Lemma 3.17
Let \( \{ R(\lambda); \lambda \in \Lambda \} \) be a pseudoresolvent. For all \( \lambda, \mu \in \Lambda \), the following assertions hold.

(a) \( R(\lambda) R(\mu) = R(\mu) R(\lambda) \).
(b) \( N(R(\lambda)) = N(R(\mu)) \).
(c) \( R(\lambda) X = R(\mu) X \).

Proof. Assertion (a) directly follows from the definition (3.8) which further yields
\[
R(\lambda) = R(\mu)(I + (\mu - \lambda) R(\lambda)) = (I + (\mu - \lambda) R(\lambda)) R(\mu).
\]
As a result, it holds $R(\lambda)X \subseteq R(\mu)X$ and $N(R(\mu)) \subseteq N(R(\lambda))$. Interchanging the roles of $\lambda$ and $\mu$, we get the converse inclusions. 

\begin{lemma}
Let $\{R(\lambda); \lambda \in \Lambda\}$ be a pseudoresolvent.
\begin{itemize}
\item[(a)] If $R(\lambda_0)$ is injective with dense range for some $\lambda_0 \in \Lambda$, then there is a closed operator $A$ with dense domain $\mathcal{D}(A) = R(\lambda_0)X$ such that $\Lambda \subseteq \rho(A)$ and $R(\lambda) = R(\lambda, A)$ for all $\lambda \in \Lambda$.
\item[(b)] If $R(\mu)$ has dense range for some $\mu \in \Lambda$ and if there are $\lambda_j \in \Lambda$ with $|\lambda_j| \to \infty$ as $j \to \infty$ such that $\|\lambda_j R(\lambda_j)\| \leq M$ for all $j \in \mathbb{N}$ and some constant $M > 0$, then $R(\lambda)$ is injective for all $\lambda \in \Lambda$.
\end{itemize}
\end{lemma}

\begin{proof}
(a) By the assumption we can define a closed operator $A = \lambda_0I - R(\lambda_0)^{-1}$ with dense domain $\mathcal{D}(A) = R(\lambda_0)X$. It then holds

$$(\lambda_0 I - A)R(\lambda_0) = R(\lambda_0)^{-1}R(\lambda_0) = I$$

as well as

$$R(\lambda_0)(\lambda_0 x - Ax) = R(\lambda_0)R(\lambda_0)^{-1}x = x$$

for all $x \in \mathcal{D}(A)$ so that $\lambda_0 \in \rho(A)$ and $R(\lambda_0) = R(\lambda_0, A)$. Lemma 3.17 shows that $R(\lambda)X = \mathcal{D}(A)$ for all $\lambda \in \Lambda$. Using the definition (3.8), we further compute

$$(\lambda I - A)R(\lambda) = [(\lambda - \lambda_0)I + (\lambda_0 I - A)]R(\lambda_0)[I - (\lambda - \lambda_0)R(\lambda)]$$

$$= I + (\lambda - \lambda_0)(R(\lambda_0)[I - (\lambda - \lambda_0)R(\lambda)] - R(\lambda))$$

$$= I.$$

Similarly one shows that $R(\lambda)(\lambda x - Ax) = x$ for each $x \in \mathcal{D}(A)$ and so assertion (a) is shown.

(b) We first note that (3.8) and the assumptions yield

$$\|(\lambda_j R(\lambda_j) - I)R(\mu)\| = \left\|\lambda_j R(\lambda_j) - \mu - \lambda_j R(\mu)\right\| \leq \left\|\lambda_j R(\lambda_j) - \mu - \lambda_j R(\mu)\right\|$$

$$\leq \frac{1}{|\mu - \lambda_j|}(M + \|R(\mu)\|) \to 0$$

as $j \to \infty$, where $\lambda_j \neq \mu$ for all sufficiently large $j$. Since $R(\mu)X$ is dense and the operators $\lambda_j R(\lambda_j)$ are uniformly bounded, it follows that $\lambda_j R(\lambda_j)x \to x$ as $j \to \infty$ for all $x \in X$. Now, if $R(\lambda)x = 0$ for some $x \in X$ and $\lambda \in \Lambda$, then Lemma 3.17 shows that $0 = R(\lambda_j)x \to x$ (as $j \to \infty$) and hence $x = 0$. 

\end{proof}

\begin{theorem}[Trotter-Kato II]
Let $A_n$ generate $C_0$-semigroups $T_n(t)$ such that $\|T_n(t)\| \leq M e^{\omega t}$ for all $t \geq 0$ and $n \in \mathbb{N}$ and some constants $M \geq 1$ and $\omega \in \mathbb{R}$. It then holds (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) among the following assertions.

\begin{itemize}
\item[(a)] There exists a densely defined operator $A$ such that $\mathcal{D}(A) \subseteq \mathcal{D}(A_n)$ for all $n \in \mathbb{N}$ and $A_n x \to Ax$ as $n \to \infty$ for all $x \in \mathcal{D}(A)$ and $(\lambda_0 I - A)\mathcal{D}(A)$ is dense in $X$ for some $\lambda_0 \in \omega + \mathbb{C}_+$.
\end{itemize}

\end{theorem}
(b) There is a \( \lambda_0 \in \omega + \mathbb{C}_+ \) such that \( R(\lambda_0, A_n) \) converges strongly to an operator \( R \in \mathcal{B}(X) \) with dense range.

(c) There is a \( C_0 \)-semigroup \( T(\cdot) \) with generator \( B \) such that \( T_n(t) \) converges strongly to \( T(t) \) for all \( t \geq 0 \) as \( n \to \infty \).

If (b) holds, then \( R = R(\lambda_0, B) \). If (a) holds, then \( B = \overline{A} \). Observe that one can apply Theorem 3.13 to the semigroups \( T_n(\cdot) \) and \( T(\cdot) \) if (a), (b) or (c) holds.

**Proof.** The implication “(c)⇒(b)” is a consequence of Theorem 3.13 with \( R = R(\lambda, B) \), since \( \|T(t)\| \leq M e^{\omega t} \) follows from the assumptions.

Let (a) hold. Take any \( x \in \mathfrak{D}(A) \) and set \( y = \lambda_0 x - Ax \). Using the uniform bound on the semigroup, we obtain that
\[
\| R(\lambda_0, A_n)y - x \| = \| R(\lambda_0, A_n)((\lambda_0 x - Ax) - (\lambda_0 I - A_n)x) \| \leq \frac{M}{\text{Re} \lambda_0 - \omega} \| Ax - A_n x \| \to 0
\]
as \( n \to \infty \). Since \( (\lambda I - A) \mathfrak{D}(A) \) is dense and \( R(\lambda_0, A_n) \) is uniformly bounded, it follows that \( R(\lambda_0, A_n) \) converges strongly to some \( R \in \mathcal{B}(X) \) whose range contains the dense set \( \mathfrak{D}(A) \). So (b) is true.

Let (b) hold. Lemma 3.16 shows that \( R(\lambda, A_n) \) converge strongly to a pseudoresolvent \( \{R(\lambda); \lambda \in \omega + \mathbb{C}_+\} \) as \( n \to \infty \). By the assumption, \( R(\lambda_0) \) has dense range. Moreover, \( (\lambda - \omega)^k R(\lambda, A_n)^k \) converges to \( (\lambda - \omega)^k R(\lambda)^k \) strongly for all \( k \in \mathbb{N} \) and all \( \lambda \in \omega + \mathbb{C}_+ \) as \( n \to \infty \). Thus \( R(\lambda) \) satisfies the Hille-Yosida condition. So Lemma 3.18 implies that \( R(\lambda) = R(\lambda, B) \) for a closed, densely defined operator \( B \) which generates a \( C_0 \)-semigroup \( T(\cdot) \) due to the Hille-Yosida theorem. Since \( R(\lambda_0, A_n) \to R(\lambda_0, B) \), Theorem 3.13 yields that \( T_n(t) \) tends to \( T(t) \) strongly for all \( t \geq 0 \) as \( n \to \infty \).

Finally, we have to show that \( B = \overline{A} \) if (a) holds. For this purpose take some \( x \in \mathfrak{D}(A) \) and note that we have
\[
x = \lim_{n \to \infty} R(\lambda_0, A_n)(\lambda_0 x - A_n x) = R(\lambda_0, B)(\lambda_0 x - Ax),
\]
which yields that \( Ax = Bx \), i.e., \( A \subseteq B \) and so \( A \) is closable with \( \overline{A} \subseteq B \). Observe that \( (\lambda_0 I - \overline{A}) \mathfrak{D}(\overline{A}) \) contains \( (\lambda_0 I - A) \mathfrak{D}(A) \) and it is thus dense in \( X \). We further obtain \( x = R(\lambda_0, B)(\lambda_0 x - \overline{A} x) \) for all \( x \in \mathfrak{D}(\overline{A}) \) by approximation so that \( \|x\| \leq \|R(\lambda_0, B)\| \cdot \|\lambda_0 x - \overline{A} x\| \).

This estimate implies that \( (\lambda_0 I - \overline{A}) \mathfrak{D}(\overline{A}) \) is closed and hence \( \lambda_0 I - \overline{A} \) is surjective. Because of \( \lambda_0 \in \rho(B) \), it follows \( \overline{A} = B \) from Lemma 1.17.

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### 3.3. The Lax-Chernoff product formula

Based on the Trotter-Kato theorem, we now discuss further approximation results for \( C_0 \)-semigroups.

**Lemma 3.20**

Let \( S \in \mathcal{B}(X) \) satisfy \( \|S^n\| \leq M \) for all \( n \in \mathbb{N} \) and some \( M > 0 \). It then holds
\[
\|e^{n(S-I)} x - S^n x\| \leq M \sqrt{n} \|Sx - x\|
\]
for all $x \in X$ and $n \in \mathbb{N}$.

**Proof.** For $n \in \mathbb{N}$, it holds

$$e^{n(S-I)} - S^n = e^{-n} \sum_{j=0}^{\infty} \frac{n^j}{j!} S^j - \sum_{j=0}^{\infty} \frac{n^j}{j!} e^{-n} S^n = e^{-n} \sum_{j=0}^{\infty} \frac{n^j}{j!} (S^j - S^n).$$

For $m > l$ and $x \in X$, we also have

$$\|S^m x - S^l x\| = \left\| \sum_{j=l+1}^{m-1} S^j(S-I)x \right\| \leq M(m-l)\|Sx - x\|.$$

We then estimate

$$\|e^{n(S-I)} x - S^n x\| \leq M e^{-n} \|Sx - x\| \sum_{j=0}^{\infty} \frac{n^j}{j!} \sqrt{j!} |n-j|$$

$$\leq M e^{-n} \|Sx - x\| \left( \sum_{j=0}^{\infty} \frac{n^j}{j!} \right)^{\frac{1}{2}} \left( \sum_{j=0}^{\infty} \frac{n^j}{j!} (n-j)^2 \right)^{\frac{1}{2}}$$

$$\leq M e^{-n} \|Sx - x\| e^2 \sqrt{n} e^2 = M \sqrt{n} \|Sx - x\|,$$

where we computed the last series in an elementary way. □

**Theorem 3.21** (Lax-Richtmyer 1957, Chernoff 1974)

Let $V : \mathbb{R}_+ \to \mathcal{B}(X)$ be a function such that $V(0) = I$ and $\|V(t)\|^2 \leq M e^{\omega t}$ for all $t \geq 0$ and $k \in \mathbb{N}$ and some $\omega \in \mathbb{R}$ and $M \geq 1$. Let $\mathcal{D}(A)$ be a dense subspace and $\lambda \in \omega + \mathbb{C}_+$ such that the limit $Ax := \lim_{t \to 0} \frac{1}{t}(V(t)x - x)$ exists for all $x \in \mathcal{D}(A)$ and such that $(\lambda I - A)\mathcal{D}(A)$ is also dense in $X$. Then $A$ is closable and $\overline{A}$ generates a $C_0$-semigroup $T(\cdot)$ such that $V\left(\frac{t}{n}\right)x$ converges to $T(t)x$ uniformly on compact subsets of $\mathbb{R}_+$ for all $x \in X$.

**Proof.** By rescaling, we may assume that $\omega = 0$. For $s > 0$ we define $A_s = \frac{1}{s}(V(s) - I) \in \mathcal{B}(X)$. We thus obtain that $A_s x \to Ax$ for all $x \in \mathcal{D}(A)$ as $s \to 0$ and that

$$\|e^{tA_s}\| = e^{-\frac{s}{n}} \|e^{\frac{1}{s}V(s)}\| \leq e^{-\frac{s}{n}} \sum_{k=0}^{\infty} \frac{t^k}{k!} \|V(s)^k\| \leq e^{-\frac{s}{n}} e^2 M = M$$

for all $t \geq 0$. Hence, Theorem 3.19 and Theorem 3.13 now show that $\overline{A}$ exists and generates a $C_0$-semigroup $T(\cdot)$ such that $e^{tA/n} x \to T(t)x$ as $n \to \infty$ uniformly for $t \in [0,t_0]$ for all $x \in X$, any null sequence $(s_n)_n$ and any $t_0 > 0$. Taking $s_n = \frac{t}{n}$, we see that $e^{tA/n}$ converges strongly to $T(t)$ uniformly for $t \in [0,t_0]$, as $n \to \infty$. On the other hand, Lemma 3.20 yields

$$\|e^{tA/n} x - V(t/n)^n x\| = \|e^{n(V(t/n) - I)} x - V(t/n)^n x\| \leq M \sqrt{n} \|V(t/n)x - x\|$$

$$= \frac{tM \sqrt{n}}{\sqrt{n}} \|A_{t/n}x\| \leq \frac{t_0 M}{\sqrt{n}} \sup_{0 \leq s \leq t_0} \|A_s x\| \to 0$$

as $n \to \infty$ for all $t \in [0,t_0]$ and all $x \in X$. □
We add two special cases of the above general approximation result.

**Corollary 3.22** (Lie-Trotter product formula, Trotter 1959, Chernoff 1974)  
Assume that $A$ and $B$ generate $C_0$-semigroups $T(\cdot)$ and $S(\cdot)$ such that
\[
\left\| \left( T \left( \frac{t}{n} \right) S \left( \frac{t}{n} \right) \right)^n \right\| \leq M e^{\omega t}
\]
holds for all $n \in \mathbb{N}$ and $t \geq 0$ and some $M \geq 1$ and $\omega \in \mathbb{R}$. Let $D = D(A) \cap D(B)$ and $(\lambda I - (A + B))D$ be dense in $X$ for some $\lambda \in \omega + \mathbb{C}_+$. Define $C := A + B$ with $D(C) := D$. Then the closure $\overline{C}$ exists and generates a $C_0$-semigroup $U(\cdot)$ satisfying
\[
U(t)x = \lim_{n \to \infty} \left( T \left( \frac{t}{n} \right) S \left( \frac{t}{n} \right) \right)^n x
\]
uniformly on all compact subsets of $\mathbb{R}_+$ and for all $x \in X$.

**Proof.** Define $V(t) = T(t)S(t)$ for $t \geq 0$. For $x \in D$, the vectors
\[
\frac{1}{t} (V(t)x - x) = T(t) \frac{1}{t} (S(t)x - x) + \frac{1}{t} (T(t)x - x)
\]
converge to $Bx + Ax$ as $t \to 0^+$. The result can now be deduced from Theorem 3.21. \qed

**Remark**  
Observe that the stability condition (3.9) holds if both semigroups are contractive. In general, one cannot find an equivalent norm for which both semigroups become contractive (cf. Remark 1.21). In fact, there are generators $A$ and $B$ such that $A + B$ exists and generates a $C_0$-semigroup, but (3.9) is violated, and thus the Lie-Trotter product formula fails, see [K-W].

Several variants of the product formula are used in numerical mathematics as “splitting methods”, see, e.g., the works by Hochbruck, Jahnke or Lubich, see also §3.6 in [P].

The Lie-Trotter formula can be used to give an alternative proof of the positivity assertion in Example 3.5. It also yields a rigorous mathematical interpretation for the “Feynman path integral formula” in quantum mechanics for the Schrödinger group $e^{it(\Delta - V)}$, see §8.13 in [G].

**Corollary 3.23** (resolvent approximation)  
Let $A$ generate the $C_0$-semigroup $T(\cdot)$. We then have
\[
T(t)x = \lim_{n \to \infty} \left( \frac{n}{t} R \left( \frac{n}{t}, A \right) \right)^n x = \lim_{n \to \infty} \left( I - \frac{t}{n} A \right)^{-n} x
\]
uniformly on all compact subsets of $\mathbb{R}_+$ and for all $x \in X$.

**Proof.** Take $M, \omega > 0$ such that $\|T(t)\| \leq M e^{\omega t}$ holds for all $t \geq 0$. Let $\delta \in (0, \frac{1}{\omega} - \frac{1}{\omega + 1})$. We then define
\[
V(t) = \begin{cases} 
I, & \text{if } t = 0, \\
\frac{1}{t} R \left( \frac{1}{t}, A \right), & \text{if } 0 < t \leq \delta, \\
0, & \text{if } t > \delta.
\end{cases}
\]
The Hille-Yosida estimate yields
\[ \|V(t)^n\| = t^{-n}\|R\left(\frac{1}{t}, A\right)^n\| \leq \frac{M}{t^n(t-1-\omega)^n} = \frac{M}{(1-\omega t)^n} = M e^{-n\log(1-\omega t)} \leq M e^{n(1+\omega)t} \]
for all \(0 < t \leq \delta < \frac{1}{\omega}\) by our choice of \(\delta\). Further,
\[ \frac{1}{t}(V(t)x - x) = \frac{1}{t} \left( \frac{1}{t} R \left( \frac{1}{t}, A \right) x - x \right) = \frac{1}{t} R \left( \frac{1}{t}, A \right) Ax \to Ax \]
as \(t \to 0\) for all \(x \in D(A)\). So the result follows from Theorem 3.21. \(\square\)

**Remark 3.24**
By Proposition 1.14, the resolvent of the generator is the Laplace transform
\[ \mathcal{L}(T(\cdot)x)(\lambda) = \int_0^\infty e^{-\lambda t} T(t)x \, dt = R(\lambda, A)x, \quad \text{Re} \lambda > \omega_0(A), \]
of the semigroup. So Corollary 3.23 is related to “real inversion formulae” of the Laplace transform (see the monograph [A-B-H-N]). There are also complex formulae such as
\[ T(t)x = \frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{\omega - i\infty}^{\omega + i\infty} e^{\lambda t} R(\lambda, A)x \, d\lambda = \frac{1}{2\pi i} \int_{\omega + i\infty}^{\omega - i\infty} e^{\lambda t} R(\lambda, A)^2x \, d\lambda \]
for \(t > 0, \omega > \omega_0(A)\) and \(x \in D(A^2)\), see Corollary III.5.16 in [E-N].

**Corollary 3.25**
Let \(\emptyset \neq U \subseteq \mathbb{R}^d\) be open and \(A\) generate a \(C_0\)-semigroup \(T(\cdot)\) on \(E = C_0(U)\) or \(E = L^p(U)\), \(1 \leq p < \infty\). Then, \(T(t)\) is positive for all \(t \geq 0\) if and only if there is an \(\omega > \omega_0(A)\) such that \(R(\lambda, A)\) is positive for all \(\lambda \geq \omega\).

**Proof.** Let the resolvent be positive and \(t > 0\). For all \(0 \leq f \in E\) and large \(n \in \mathbb{N}\), the functions \(\left(\frac{t}{n} R\left(\frac{n}{t}, A\right)\right)^n f\) are positive and hence their limit \(T(t)f\) is positive, too. For \(\lambda > \omega_0(A)\), the converse follows in a similar way from the formula
\[ R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t)f \, dt \]
shown in Proposition 1.14. \(\square\)

**Example 3.26**
Let \(\emptyset \neq U \subseteq \mathbb{R}^d\) be open and bounded with \(C^2\) boundary, \(E_\infty = C_0(U)\) and \(E_p = L^p(U)\) for \(1 < p < \infty\), and \(A_p = \Delta\) for \(p \in (1, \infty)\) with \(D(A_\infty) = \{ u \in \bigcap_{p>1} W^2_p(U); u, \Delta u \in E_\infty \}\) on \(E_\infty\) and \(D(A_p) = W^2_p(U) \cap W^1_p(U)\) on \(E_p\) for \(1 < p < \infty\). These operators generate (analytic) \(C_0\)-semigroups \(T_p(\cdot)\) on \(E_p\), see Examples 1.37, 2.16 and 2.19. Let \(\lambda > 0\) and \(0 \leq f \in C_0(U) \subseteq E_p\) for all \(1 < p \leq \infty\). Set \(u = R(\lambda, A_\infty)f \in D(A_\infty)\). Observe that \(u \in D(A_p)\) for all \(p \in (1, \infty)\)
and \( \lambda u - \Delta u = f \) on \( U \) so that \( u = R(\lambda, A_p)f \) for all \( p \in (1, \infty] \). Clearly, \( v = \text{Im} \ u \) belongs to \( \mathcal{D}(A_{\infty}) \) and \( \lambda v - \Delta v = \text{Im} \ f = 0 \) so that \( v = 0 \) and \( u \) is real-valued. Suppose there is an \( x_0 \in U \) such that \( u(x_0) < 0 \). Since \( u = 0 \) on \( \partial U \), \( u \) has a minimum \( u(x_1) < 0 \) for some \( x_1 \in U \). Then, \( \Delta u(x_1) \geq 0 \) by Proposition 3.1.10 in [L], implying \( f(x_1) = \lambda u(x_1) - \Delta u(x_1) < 0 \) which is impossible. Hence, \( u = R(\lambda, A_p)f \geq 0 \). Since \( C_0(U) \) is dense in \( E_p \), we obtain that \( R(\lambda, A_p) \geq 0 \) and thus \( T_p(t) \) is positive for all \( t \geq 0 \) and \( p \in (1, \infty] \) by Corollary 3.25.

There is a related characterization of positive contraction semigroups by Phillips, see Theorem C-II-1.2 in [N-ed].
CHAPTER 4

Asymptotic behaviour

4.1. Exponential stability and dichotomy

Definition 4.1
A $C_0$-semigroup $T(\cdot)$ is called (uniformly) exponentially stable if there exist constants $M, \epsilon > 0$ such that

$$\|T(t)\| \leq M e^{-\epsilon t} \quad \text{for all } t \geq 0$$

(or equivalently $\omega_0(T) < 0$ or equivalently $\|T(t)x\| \leq M e^{-\epsilon t} \|x\|$ for all $x \in X$ and $t \geq 0$).

Let $A$ generate the $C_0$-semigroup $T(\cdot)$ and $\epsilon > 0$. Observe that we have $\|T(t)\| \leq e^{-\epsilon t}$ for all $t \geq 0$ if and only if $A - \epsilon I$ is dissipative by the Lumer-Phillips theorem and rescaling.

Recall that for an operator $T \in B(X)$ we have

$$r(T) := \max\{|\lambda|; \lambda \in \sigma(T)\} = \lim_{n \to \infty} \frac{1}{n} \log \|T^n\| = \inf_{n \in \mathbb{N}_0} \frac{1}{n} \|T^n\| \leq \|T\|,$$

see, e.g., Theorem 1.16 in [ST].

Proposition 4.2
Let $T(\cdot)$ be a $C_0$-semigroup with generator $A$. Then the following assertions are equivalent.

(a) The semigroup $T(\cdot)$ is exponentially stable.

(b) We have $\|T(t_0)\| < 1$ for some $t_0 > 0$.

(c) We have $r(T(t_1)) < 1$ for some $t_1 > 0$.

(d) It holds that $\omega_0(A) < 0$.

If this is the case, then (b) is valid for all sufficiently large $t_0 > 0$, assertion (c) is true for all $t_1 > 0$ and we have $s(A) < 0$. It further holds

$$e^{ts(A)} \leq e^{t\omega_0(A)} = r(T(t))$$

for all $t > 0$ and

$$\omega_0(A) = \lim_{t \to \infty} \frac{1}{t} \log \|T(t)\| = \inf_{t > 0} \frac{1}{t} \log \|T(t)\|$$

(with $\log(0) := -\infty$).

PROOF. Since $\log \|T(t+s)\| \leq \log \|T(t)\| + \log \|T(s)\|$, the elementary Lemma IV.2.3 in [E-N] shows that the limit $\lim_{t \to \infty} \frac{1}{t} \log \|T(t)\|$ exists and equals $\omega := \inf_{t > 0} \frac{1}{t} \log \|T(t)\|$. Hence, $e^{\omega \omega} \leq \|T(t)\|$ for all $t \geq 0$ and $\omega \leq \omega_0(A)$. Take any $\omega_1 > \omega$. Then there is a $t_0 \geq 0$ such that
\[ \|T(t)\| \leq e^{\omega_1 t} \text{ for all } t \geq t_0 \text{ so that } \|T(t)\| \leq M e^{\omega_1 t} \text{ for all } t \geq 0 \text{ and } M := \max\{e^{\omega_1 t}\|T(t)\|; 0 \leq t \leq t_0\} \geq 1. \] 

This means that \( \omega_1 \geq \omega_0(A) \) and so \( \omega = \omega_0(A) \). Using (4.1), we infer

\[ r(T(t)) = \lim_{n \to \infty} \exp \left( \frac{1}{nt} \log \|T(nt)\| \right) = \exp \left( t \lim_{n \to \infty} \frac{1}{nt} \log(\|T(nt)\|) \right) = e^{\lambda_0(A)}, \]

for all \( t > 0 \). All other assertions about \( T(\cdot) \) now follow. Proposition 1.14 says that \( s(A) \leq \omega_0(A) \), which yields the remaining inequality \( e^{ts(A)} \leq e^{\omega_0(A)} \).

If \( X = \mathbb{C}^d \), a theorem by Lyapunov says that \( s(A) = \omega_0(A) \) so that \( T(\cdot) \) is exponentially stable if and only if \( s(A)<0 \). This equivalence fails in infinite-dimensional spaces as the next example shows.

**Example 4.3** (Greiner-Voigt-Wolff, 1981)

Let \( X = C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^s \, ds) \) be endowed with the norm

\[ \|f\| = \|f\|_\infty + \int_0^\infty |f(s)| e^s \, ds =: \|f\|_\infty + \|f\|_1. \]

Define \( T(t)f = f(\cdot + t) \) for \( t \geq 0 \) and \( f \in X \). Observe that

\[ \|T(t)f\| = \sup_{s \geq 0} |f(s + t)| + \int_0^\infty |f(s + t)| e^s \, ds \leq \|f\|_\infty + e^{-t} \int_t^\infty |f(\tau)| e^\tau \, d\tau \]

\[ \leq \|f\|_\infty + e^{-t} \|f\|_1 \leq \|f\|. \]

Hence, \( T(t) \) is a contraction on \( X \). It is easy to see that \( T(\cdot) \) is a semigroup on \( X \) and strongly continuous on \( C_c(\mathbb{R}_+) \) for \( \|\cdot\| \), and that \( C_c(\mathbb{R}_+) \) is dense in \( X \). As a result, \( T(\cdot) \) is a positive contraction semigroup on \( X \) with generator \( A \). Given \( t > 0 \) and \( q \in (0,1) \), take \( f \in X \) with \( \|f\| = 1 \) and \( \|f\|_\infty = |f(t)| = q \). Then \( \|T(t)f\| \geq \|T(t)f\| \geq |T(t)f(0)| = |f(t)| = q \). So we obtain that \( \|T(t)f\| = 1 \) and \( \omega_0(A) = 0 \).

We want to compute \( s(A) \). Let \( \Re \lambda < -1 \). We then have \( e_\lambda \in X \) and \( T(t)e_\lambda = e^{\lambda t}e_\lambda \), for all \( t \geq 0 \) so that \( \frac{1}{2}(T(t)e_\lambda - e_\lambda) \) tends to \( \lambda e_\lambda \) as \( t \to 0^+ \). Hence, \( e_\lambda \in D(A) \) with \( A e_\lambda = \lambda e_\lambda \) and \( \{ \lambda \in \mathbb{C}; \Re \lambda \leq -1\} \subseteq \sigma(A) \) since \( \sigma(A) \) is closed. Note that \( s(A) \leq \omega_0(A) = 0 \). Next, let \( \Re \lambda \in (-1,0] \) and \( f \in X \). The above estimate shows that \( \|T(t)f\|_1 \leq e^{-t} \|f\|_1 \) for all \( t \geq 0 \) so that the integrals \( \int_0^b e^{-\lambda t} T(t)f \, dt =: J(b) \) converge in \( L^1(\mathbb{R}_+, e^s \, ds) \) as \( b \to \infty \). On the other hand, for \( b' > b \geq 0 \) we have

\[ \left\| \int_b^{b'} e^{-\lambda t} T(t)f \, dt \right\| \leq \sup_{s \geq 0} \int_b^{b'} e^{-\lambda(s + t)} |f(s + t)| \, dt = \sup_{s \geq 0} \int_{b + s}^{b'} e^{\Re \lambda s} e^{-\Re \lambda \tau} |f(\tau)| \, d\tau \]

\[ \leq \int_b^\infty e^\tau |f(\tau)| \, d\tau. \]

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As a consequence, \( J(b) \) also converges in \( C_0(\mathbb{R}_+) \) as \( b \to \infty \). Proposition 1.14 now shows that \( \lambda \in \rho(A) \), and hence \( s(A) = -1 < \omega_0(A) = 0 \).

We note that the above example can be modified such that \( s(A) = -\infty \) (see Exercise IV.2.13 (5) in [E-N]). There are analogous examples on Hilbert spaces using growing Jordan blocks on \( X = \ell^2 \) (see [Z]) or a perturbed wave equation (see [R]). However, in infinite dimensions it is often more appropriate to complement spectral conditions by resolvent estimates. To establish a corresponding stability theorem, we need some preparations on the Bochner integral.

Let \( X \) be a Banach space and \( J \subseteq \mathbb{R} \) be an interval. Simple functions \( f : J \to X \) and their integral are defined as in the case \( X = \mathbb{R} \). A function \( f : J \to X \) is called strongly measurable if there are simple functions \( f_n : J \to X \) converging to \( f \) pointwise almost everywhere. Observe that then the function \( t \mapsto \| f(t) \| \) is measurable. Hence, we can define

\[
L^p(J, X) := \{ f : J \to X; f \text{ is strongly measurable and } \| f(\cdot) \|_X \in L^p(J) \}
\]

for \( p \in [1, \infty) \), where we identify functions that coincide almost everywhere. A theorem by Bochner says that \( f \in L^1(J, X) \) if and only if there are simple functions converging to \( f \) pointwise almost everywhere such that the sequence \( (f_n) \) is a Cauchy sequence for \( \| \cdot \| \). This fact implies that the integrals \( \int_J f_n(t) \, dt \) converge in \( X \) and that their limit is independent of the choice of such a sequence \( (f_n) \). This limit is denoted by \( \int_J f(t) \, dt \) and called the (Bochner) integral of \( f \). It can be shown that \( L^p(J, X) \) is a Banach space and the analogues of H"older's inequality and the theorems of Riesz-Fischer, Lebesgue and Fubini hold for the Bochner integral, see, e.g., [A-E III]. For \( f \in L^p(\mathbb{R}, X) \) we define the Fourier transform

\[
\hat{f}(\tau) = \mathcal{F}f(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\tau t} f(t) \, dt, \quad \tau \in \mathbb{R}.
\]

As in the scalar case one shows that \( \hat{f} \in C_0(\mathbb{R}, X) \). If \( X \) is Hilbert space, then it can be shown as in the case \( X = \mathbb{C} \) that \( \mathcal{F} \) on \( L^1(\mathbb{R}, X) \cap L^2(\mathbb{R}, X) \) extends to a unitary operator

\[
\mathcal{F} : L^2(\mathbb{R}, X) \to L^2(\mathbb{R}, X)
\]

where \( L^2(\mathbb{R}, X) \) is a Hilbert space with the inner product

\[
(f|g) := \int_{\mathbb{R}} (f(t)|g(t))_X \, dt, \quad f, g \in L^2(\mathbb{R}, X),
\]

see Theorem C.14 in [E-N].

Theorem 4.4 (Gearhart’s stability theorem)
Let $X$ be a Hilbert space. A $C_0$-semigroup $T(\cdot)$ with generator $A$ is exponentially stable if and only if $s(A) < 0$ and $\sup_{\lambda \in \mathbb{C}_+} \| R(\lambda, A) \| < \infty$.

In a general Banach space $X$ the boundedness of the resolvent $R(\cdot, A)$ on $\mathbb{C}_+$ only implies the existence of some constants $M, \epsilon > 0$ such that we have
\[ \| T(t)x \| \leq M e^{-\epsilon t} \| x \|_A \]
for all $t \geq 0$ and $x \in \mathcal{D}(A)$. See [WW] for this and related results and examples indicating their optimality.

**Proof.** The necessity of the condition was shown in Proposition 1.14 for a general Banach space $X$. Assume that the resolvent exists and is bounded on $\mathbb{C}_+$. Fix $\omega > \max \{ \omega_0(A), 0 \}$ and take $\omega \geq \omega$. We set $T_\omega(t) = e^{-\omega t} T(t)$ for $t \geq 0$ and $T_\omega(t) = 0$ for $t < 0$. Then $u = T_\omega(\cdot)x$ belongs to $L^2(\mathbb{R}, X) \cap L^1(\mathbb{R}, X)$ for all $x \in X$ and $\| u \|_2 \leq c_0 \| x \|$ for some $c_0 > 0$. Since
\[ (\mathcal{F}u)(\tau) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\tau t} e^{-\omega t} T(t)x \, dt = \frac{1}{\sqrt{2\pi}} R(\omega + i\tau, A)x, \quad \tau \in \mathbb{R}. \]
Plancherel’s theorem implies that $\tau \mapsto R(\omega + i\tau, A)x$ belongs to $L^2(\mathbb{R}, \mathbb{X})$ with norm less than or equal to $\sqrt{2\pi}c_0 \| x \|$ for all $\omega \geq \omega$. The resolvent equation further yields that
\[ r_\lambda(\tau) := R(\lambda + i\tau, A)x = R(\omega + i\tau, A)x + (\omega - \lambda) R(\lambda + i\tau, A) R(\omega + i\tau, A)x \]
for all $\lambda \in \mathbb{C}_+$ and $\tau \in \mathbb{R}$. The assumption thus yields that $\| r_\lambda \|_2 \leq \tau(1 + |\lambda - \omega|) \| x \|$ for all $\lambda \in \mathbb{C}_+$ and $x \in X$ and some $\tau > 0$. One can further see that $\frac{d}{d\lambda} r_\lambda = -R(\lambda + i\tau, A)^2 x$ exists in $L^2(\mathbb{R}, X)$ for all $\lambda \in \mathbb{C}_+$. For any fixed $g \in L^\infty(\mathbb{R}, X)$ with compact support the functional
\[ \Phi(f) = \int_\mathbb{R} (f(t)|g(t)) \, dt \]
belongs to $L^2(\mathbb{R}, X^*)$ and $\| \Phi \| \leq \| g \|_2$. Observe that the scalar functions $\omega \mapsto \Phi(\sqrt{2\pi} T_\omega(\cdot)x)$ and $\omega \mapsto \Phi(\mathcal{F}^{-1} r_\omega)$ for $\omega \geq \omega$ have holomorphic extensions to $\mathbb{C}_+$. (Use that $\Phi(T_\omega(\cdot)x) = \int_0^b e^{-\omega t}(T(t)x)g(t)) \, dt$, where supp $g \subseteq [0, b]$.) Since these two functions coincide for $\omega \geq \omega$, they coincide on $\mathbb{C}_+$ by the identity theorem. We now deduce that
\[ \sqrt{2\pi} \int_0^\infty e^{-at}(T(t)x)g(t)) \, dt = \| \Phi(\sqrt{2\pi} T_\omega(\cdot)x) \| = \| \Phi(\mathcal{F}^{-1} r_\omega) \| \leq \| \Phi \| \cdot \| \mathcal{F}^{-1} r_\omega \|_2 \leq c_1 \| g \|_2 \cdot \| x \| \]
for all $a \in (0, 1]$ and $x \in X$ and the constant $c_1 := \tau(1 + \omega)$. Let $x \in X$. For each $n \in \mathbb{N}$, we define $g_n(t) = T(t)x$ for $0 \leq t \leq n$ and $g_n(t) = 0$ otherwise. Inserting $g_n$ in the above estimate, we infer
\[ \int_0^n e^{-at} \| T(t)x \|^2 \, dt = \int_0^\infty e^{-at}(T(t)x)g_n(t)) \, dt \leq \frac{c_1}{\sqrt{2\pi}} \| g_n \|_2 \cdot \| x \| = \frac{c_1}{\sqrt{2\pi}} \| T(\cdot)x \|_{L^2([0,n],X)} \cdot \| x \| \cdot \]
4.1. EXPONENTIAL STABILITY AND DICHOTOMY

Letting \( a \to 0 \) and using Lebesgue’s theorem, we obtain
\[
\|T(\cdot)x\|_{L^2([0,n],X)}^2 \leq \frac{c_1}{\sqrt{2\pi}} \|x\| \cdot \|T(\cdot)x\|_{L^2([0,n],X)}
\]
and thus
\[
\|T(\cdot)x\|_{L^2([0,n],X)} \leq \frac{c_1}{\sqrt{2\pi}} \|x\|
\]
for all \( n \in \mathbb{N} \) and \( x \in X \). Fatou’s lemma shows \( T(\cdot)x \in L^2(\mathbb{R}_+,X) \) for all \( x \in X \). Datko’s lemma 4.5 now implies the assertion.

**Lemma 4.5** (Datko, 1970)

Let \( T(\cdot) \) be a \( C_0 \)-semigroup on a Banach space \( X \) and \( 1 \leq p < \infty \). If \( T(\cdot)x \in L^p(\mathbb{R}_+,X) \) for all \( x \in X \), then \( T(\cdot) \) is exponentially stable.

**Proof.** Define the bounded operator
\[
\Phi_n : X \to L^p(\mathbb{R}_+,X); \quad x \mapsto 1_{[0,n]}T(\cdot)x
\]
for each \( n \in \mathbb{N} \). The assumption shows that \( \sup_{n \in \mathbb{N}} \|\Phi_n(x)\| < \infty \) holds for all \( x \in X \), and hence \( C := \sup_{n \in \mathbb{N}} \|\Phi_n\| \) is finite thanks to the principle of uniform boundedness. As a result, \( \int_0^t \|T(s)x\|^p \, ds \leq C^p \|x\|^p \) for all \( t \geq 0 \) and \( x \in X \). Fix constants \( M \geq 1 \) and \( \omega > 0 \) such that \( \|T(t)\| \leq M e^{\omega t} \) for all \( t \geq 0 \). We can then calculate
\[
\frac{1 - e^{-p\omega t}}{p\omega} \|T(t)x\|^p = \int_0^t e^{-p\omega s} \|T(s)T(t-s)x\|^p \, ds \leq \int_0^t M^p e^{\omega ps} e^{-\omega sp} \|T(t-s)x\|^p \, ds
\]
\[
=M^p \int_0^t \|T(s)x\| \, ds \leq (CM)^p \|x\|^p
\]
for all \( t \geq 0 \) and \( x \in X \) so that \( \|T(t)\| \leq N \) for all \( t \geq 0 \), where \( N := \max\{M e^\omega, (p\omega)^{1/p} CM (1-e^{-p\omega})^{-1/p}\} \). It follows
\[
t\|T(t)x\|^p = \int_0^t \|T(t-s)T(s)x\|^p \, ds \leq N^p \int_0^t \|T(s)x\|^p \, ds \leq (CN)^p \|x\|^p,
\]
and hence \( \|T(t)\| \leq \frac{CN}{t^{1/p}} \). Proposition 4.2 now implies the assertion.

**Example 4.6** (damped wave equation)

Let \( \emptyset \neq U \subset \mathbb{R}^d \) be open and bounded with \( C^2 \) boundary and let \( b \in L^\infty(U) \) satisfy \( b(x) \geq \beta \) for almost every \( x \in U \) and some \( \beta > 0 \). As in Example 3.7 one sees that the operator
\[
A = \begin{pmatrix} 0 & I \\ \Delta_D & -b \end{pmatrix} \quad \text{with} \quad \mathcal{D}(A) = \mathcal{D}(\Delta_D) \times \dot{W}_2^1(U)
\]

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generates a contraction $C_0$-semigroup $T(\cdot)$ on $E = \dot{W}^1_2(U) \times L^2(U)$ where $\Delta Du = \Delta u$ and $\mathcal{D}(\Delta D) = W^2_2(U) \cap W^1_2(U)$. Moreover, the Cauchy Problem

$$\begin{cases}
u''(t) = \Delta Du(t) - bu'(t), & t \geq 0 \\
u(0) = u_0, & u'(0) = u_1
\end{cases}$$

is uniquely solved by $(u(t), u'(t))^T = T(t)(u_0, u_1)^T$ for all $(u_0, u_1) \in \mathcal{D}(A)$. We assert that $T(\cdot)$ is exponentially stable, and thus the “energy”

$$\|T(t)(u_0, u_1)\|^2_E = \|\nabla u\|^2_2 + \|u'(t)\|^2_2$$

of the solution decays as $c e^{-2\epsilon t} \|(u_0, u_1)\|_E^2$ for some $c, \epsilon > 0$. In order to show this, we use Theorem 4.4. First, we note that

$$\text{R} \left( \frac{f}{g} \right) = \left( \frac{\Delta_D^{-1}(b f + g)}{f} \right), \quad (f, g) \in E,$$

defines the bounded inverse of $A$. Next, we show that

$$\text{R}(i\tau, A) \in C < \infty.$$  
(4.2)

If (4.2) holds, it will follow that $\lambda \in \rho(A)$ and $\|\text{R}(\lambda, A)\| \leq 2C$ whenever $|\text{Re} \lambda| \in [0, \frac{1}{2C}]$, due to Lemma 3.3 (with $\lambda$ replaced by $i\tau$ and $B$ by $|\text{Re} \lambda I$). Combining this inequality with the Hille-Yosida estimate, we have checked the assumption of Theorem 4.4.

We show (4.2). Because of $s(A) \leq 0$, any $i\tau \in \sigma(A)$ would belong to $\partial \sigma(A)$ and thus it would follow that

$$m(\tau) := \inf \{\|i\tau w - Aw\|_E; w \in E, \|w\|_E = 1\} = 0$$

due to Proposition 1.19 of [ST] or (4.6) below. So our claim (4.2) holds if $\inf_{\tau \in \mathbb{R}} m(\tau) =: m_0 > 0$, and then $C = \frac{1}{m_0}$. Since $0 \in \rho(A)$ and $\rho(A)$ is open, there is a $\tau_0 > 0$ such that $[-i\tau_0, i\tau_0] \subseteq \rho(A)$, and so $m(\tau) \geq \delta := (\max_{|\tau| \leq \tau_0} \|\text{R}(i\tau, A)\|)^{-1} > 0$ for all $\tau \in [-\tau_0, \tau_0]$. Fix $\epsilon \in (0, \frac{\delta}{2})$ such that $0 < \frac{3\delta}{\beta - 2\epsilon} < \tau_0$. Suppose there are $|\tau| \geq \tau_0$ and $w = (u, v) \in \mathcal{D}(A)$ such that $\|w\|^2_E = \|\nabla u\|^2_2 + \|v\|^2_2 = 1$ and $\|i\tau w - Aw\|_E < \epsilon$. It follows that

$$\epsilon \geq \left| (i\tau I - A) \begin{pmatrix} u \\ v \end{pmatrix} \right| = \left| \int_U \nabla (i\tau u - v) \cdot \nabla \bar{\nu} \, dx + \int_U (-\Delta u \bar{\nu} + (i\tau + b)v\bar{\nu}) \, dx \right|$$

$$= \left| i\tau (\|\nabla u\|^2_2 + \|v\|^2_2) - \int_U \nabla u \cdot \nabla \bar{\nu} \, dx + \int_U \nabla v \cdot \nabla \bar{\nu} \, dx + \int_U b|v|^2 \, dx \right|$$

$$= \left| i(\tau + 2 \text{Im} \int_U \nabla u \cdot \nabla \bar{\nu} \, dx) + \int_U b|v|^2 \, dx \right|,$$

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where we integrated by parts. Considering imaginary and real parts, we infer that

\[ \epsilon \geq \left| \tau + 2 \text{Im} \int_U \nabla u \cdot \nabla v \, dx \right| \quad \text{and} \quad \epsilon \geq \int_U |v|^2 \, dx \geq \beta \|v\|^2, \]

so that \( \|\nabla u\|_2^2 = 1 - \|v\|_2^2 \geq 1 - \epsilon \) and \( 1 - 2\|\nabla u\|_2^2 \leq \frac{2\epsilon}{\beta} - 1 < 0. \) We conclude that

\[ |\tau| \left(1 - \frac{2\epsilon}{\beta}\right) \leq |\tau| \cdot |1 - 2 \cdot \|\nabla u\|^2_2| = \left| \tau + 2 \text{Im} \int_U \nabla u \cdot i\tau \nabla v \, dx \right| \]

\[ \leq \left| \tau + 2 \text{Im} \int_U \nabla u \cdot \nabla v \, dx \right| + \left| 2 \text{Im} \int_U \nabla u \cdot i\tau \nabla u - \nabla v \, dx \right| \]

\[ \leq \epsilon + 2 \|\nabla u\|_2^2 \cdot \|\nabla (i\tau u - v)\|_2 \leq \epsilon + 2 \| (i\tau I - A)w \|_E \leq 3\epsilon. \]

As a result, \( |\tau| \leq \frac{3\beta\epsilon}{\beta - 2\epsilon} \) which is impossible. We have shown that \( m(\tau) \geq \epsilon > 0 \) for all \( |\tau| \geq \tau_0, \) as needed.

We refer to Theorem VI.3.18 in [E-N] for a generalization of the above example.

**Definition 4.7**

A \( C_0 \)-semigroup \( T(\cdot) \) has an exponential dichotomy if there are constants \( N, \delta > 0 \) and a projection \( P \in \mathcal{B}(X) \) such that \( T(t)P = PT(t), T(t) : N(P) \to N(P) \) has an inverse denoted by \( T_u(-t), \|T(t)P\| \leq Ne^{-\delta t} \) and \( \|T_u(-t)Q\| \leq Ne^{-\delta t} \) for all \( t \geq 0, \) where \( Q = I - P. \)

Recall that \( N(P) = QX. \) Observe that exponential dichotomy coincides with exponential stability if \( P = I. \) Moreover, exponential dichotomy means that \( T(t)X_j \subseteq X_j \) for all \( t \geq 0 \) where \( j = s \) or \( j = u, \) \( X_s := PX \) and \( X_u := QX, \) that \( T_s(\cdot) := T(\cdot)|X_s \) is an exponentially stable \( C_0 \)-semigroup on \( X_s \) and that \( T(\cdot) \) induces a \( C_0 \)-group \( T_u(\cdot) \) on \( X_u \) which is exponentially stable in backwards time (use Lemma 1.22 for the group property).

**Proposition 4.8**

A \( C_0 \)-semigroup \( T(\cdot) \) has an exponential dichotomy if and only if \( \mathbb{T} := \{ \lambda \in \mathbb{C}; |\lambda| = 1 \} \subseteq \rho(T(t)) \) for some (and hence all) \( t > 0. \)

**Proof.** Let \( T(\cdot) \) have an exponential dichotomy. Take \( t > 0 \) and \( \lambda \in \mathbb{T}. \) Then the series

\[ R_\lambda = \lambda^{-1} \sum_{n=0}^\infty \lambda^{-n}T(nt)P - \lambda^{-1} \sum_{n=1}^\infty \lambda^nT_u(-nt)Q \]

converges in \( \mathcal{B}(X). \) It holds

\[ (\lambda I - T(t)) R_\lambda = (I - \lambda^{-1}T(t)) \left( \sum_{n=0}^\infty (\lambda^{-1}T(t))^n P - \sum_{n=1}^\infty (\lambda^{-1}T_u(t))^{-n} Q \right) \]

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Similarly one sees that \( R_\lambda(\lambda I - T(t)) = I \). As a result, \( T \) is contained in \( \rho(T(t)) \) for all \( t > 0 \). Next, assume that \( T \subseteq \rho(T(t)) \) for some \( t > 0 \). We define

\[
P := \frac{1}{2\pi i} \int T(\lambda) \, d\lambda.
\]

It is known that \( P^2 = P \in \mathcal{B}(X) \) commutes with \( T(t) \) and we have \( \sigma(T_s(t)) = \sigma(T(t)) \cap B(0,1) \) and \( \sigma(T_u(t)) = \sigma(T(t)) \setminus \overline{B}(0,1) \) (see, e.g., Theorem 5.5 in [ST]). Since \( r(T_s(t)) < 1 \), the \( C_0 \)-semigroup \( T_u(t) \) is exponentially stable on \( X_s = PX \) by Proposition 4.2. We further have that \( T_u(t) \) is invertible and \( \sigma(T_u(t)^{-1}) = \sigma(T_u(t))^{-1} \) so that \( r(T_u(t)^{-1}) < 1 \). Thus the \( C_0 \)-semigroup \( (T_u(t)^{-1})_{t\geq 0} \) is also exponentially stable on \( X_u = QX \) for the same reason. Consequently, \( T(\cdot) \) has an exponential dichotomy.

For \( A \in \mathcal{B}(X) \) it is known that the spectral mapping theorem \( \sigma(e^{tA}) = e^{t\sigma(A)} \) holds for all \( t \geq 0 \), see, e.g., Theorem 5.3 in [ST]. As a result, \( e^{tA} \) has an exponential dichotomy if and only if \( i\mathbb{R} \subseteq \rho(A) \). As before, the important implication \( \Leftarrow \) fails in general for unbounded \( A \) even on a Hilbert space, see [M-S]. It follows from “Gearhart’s spectral mapping theorem” that a \( C_0 \)-semigroup on a Hilbert space has an exponential dichotomy if and only if \( i\mathbb{R} \subseteq \rho(A) \) and \( \sup \tau \in \mathbb{R} \| R(i\tau, A) \| < \infty \); see, e.g., Theorem 2.6.3 in [vN] and see [S] for a Banach space version.

### 4.2. Spectral mapping theorems

Let \( A \) generate a \( C_0 \)-semigroup \( T(\cdot) \). We say that the spectral mapping theorem holds if

\[
\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} \quad \text{for all } t \geq 0.
\]

Observe that we need to exclude 0 on the left hand side since 0 does not belong to \( e^{t\sigma(A)} \). As seen above, the spectral mapping theorem fails in general. We first explore which partial results are still true. For this purpose, we recall the following concepts and results from spectral theory, where \( A \) is a densely defined closed operator. We define by

\[
\begin{align*}
\sigma_p(A) &= \{ \lambda \in \mathbb{C}; \lambda I - A \text{ is not injective} \}, \\
\sigma_{ap}(A) &= \{ \lambda \in \mathbb{C}; \lambda I - A \text{ is not injective or } (\lambda I - A)D(A) \text{ is not closed} \}, \\
\sigma_r(A) &= \{ \lambda \in \mathbb{C}; (\lambda I - A)D(A) \text{ is not dense} \}
\end{align*}
\]

the point spectrum, the approximate point spectrum and the residual spectrum of \( A \), respectively. It clearly holds that \( \sigma_p(A) \subseteq \sigma_{ap}(A) \) and

\[
(4.3) \quad \sigma(A) = \sigma_{ap}(A) \cup \sigma_r(A).
\]

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One can further show that
\[(4.4) \quad \sigma_{ap}(A) = \left\{ \lambda \in \mathbb{C} ; \exists (x_n)_n \in \mathcal{D}(A)^\mathbb{N} : \|x_n\| = 1 \text{ and } \lim_{n \to \infty} \|(\lambda I - A)x_n\| = 0 \right\},\]
\[(4.5) \quad \sigma_r(A) = \sigma_p(A^*) \quad \text{and} \quad R(\lambda, A^*) = R(\lambda, A^*) \quad \text{for all } \lambda \in \rho(A) = \rho(A^*)\]
as well as
\[(4.6) \quad \partial \sigma(A) \subseteq \sigma_{ap}(A),\]
see, e.g., Proposition 1.19 and Theorem 1.24 of [ST]. We call the vectors in (4.4) approximate eigenvectors and the elements of \(\sigma_{ap}(A)\) approximate eigenvalues.

**Proposition 4.9** (spectral inclusion)

Let \(A\) generate the \(C_0\)-semigroup \(T(\cdot)\). It then holds
\[e^{t\sigma(A)} \subseteq \sigma(T(t)) \quad \text{and} \quad e^{t\sigma_j(A)} \subseteq \sigma_j(T(t))\]
for all \(t \geq 0\) and \(j \in \{p, ap, r\}\). (Approximate) Eigenvectors of \(A\) for the (approximate) eigenvalue \(\lambda\) are (approximate) eigenvectors of \(T(t)\) for the (approximate) eigenvalue \(e^{t\lambda}\).

**PROOF.** Recall from Lemma 1.12 that
\[e^{\lambda t} x - T(t)x = (\lambda I - A) \int_0^t e^{\lambda(t-s)} T(s)x \, ds \quad \text{for } x \in X\]
\[= \int_0^t e^{\lambda(t-s)} T(s)(\lambda I - A)x \, ds \quad \text{for } x \in \mathcal{D}(A)\]
and all \(\lambda \in \mathbb{C}\) and \(t \geq 0\). So if \(\lambda x = Ax\) and \(x \in \mathcal{D}(A) \setminus \{0\}\), then \(e^{\lambda t} x = T(t)x\) and \(x\) is an eigenvector of \(T(t)\) for the eigenvalue \(e^{\lambda t} \in \sigma_p(T(t))\). If \((\lambda I - A)\mathcal{D}(A)\) is not dense or not equal to \(X\), then \(R(e^{\lambda t} I - T(t))\) has the same property. Finally, let \((x_n)_n\) be approximate eigenvectors of \(A\) for \(\lambda \in \sigma_{ap}(A)\). It follows that
\[\|e^{\lambda t} x_n - T(t)x_n\| \leq c\|\lambda x_n - Ax_n\| \to 0\]
as \(n \to \infty\) and so \(x_n\) are approximate eigenvectors for the approximate eigenvalue \(e^{\lambda t} \in \sigma_{ap}(T(t))\).☐

**Corollary 4.10**

For a \(C_0\)-semigroup \(T(\cdot)\) with generator \(A\) having an exponential dichotomy, it holds that \(i\mathbb{R} \subseteq \rho(A)\) since \(\mathbb{T} \subseteq \rho(T(1)) \subseteq \mathbb{C} \setminus e^{\sigma(A)}\) by Propositions 4.8 and 4.9.

In the following example we use the spectral inclusion to compute the spectrum of the periodic translation. Here the spectral mapping theorem fails for irrational \(t\), but a variant with an additional closure holds.
Example

Let \( X = \{ f \in C(\mathbb{R}); f(t) = f(t+1) \text{ for all } t \in \mathbb{R} \} \) be endowed with the supremum norm and \( T(t)f = f(t+1) \text{ for } t \in \mathbb{R} \text{ and } f \in X \). It is easy to see that \( T(\cdot) \) is an isometric \( C_0 \)-group on \( X \) using that any \( f \in X \) is uniformly continuous. Note that \( T(n) = I \) for all \( n \in \mathbb{N}_0 \).

As in Example 1.15 one can verify that the generator \( A \) of \( T(\cdot) \) is given by \( Af = f' \) with \( \mathcal{D}(A) = C^1(\mathbb{R}) \cap X \). Clearly, \( e^{2\pi in} \) belongs to \( \mathcal{D}(A) \) and \( Ae^{2\pi in} = 2\pi in e^{2\pi in} \) for all \( n \in \mathbb{Z} \). On the other hand, Proposition 4.9 shows that \( e^{\sigma(A)} \subseteq \sigma(T(1)) = \{1\} \) so that \( \sigma(A) \subseteq 2\pi i\mathbb{Z} \). As a result, \( \sigma(A) = \sigma_p(A) = 2\pi i\mathbb{Z} \). If \( t \in \mathbb{R}_+ \setminus \mathbb{Q} \), it is known that \( e^{t\sigma(A)} = e^{t2\pi i\mathbb{Z}} \) is dense in \( \mathbb{T} \).

Since \( T(t) \) is isometric and invertible, we have \( \sigma(T(t)) \subseteq \mathbb{T} \) so that

\[
\mathbb{T} = e^{t\sigma(A)} \subseteq \sigma(T(t)) \subseteq \mathbb{T}
\]

and thus

\[
e^{t\sigma(A)} = \sigma(T(t)) = \mathbb{T}
\]

if \( t \in \mathbb{R}_+ \setminus \mathbb{Q} \). If \( t = \frac{k}{l} \) for some \( k, l \in \mathbb{N} \) without common divisors, we have

\[
\sigma(T(t))' = \sigma \left( T \left( \frac{k}{l} \right) \right) = \sigma(T(k)) = \{1\}
\]

using the spectral mapping theorem for bounded operators, see, e.g., Theorem 5.3 of [ST]. As a result, \( \sigma(T(t)) \) belongs to the group \( \Gamma_l \) of the \( l \)th unit roots. On the other hand, the set \( e^{t\sigma(A)} = \exp(2\pi i\frac{k}{l}\mathbb{Z}) \) is equal to \( \Gamma_l \) and contained in \( \sigma(T(t)) \) so that

\[
e^{t\sigma(A)} = \sigma(T(t)) = \{z \in \mathbb{C}; z^l = 1\}
\]

if \( t = \frac{k}{l} \) for some \( k, l \in \mathbb{N} \) without common divisors.

**Theorem 4.11** (spectral mapping theorem for the point spectrum)

*If \( A \) generates the \( C_0 \)-semigroup \( T(\cdot) \), then it holds*

\[
\sigma_p(T(t)) \setminus \{0\} = e^{t\sigma_p(A)}
\]

*for all \( t \geq 0 \).*

**Proof.** Let \( t > 0, \lambda \in \mathbb{C} \) and \( x \in X \setminus \{0\} \) such that \( e^{\lambda t} x = T(t)x \). Hence, the function \( u(s) := e^{-\lambda s} T(s)x \) has period \( t > 0 \). Suppose that all Fourier coefficients

\[
\frac{1}{\sqrt{t}} \int_0^t e^{-2\pi ins} u(s) \, ds, \quad n \in \mathbb{Z},
\]

would vanish. Then all Fourier coefficients of the scalar function \( \varphi(t) = \langle u(t), x^* \rangle \) would vanish for any \( x^* \in X^* \). Due to Example 2.17 of [FA], it would follow that \( \varphi = 0 \), and so \( u = 0 \) by
Lemma 1.12 shows that the Hahn-Banach theorem. This is wrong and thus there exists an \( m \in \mathbb{Z} \) such that
\[
y := \int_0^t e^{-\frac{2\pi im}{t}} u(s) \, ds \neq 0.
\]
As a result, \( \mu := \lambda + \frac{2\pi im}{t} \) belongs to \( \sigma_p(A) \) so that \( e^{\lambda t} = e^{\mu t} \in e^t \sigma_p(A) \) and \( \sigma_p(T(t)) \subseteq e^t \sigma_p(A) \) holds. The other inclusion was shown in Proposition 4.9.

Unfortunately, in general (4.5) to deduce the spectral mapping theorem for the residual spectrum from Theorem 4.11. It is now natural to ask whether or not a relation like in 4.11 is also valid for other interesting parts of the spectrum. In what follows, we want to use the dual semigroup \( (T(t^*))^t \geq 0 \) and to deduce the spectral mapping theorem for the residual spectrum from Theorem 4.11. Unfortunately, in general \( T(\cdot)^* \) fails to be strongly continuous. (Consider, e.g., the left shift \( T(\cdot) \) on \( L^1(\mathbb{R}) \), where \( T(\cdot)^* \) is the right shift on \( L^\infty(\mathbb{R}) \) which is not strongly continuous due to Example 1.6.)

Let \( A \) generate the \( C_0 \)-semigroup \( T(\cdot) \). We define the sun dual by setting
\[
X^\odot := \{ x^* \in X^* ; T(t)^* x^* \rightarrow x^* \text{ as } t \rightarrow 0 \} \quad \text{and} \quad T(t)^\odot := T(t)|_{X^\odot}.
\]
By using Lemma 1.5 and recalling \( \|T(t)\| = \|T(t)^*\| \), one sees that \( X^\odot \) is a closed subspace of \( X^* \) and \( T(\cdot)^\odot \) is a \( C_0 \)-semigroup on this space, see the exercises. Its generator is denoted by \( A^\odot \). If \( x^* \in \mathcal{D}(A^\odot) \), then
\[
\langle x, A^\odot x^* \rangle = \lim_{t \rightarrow 0} \left\langle x, \frac{1}{t} (T(t)^* - I) x^* \right\rangle = \lim_{t \rightarrow 0} \left\langle \frac{1}{t} (T(t) - I) x, x^* \right\rangle = \langle Ax, x^* \rangle
\]
for all \( x \in \mathcal{D}(A) \); i.e., \( A^\odot \subseteq A^* \). For \( x^* \in \mathcal{D}(A^*) \), we further have
\[
\|T(t)^* x^* - x^*\| = \sup_{x \in X, \|x\| \leq 1} |\langle x, T(t)^* x^* - x^* \rangle| = \sup_{\|x\| \leq 1} |\langle T(t)x - x, x^* \rangle| = \sup_{\|x\| \leq 1} \left| \int_0^t T(s)x \, ds, A^* x^* \right| \leq c \|A^* x^*\| t
\]
for all \( 0 \leq t \leq 1 \) and a constant \( c > 0 \), where we used Lemma 1.12. Hence, \( x^* \in X^\odot \) and \( \mathcal{D}(A^*) \subseteq X^\odot \). We clearly have \( \sigma_p(A^\odot) \subseteq \sigma_p(A^*) \) and \( \sigma_p(T(t)^\odot) \subseteq \sigma_p(T(t)^*) \) for all \( t \geq 0 \). Let \( T(t)^* x^* = e^{\lambda t} x^* \) for some \( x^* \in X^\star \setminus \{0\} \) and \( t \geq 0 \). Note that \( R(\mu, A)^* = R(\mu, A^*) \) is injective and maps \( X^\star \) into \( \mathcal{D}(A^*) \subseteq X^\odot \) for \( \mu \in \rho(A^*) = \rho(A) \) and it commutes with \( T(t)^* \). Hence, \( R(\mu, A^*) x^* \) is an eigenvector for \( T(t)^\odot \) and the eigenvalue \( e^{\lambda t} \). Let \( x^* \in \mathcal{D}(A^*) \) with \( A^* x^* = \lambda x^* \).

As above, we compute
\[
\left\| \frac{1}{t} (T(t)^\odot x^* - x^*) - \lambda x^* \right\| = \sup_{x \in X, \|x\| \leq 1} \left| \left\langle A^* \frac{1}{t} \int_0^t T(s)x \, ds, x^* \right\rangle - \langle x, \lambda x^* \rangle \right|
\]
Proposition 4.9, Theorem 4.12 and the formulae (4.3) and (4.4), it suffices to show

\[
\lim_{t \to 0} \left\| x, \frac{1}{t} \int_0^t T(s)A^*x^* \, ds - \lambda x^* \right\| = 0
\]

as \(t \to 0\), where we used \(\mathcal{D}(A^*) \subseteq X^\circ\) as well as \(A^*x^* = \lambda x^*\). We have thus shown

\[
(4.7) \quad \sigma_p(A^\circ) = \sigma_p(A^*) \quad \text{and} \quad \sigma_p(T(t)^\circ) = \sigma_p(T(t)^*) \quad \text{for all} \quad t \geq 0.
\]

These equalities also hold for the full spectra. For this and further information we refer to Proposition IV.2.18 and §II.2.6 of [E-N].

**Theorem 4.12** (spectral mapping theorem for the residual spectrum)

Let \(A\) generate the \(C_0\)-semigroup \(T(\cdot)\). Then it holds \(\sigma_r(T(t)) \setminus \{0\} = e^{t\sigma_r(A)}\) for all \(t \geq 0\).

**Proof.** Combining (4.5), (4.7) and Theorem 4.11, we obtain

\[
\sigma_r(T(t)) \setminus \{0\} = \sigma_p(T(t)^*) \setminus \{0\} = \sigma_p(T(t)^\circ) \setminus \{0\} = e^{t\sigma_p(A^\circ)} = e^{t\sigma_p(A^*)} = e^{t\sigma_r(A)}
\]

for all \(t \geq 0\).

As a result, the spectral mapping theorem can only fail if we are not able to transport approximate eigenvectors from \(T(t)\) to \(A\). This can be done if the semigroup has some additional regularity.

**Theorem 4.13** (spectral mapping theorem for eventually norm continuous semigroups)

Let \(A\) generate the \(C_0\)-semigroup \(T(\cdot)\) and let

\[
(t_0, \infty) \to B(X); \quad t \mapsto T(t)
\]

be continuous (in operator norm) for some \(t_0 \geq 0\). Then the spectral mapping theorem

\[
\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}
\]

holds for all \(t \geq 0\). The semigroup is norm continuous for \(t > t_0\) if it is analytic (then \(t_0 = 0\)) or if \(T(t_0)\) is compact for some \(t_0 > 0\).

**Proof.** If \(T(t_0)\) is compact, then \(\overline{T(t_0)\mathcal{B}_X(0,1)}\) is compact. Hence, the map

\[
[t_0, \infty) \to B(X); \quad t \mapsto T(t)x = T(t-t_0)T(t_0)x
\]

is uniformly continuous for \(x \in \overline{\mathcal{B}_X(0,1)}\) and thus continuous for the operator norm. In view of Proposition 4.9, Theorem 4.12 and the formulae (4.3) and (4.4), it suffices to show

\[
\sigma_{ap}(T(t)) \setminus \{0\} \subseteq e^{t\sigma_{ap}(A)}
\]

for all \(t > 0\), or: If \(\lambda \in \mathbb{C}\), \(\tau > 0\) and \(x_n \in X\) satisfy \(\|x_n\| = 1\) for all \(n \in \mathbb{N}\) and \(\lambda x_n - T(\tau)x_n \to 0\) as \(n \to \infty\), then there is a \(\mu \in \sigma_{ap}(A)\) such that \(\lambda = e^{\tau\mu}\). Considering the semigroup \((e^{-\nu s}T(s\tau))_{s \geq 0}\) with \(\lambda = e^{\nu}\) and its generator \(B = \tau A - \nu I\), we can assume \(\lambda = 1\) and \(\tau = 1\). Fix some \(k \in \mathbb{N}\) with \(k > t_0\). Then the map \([0,1] \to X; \quad s \mapsto\)
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\(T(s)T(k)x_n\) is continuous uniformly for \(n \in \mathbb{N}\) by our assumption. Moreover, 
\[\|T(k)x_n - x_n\| \leq \|T(k-1)(T(1)x_n - x_n)\| + \|T(k-2)(T(1)x_n - x_n)\| + \ldots + \|T(1)x_n - x_n\|\]
tends to 0 as \(n \to \infty\). This fact implies that the map \([0, 1] \to X; s \mapsto T(s)(T(k)x_n - x_n)\)
is continuous uniformly for \(n \in \mathbb{N}\). Hence, the map \([0, 1] \to X; s \mapsto T(s)x_n\) is continuous uniformly for \(n \in \mathbb{N}\). Next, choose \(x_n^* \in X^*\) such that \(\|x_n^*\| \leq 1\) and \(\langle x_n, x_n^* \rangle \geq \frac{1}{2}\) for all \(n \in \mathbb{N}\), using the Hahn-Banach theorem. Since the functions \(\varphi_n(s) := \langle T(s)x_n, x_n^* \rangle\) are continuous from \([0, 1]\) to \(\mathbb{C}\) uniformly in \(n \in \mathbb{N}\) and uniformly bounded, the Arzelà-Ascoli theorem (see, e.g., Theorem 1.45 in [FA]) shows that a subsequence \(\varphi_{n_j}\) converges in \(C([0,1])\) to a function \(\varphi\). Observe that
\[
\|\varphi\| = \lim_{j \to \infty} |\varphi(0)| = \lim_{j \to \infty} |\varphi_{n_j}(0)| = \lim_{j \to \infty} \langle x_{n_j}, x_{n_j}^* \rangle \geq \frac{1}{2}
\]
showing that \(\varphi \neq 0\). Therefore, Example 2.17 of [FA] implies that \(\varphi\) has a nonzero Fourier coefficient, i.e., there exists an \(m \in \mathbb{Z}\) such that for \(\mu := 2\pi im\) we have \(\int_0^1 e^{-\mu s} \varphi(s)\,ds \neq 0\). We now set \(z_n := \int_0^1 e^{-\mu s} T(s)x_n\,ds\). Using Lemma 1.12, we obtain that \(z_n \in D(A)\) and
\[(\mu I - A)z_n = (I - e^{-\mu T(1)})x_n = x_n - T(1)x_n \to 0\]
as \(n \to \infty\). It further holds
\[
\liminf_{j \to \infty} \|z_{n_j}\| \geq \liminf_{j \to \infty} \|\varphi_{n_j}(0)\| = \liminf_{j \to \infty} \langle x_{n_j}, x_{n_j}^* \rangle = \left|\int_0^1 e^{-\mu s} \langle T(s)x_{n_j}, x_{n_j}^* \rangle\,ds\right| = \left|\int_0^1 e^{-\mu s} \varphi(s)\,ds\right| > 0
\]
so that \(\mu \in \sigma_{ap}(A)\), completing the proof. \(\square\)

**Corollary 4.14**

If \(A\) generates a \(C_0\)-semigroup \(T(\cdot)\) such that the map \((t_0, \infty) \to B(X); t \mapsto T(t)\) is continuous for some \(t_0 \geq 0\), then the following equivalences hold.

(a) The semigroup \(T(\cdot)\) is exponentially stable if and only if \(s(A) < 0\).
(b) The semigroup \(T(\cdot)\) has an exponential dichotomy if and only if \(i\mathbb{R} \subseteq \rho(A)\).

**Proof.** We have \(e^{s0(A)} = r(T(1)) = e^{s(A)}\) by Proposition 4.2 and Theorem 4.13, which yields (a). Assertion (b) follows similarly from Proposition 4.8 and Theorem 4.13. \(\square\)
CHAPTER 5

Semilinear equations

We refer to Chapter 5 and 7 of [P] and Chapter 7 of [L] for a more detailed discussion.

5.1. The case of a generator

Let $A$ generate the $C_0$-semigroup $T(\cdot)$ on the Banach space $X$ satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and some $M, \omega > 0$ and let $F : X \to X$ be locally Lipschitz, i.e., for all $r > 0$ there exists an $L(r) > 0$ such that

$$\|F(x) - F(y)\| \leq L(r)\|x - y\|$$

holds whenever $x, y \in B_X(0, r)$ is fulfilled. For any given $u_0 \in X$ and an interval $J$ with $0 = \inf J < \sup J$ we study the problem

$$\begin{cases}
u'(t) = Au(t) + F(u(t)), & t \in J, \\ u(0) = u_0.\end{cases}$$

A (classical) solution of (5.1) on $J$ is a function $u \in C^1(J, X) \cap C(\tilde{J}, X)$ such that $u(t) \in D(A)$ for all $t \in J$ and (5.1) holds, where $\tilde{J} := J \cup \{0\}$. If $u$ is a classical solution, then $F \circ u \in C(\tilde{J}, X)$. Hence, Proposition 2.6 implies that $u$ is a mild solution, i.e., a function $u \in C(\tilde{J}, X)$ such that

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s)) \, ds, \quad t \in J.$$

We recall that already in the case $X = \mathbb{R}$, $A = 0$ and $F(u) = u^2$ problem (5.1) has a unique solution, namely $u(t) = \frac{1}{(u_0^2 - t)}$ if $u_0 > 0$, which exists only on the finite interval $J = J(u_0) = [0, \frac{1}{u_0})$.

**Theorem 5.1**

Let $A$ generate the $C_0$-semigroup $T(\cdot)$ on the Banach space $X$ satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and some $M, \omega > 0$ and let $F : X \to X$ be locally Lipschitz. Let $\rho > 0$ be given and take $u_0 \in \overline{B}(0, \rho)$. Then there is a number $b_0(\rho) > 0$, a maximal existence time $b(u_0) \in (b_0(\rho), \infty]$ and a unique mild solution $u = u(\cdot; u_0) \in C([0, b(u_0)), X)$ of (5.1). If $b(u_0) < \infty$, then $\lim \sup_{t \to b(u_0)} \|u(t)\|_X = \infty$. Furthermore, the map

$$\overline{B}_X(0, \rho) \to C([0, b_0(\rho)], X); \ u_0 \mapsto u(\cdot; u_0)|_{[0, b_0(\rho)]}$$

is Lipschitz continuous.
CHAPTER 5. SEMILINEAR EQUATIONS

Proof. 1) Fix \( \rho > 0 \) and take \( u_0 \in X \) with \( \|u_0\| \leq \rho \). Fix any \( r > M e^{\omega} \rho \) and take \( 0 < b \leq 1 \) to be chosen below. Define

\[
E(b) := \{ u \in C([0, b], X); \|u\|_{\infty} \leq r \}
\]

and

\[
[\Phi(u)](t) := [\Phi_{u_0}(u)](t) := T(t)u_0 + \int_0^t T(t-s)F(u(s)) \, ds
\]

for \( t \in [0, b] \) and \( u \in E(b) \). Clearly, \( \Phi(u) \in C([0, b], X) \) and \( E(b) \) is a complete metric space for the metric induced by \( \| \cdot \|_{\infty} \). For \( u, v \in E(b) \) we estimate

\[
\|\Phi(u)(t)\| \leq M e^{\omega t} \|u_0\| + \int_0^t M e^{\omega(t-s)}(\|F(u(s)) - F(0)\| + \|F(0)\|) \, ds
\]

\[
\leq M e^{\omega} \rho + bM e^{\omega}(L(r)r + \|F(0)\|)
\]

as well as

\[
\|\Phi(u)(t) - \Phi(v)(t)\| \leq \int_0^t M e^{\omega(t-s)}\|F(u(s)) - F(v(s))\| \, ds
\]

\[
\leq bM e^{\omega} L(r)\|u - v\|_{\infty}
\]

for all \( 0 \leq t \leq b \leq 1 \). Choosing \( b_0 := b_0(\rho) \in (0, 1] \) small enough\(^1\), we conclude that \( \Phi(u) \in E(b) \) and \( \Phi \) is Lipschitz with Lipschitz constant smaller than or equal to \( \frac{1}{2} \) if \( 0 < b \leq b_0 \). Banach’s fixed point theorem then gives a unique fixed point \( u = \Phi(u) \in E(b) \) for all \( 0 < b \leq b_0 \) which is a mild solution by definition. Let \( u_0, v_0 \in \overline{B}_X(0, \rho) \) with solutions \( u = \Phi_{u_0}(u) \) and \( v = \Phi_{v_0}(v) \), respectively, in \( E(b_0) \). Then the calculation

\[
\|u - v\|_{\infty} \leq \|\Phi_{u_0}(u) - \Phi_{u_0}(u)\|_{\infty} + \|\Phi_{u_0}(u) - \Phi_{v_0}(v)\|_{\infty} \leq \frac{1}{2}\|u - v\|_{\infty} + \|T(\cdot)(u_0 - v_0)\|_{\infty}
\]

shows that

\[
\|u - v\|_{\infty} \leq 2M e^{\omega b_0} \|u_0 - v_0\|,
\]

which gives the required Lipschitz property.

2) Let \( u \in C([0, t_0], X) \) be a mild solution of (5.1). By step 1) there is a \( t_1 > 0 \) and a solution \( v \in C([0, t_1], X) \) of (5.1) with \( v(0) = u(t_0) \). We set \( w(t) = u(t) \) for \( 0 \leq t \leq t_0 \) and \( w(t) = v(t - t_0) \) for \( t_0 < t \leq t_0 + t_1 =: b_1 \). Then \( w \in C([0, t_0 + t_1], X) \) satisfies (5.2) on \([0, t_0]\). Let \( t_0 < t \leq b_1 \). Since \( u \) fulfills (5.2) on \([0, t_0]\), we obtain

\[
w(t) = v(t - t_0) = T(t - t_0)u(t_0) + \int_0^{t-t_0} T(t - t_0 - s)F(v(s)) \, ds
\]

\(^1\)e.g., \( b_0(\rho) = \min \left\{ \frac{1}{M e^{\omega}(L(r)r + \|F(0)\|)}; \frac{2M e^{\omega} L(r)}{1 - M e^{\omega}(L(r)r + \|F(0)\|)} \right\} \) will do.
\[= T(t-t_0)T(t_0)u_0 + T(t-t_0) \int_0^{t_0} T(t_0-s)F(u(s)) \, ds + \int_{t_0}^{t} T(t-r)F(v(r-t_0)) \, dr\]

\[= T(t)u_0 + \int_0^{t} T(t-s)F(w(s)) \, ds.\]

Thus \(w\) is a mild solution of (5.1). We iterate this procedure and obtain a mild solution of (5.1) on \([0,b(u_0))\) where \(b(u_0) := \lim_{n \to \infty} b_n \in (0, \infty]\) and \(b_n := \sum_{j=0}^{n} t_j\). Suppose that \(b(u_0) < \infty\) and \(\|u(t)\| \leq R\) for all \(0 \leq t < b(u_0)\) and some \(R > 0\). By step 1), any solution with initial value in \(\mathcal{B}_{X}(0, R)\) has a minimal existence time \(b_0(R) > 0\) so that we can choose \(t_j \geq b_0(R)\) for all \(j\) so that \(b_n \geq nb_0(R) \to \infty\) as \(n \to \infty\), contradicting \(b(u_0) < \infty\).

3) It remains to check uniqueness. Let \(u, v\) be mild solutions of (5.1) on the same interval \(J\). If \(u \neq v\), there is a \(0 < \tau_0 \in J\) and a sequence \(\tau_n \in J \cap (\tau_0, \infty)\) converging to \(\tau_0\) such that \(x_0 = u(\tau_0) = v(\tau_0)\) and \(u(\tau_n) \neq v(\tau_n)\) for all \(n \in \mathbb{N}\). By step 1), the problem (5.1) with initial value \(x_0\) has a unique mild solution in a ball with some radius \(r > \|x_0\|\) in \(C([0,b], X)\) for all sufficiently small \(b \in (0, b(x_0)]\). Choosing \(b\) small enough, the functions \(u(\cdot + \tau_0)\) and \(v(\cdot + \tau_0)\) are defined on \([0,b]\), take values in \(\mathcal{B}_{X}(0, r)\) on the interval \([0,b]\) and are mild solutions to the problem (5.1) on \([0,b]\) with initial value \(x_0\), as one easily verifies by a computation similar to the one carried out at the beginning of the second step or as in the proof of Corollary 5.2 below. Thus these functions coincide, and in particular, \(u\) and \(v\) coincide in the points \(\tau_n\) for sufficiently large \(n\), which contradicts the choice of the sequence \((\tau_n)_n\). \(\square\)

**Corollary 5.2**

Let \(u_0 \in \mathcal{D}(A)\), \(A\) generate a \(C_0\)-semigroup \(T(\cdot)\) and \(F : X \to X\) be locally Lipschitz continuous. Then the solution \(u\) constructed in Theorem 5.1 is locally Lipschitz on \([0,b(u_0))\).

**Proof.** Fix an \(r \in (0,b(u_0))\), where \(b(u_0)\) is the maximal existence time of \(u\), and set

\[\rho := \sup_{0 \leq t \leq r} \sup_{0 \leq h \leq r-t} \|u(t+h)\| \in \|u_0\|, \infty)\]

and

\[C := \sup_{0 \leq t \leq r} \|F(u(t))\| < \infty.\]

Moreover, let \(b_0(\rho) > 0\) be the number we obtained in Theorem 5.1 and put \(\tau_1 := \min\{r, b_0(\rho)\}\). In what follows we set as usual \(\infty + a := a + \infty := \infty\) for any \(a \in \mathbb{R}\). Next, observe that the function

\[v_h : [0,b_0(\rho) - h) \to X; t \mapsto u(t + h)\]

is a mild solution of (5.1) on \([0,b_0(\rho) - h)\) with initial value \(u(h)\) for each \(h \in (0,b_0(\rho))\). This
can be seen by the following calculation:

\[
v(t) = T(t + h)u_0 + \int_0^{t+h} T(t + h - s)F(u(s)) \, ds
\]

\[
= T(t) \left( T(h)u_0 + \int_0^h T(h - s)F(u(s)) \, ds \right) + \int_h^{t+h} T(t + h - s)F(u(s)) \, ds
\]

\[
= T(t)u(h) + \int_0^t T(t - r)F(v(r)) \, dr,
\]

where we used the substitution \( v = s - h \) and that \( u \) is a mild solution of (5.1). In addition, note that \( b(u_0) - h = b(u(h)) \), which is clear for \( b(u_0) = \infty \) and which elsewise follows from \( \lim_{t \to \infty} \|u(t + h)\| = \infty \). The uniqueness in Theorem 5.1 thus implies \( v_h = u(\cdot ; u(h)) \). Moreover, we have \( b_0(\rho) < b(u_0) - h \) for \( h \in [0, \tau_1] \) since then \( \|u(h)\| \leq \rho \). Consequently, the Lipschitz property in Theorem 5.1 yields

\[
\|u(t + h) - u(t)\| = \|v_h(t) - u(t)\| \leq \sup_{0 \leq \tau \leq b_0(\rho)} \|u(\tau; u(h)) - u(\tau; u_0)\|
\]

\[
\leq L\|u(h) - u_0\| \leq L\|T(h)u_0 - u_0\| + L\left| \int_0^h T(h - s)F(u(s)) \, ds \right|\]

\[
\leq L\left| \int_0^h T(\tau)Au_0 \, d\tau \right| + LCM e^{\omega h} h \leq LM e^{\omega(\tau + L\|A\|)h}
\]

for all \( 0 \leq t \leq \tau \leq \tau_1 \), where \( M \) and \( \omega \) are as in Theorem 5.1 and \( L = 2M e^{\omega b_0(\rho)} \).

With a similar procedure as in step 2) of the proof of Theorem 5.1 we can thus now find a strictly increasing sequence \( \{\tau_{n}\}_{n=0}^{\infty} \) in \( [0, b(u_0)) \) converging to \( b(u_0) \) with \( \tau_0 = 0 \) such that \( u \) is Lipschitz-continuous on \( [\tau_{j-1}, \tau_j] \) with some Lipschitz constant \( L_j \). Consider \( 0 \leq t \leq s < b(u_0) \).

There are uniquely determined integers \( 0 \leq j < n \) such that \( \tau_j \leq t < \tau_{j+1} \leq \tau_n \leq s < \tau_{n+1} \).

We then conclude

\[
\|u(t) - u(s)\| \leq \|u(t) - u(\tau_{j+1})\| + \sum_{k=j+1}^{n-1} \|u(\tau_k) - u(\tau_{k+1})\| + \|u(\tau_n) - u(s)\|
\]

\[
\leq L_{j+1}|t - \tau_{j+1}| + \sum_{k=j+1}^{n-1} L_{k+1}|\tau_k - \tau_{k+1}| + L_{n+1}|\tau_n - s|
\]

\[
\leq \max_{k=j+1,\ldots,n+1} L_k \left( \tau_{j+1} - t + \sum_{k=j+1}^{n-1} (\tau_{k+1} - \tau_k) + s - \tau_n \right) = \max_{k=j+1,\ldots,n+1} L_k |t - s|
\]

which finally shows the required local Lipschitz property of \( u \). \qed
Let $X$ and $Y$ be Banach spaces and $\emptyset \neq D \subseteq X$ be open. A map $F : D \rightarrow Y$ is called (Fréchet) differentiable at $x_0 \in D$ if there is an operator $S \in \mathcal{B}(X,Y)$ such that

$$
\lim_{D-x_0 \ni h \to 0} \frac{1}{\|h\|} \|F(x_0 + h) - F(x_0) - Sh\| = 0
$$

holds. Then we set $F'(x_0) := S$ and call it the derivative of $F$ at $x_0$. We say that $F$ is continuously differentiable on $D$ if $F$ is differentiable at each point of $D$ and the function

$$
F' : D \rightarrow \mathcal{B}(X,Y); \quad x \mapsto F'(x)
$$

is continuous. In this case we write $F \in \mathcal{C}^1(D,Y)$. The usual rules of calculus (including the implicit function theorem) hold in this setting with analogous proofs and straightforward modifications. (See, e.g., [A-E II] or III.5 in [W]). If $D$ is convex and $F \in \mathcal{C}^1(D,Y)$, we then have

$$
F(z) - F(x) = \int_0^1 \frac{d}{dt} F(x + t(z - x)) \, dt = \int_0^1 F'(x + t(z - x))(z - x) \, dt
$$

and we thus obtain in this situation

$$
\|F(z) - F(x)\| \leq \max_{0 \leq t \leq 1} \|F'(x + t(z - x))\| \cdot \|z - x\|
$$

for all $z, x \in D$. As a result, a function $F \in \mathcal{C}^1(D,Y)$ is locally Lipschitz continuous provided that its derivative is locally bounded.

**Lemma 5.3**

Let $u : [a,b) \rightarrow X$ be differentiable from the right with continuous right-hand side derivative $v : [a,b) \rightarrow X$. Then $u \in \mathcal{C}^1([a,b), X)$ (with $u' = v$).

**Proof.** Let $x^* \in X^*$. Then $\varphi(t) = \langle u(t), x^* \rangle$ satisfies the assumptions for $X = \mathbb{C}$ (with right-hand derivative $x^* \circ v$). Corollary 2.1.2 of [P] (see also part (b) in Example 1.15) implies $\varphi \in \mathcal{C}^1([0,b))$ with $\varphi' = x^* \circ v$. Fix $h \in (0, b-a)$ and take $t \in [a+h, b)$. Due to the Hahn-Banach theorem, there exists a $x_h^* \in X^*$ such that $\|x_h^*\| \leq 1$ and

$$
\left| - \frac{1}{h} (u(t) - u(t-h)) - v(t), x_h^* \right| = \left| - \frac{1}{h} (u(t) - u(t-h)) - v(t) \right| = : D_h(t).
$$

Setting $\varphi_h(t) := \langle u(t), x_h^* \rangle$, we then compute

$$
D_h(t) = \left| \frac{1}{h} (\varphi_h(t-h) - \varphi_h(t)) - \varphi'_h(t) \right| = \left| \frac{1}{h} \int_{t-h}^t (\varphi'_h(\tau) - \varphi'_h(t)) \, d\tau \right| = \left| \frac{1}{h} \int_{t-h}^t (v(\tau) - v(t), x_h^*) \, d\tau \right|
$$

$$
\leq \frac{h}{h} \max_{t-h \leq \tau \leq t} \|v(\tau) - v(t)\|.
$$

Since the right side tends to 0 as $h \to 0$, we obtain that $u$ is differentiable at any $t \in [a,b)$ with (continuous) derivative $v$. \qed
Theorem 5.4
Let $A$ generate the $C_0$-semigroup $T(\cdot)$ and let $F \in \mathcal{C}^1(X)$ have a locally bounded derivative. For each $u_0 \in \mathcal{D}(A)$, the mild solution of (5.1) on $[0, b(u_0))$ given by Theorem 5.1 is a classical one.

Proof. 1) We have to show that $u \in \mathcal{C}^1([0, b(u_0)), X)$ since then $F \circ u \in \mathcal{C}^1([0, b(u_0)), X)$ and thus the assertion will follow from Lemma 2.8 and (5.2). Fix any $b \in (0, b(u_0))$. The operators $B(s) := F'(u(s))$ depend continuously on $s \in [0, b]$ and $L := \sup_{0 \leq s \leq b} \|B(s)\| < \infty$. The (linear nonautonomous) problem
\begin{equation}
(5.6) \quad v(t) = T(t)(F(u_0) + Au_0) + \int_0^t T(t - s)B(s)v(s) \, ds
\end{equation}
can be solved as in part 1) of the proof of Theorem 5.1 for $x \in [0, b]$ by a fixed point argument on $\mathcal{C}([0, b], X)$ (using that $B(s)$ is globally Lipschitz with Lipschitz constant $L$). Since (5.3) is not needed here, equation (5.4) allows to choose $b_0 = \min\{1, \frac{1}{2M\omega_L^{1/2}}\}$ independently of the initial value. As a result, we can solve (5.6) on $[0, b_0]$ with $F(u_0) + Av_0$ replaced by $v(b_0)$ and thus obtain a solution of (5.6) on $[0, 2b_0]$ as in part 2) of the proof of the Theorem 5.1. In finitely many steps we then construct a solution $v \in \mathcal{C}([0, b], X)$ of (5.6).

2) We now show that $v$ is the derivative of $u$. Let $0 \leq t \leq t + h \leq b$ for some $h > 0$. Equations (5.2) and (5.6) imply that
\begin{align*}
w_h(t) := & \frac{1}{h}(u(t + h) - u(t)) - v(t) \\
= & \left[ T(t)\frac{1}{h}T(h) - I \right]u_0 - T(t)Au_0 + \left[ \frac{1}{h} \int_0^h T(t + h - s)F(u(s)) \, ds - T(t)F(u_0) \right] \\
& + \int_0^t T(t - s) \left( \frac{1}{h} (F(u(s + h)) - F(u(s))) - F'(u(s))v(s) \right) \, ds \\
=: & S_1(h, t) + S_2(h, t) + S_3(h, t).
\end{align*}
First, we observe that
\[ \|S_1(h, t)\| \leq M e^{\omega b} \left\| \frac{1}{h}T(h) - I \right\| u_0 - Au_0 \] =: \alpha_1(h) \to 0
and that
\[ \|S_2(h, t)\| \leq \frac{1}{h} \left\| T(t) \int_0^h (T(h - s)F(u(s)) - F(u_0)) \, ds \right\| \]
\[ \leq M e^{\omega b} \frac{1}{h} \sup_{0 \leq s \leq h} \|T(h - s)F(u(s)) - F(u_0)\| \] =: \alpha_2(h) \to 0

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as \( h \to 0^+ \), where we used \( u_0 \in \mathcal{D}(A) \) for the computation of the first limit and Lemma 1.8 for the second one. Next, we write

\[
S_3(h,t) = \int_0^t T(t-s) \frac{1}{h} [F(u(s+h)) - F(u(s)) - F'(u(s))(u(s+h) - u(s))] \, ds \\
+ \int_0^t T(t-s) F'(u(s)) w_h(s) \, ds =: S_{3,1}(h,t) + S_{3,2}(h,t).
\]

Using Corollary 5.2 as well as (5.5), we infer that

\[
\| S_{3,1}(h,t) \| \leq M e^{\omega b} \max_{0 \leq \tau \leq 1} \left\| \int_0^1 \left[ F'(u(s) + \tau(u(s+h) - u(s))) - F'(u(s)) \right] (u(s+h) - u(s)) \, d\tau \right\|
\]

\[
\leq C(b) \frac{h}{h} \max_{0 \leq \tau \leq 1} \| F'(u(s) + \tau(u(s+h) - u(s))) - F'(u(s)) \| =: \alpha_3(h)
\]

which tends to 0 as \( h \to 0^+ \) since \( F' \) is uniformly continuous on the compact set

\[
\{ u(s) + \tau(u(r) - u(s)) ; 0 \leq \tau \leq 1, 0 \leq r, s \leq b \}.
\]

Hence,

\[
\| w_h(t) \| \leq \alpha_1(h) + \alpha_2(h) + \alpha_3(h) + M e^{\omega b} L \int_0^t \| w_h(s) \| \, ds,
\]

which leads to

\[
\| w_h(t) \| \leq (\alpha_1(h) + \alpha_2(h) + \alpha_3(h)) e^{M e^{\omega b} L}
\]

for all \( t \in [0,b] \) thanks to Gronwall's inequality. Letting \( h \to 0^+ \) now shows that \( u \) is differentiable from the right and the right-hand side derivative coincides with \( v \). Since \( v \) is continuous on \([0,b], \) Lemma 5.3 yields \( u \in C^1([0,b), X) \). The assertion now follows since \( b \in (0,b(u_0)) \) is arbitrary.

**Example 5.5** (scalar reaction diffusion equation)

Let \( \emptyset \neq U \subseteq \mathbb{R}^d \) be open and bounded with \( C^2 \) boundary. Let \( E = C_0(U) \) and \( Au = \Delta u \) with \( \mathcal{D}(A) = \{ u \in \bigcap_{p>1} W^2_p(U) ; u, \Delta u \in E \} \). Let \( f : \mathbb{C} \to \mathbb{C} \) be locally Lipschitz with constant \( L(r) \) on \( \overline{B}(0,r) \). Take \( u, v \in E \) with \( \| u \|_\infty, \| v \|_\infty \leq r \) for some \( r > 0 \). Define \( F(u) = f \circ u : U \to \mathbb{C} \). Then \( F(u) \in C_0(U) \) and

\[
\| F(u) - F(v) \|_\infty = \sup_{x \in U} | f(u(x)) - f(v(x)) | \leq \sup_{x \in U} L(r) | u(x) - v(x) | = L(r) \| u - v \|_\infty
\]

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so that $F : E \to E$ is locally Lipschitz. Next, let $f$ be (real) differentiable with a continuous derivative $f' : \mathbb{C} \to \mathbb{C}$. Define $S_u v = f'(u)v$. We then compute
\[
D(v) := \|F(u + v) - F(u) - S_u v\|_{\infty} = \sup_{x \in U} |f(u(x) + v(x)) - f(u(x)) - f'(u(x))v(x)|
\]
\[
= \sup_{x \in U} \left| \int_0^1 (f'(u(x) + \tau v(x)) - f'(u(x)))v(x) \, d\tau \right| \leq \|v\|_{\infty} \sup_{|s| \leq r} \|f'(s+t) - f'(s)\|.
\]
Since $f'$ is uniformly continuous on compact sets, we deduce that \( \frac{1}{\|v\|_{\infty}} D(v) \to 0 \) as $\|v\|_{\infty} \to 0$, and so $F$ is differentiable with $F'(u)v = f'(u)v$. The uniform continuity of $f'$ on $\overline{B}(0, r)$ further implies that
\[
\|F'(u)w - F'(v)w\|_{\infty} = \sup_{x \in U} |f'(u(x)) - f'(v(x))| \cdot \|w\|_{\infty}
\]
tend to 0 as $v \to u$ in $E$, uniformly for $w \in E$ with $\|w\|_{\infty} \leq 1$. Hence, $F \in \mathcal{C}^1(E)$. Similarly, one sees that $F'$ is locally bounded. As a result, we can apply Theorem 5.1 and 5.4 to the problem
\[
\begin{cases}
u'(t) = \Delta u(t) + f(u(t)), & \text{on } U, \quad t > 0, \\
u(t) = 0, & \text{on } \partial U, \quad t > 0, \\
u(0) = u_0 \in E.
\end{cases}
\]
(5.7)

Hence, the evolution equation (5.1) corresponding to (5.7) has a unique local mild solution which is classical if $u_0 \in \mathcal{D}(A)$. In fact, Proposition 5.8 below shows that the mild solution is a classical one on $(0, b(u_0))$ for every $u_0 \in E$.

In order to establish global existence, we have to turn our attention to specific situations. As a simple example we consider $g(s) = s(1 - s)$. Let $0 \leq u_0 \leq 1$. We claim that then $0 \leq u(t) \leq 1$ as long as the mild solution of (5.7) with $f = g$ exists. Theorem 5.1 then shows that (5.7) has a global positive solution if $f = g$ and $0 \leq u_0 \leq 1$. To prove positivity, suppose that $u(t_0, x_0) < 0$ for some $t_0 \in [0, b(u_0))$ and $x \in \overline{U}$. Fix some $b \in (t_0, b(u_0))$ and $\lambda > \max\{1 - u(t, x); t \in [t_0, b], x \in \overline{U}\}$. Set $V := \{(t, x) \in (0, b] \times U; u(t, x) < 0\}$ and $v := -e_{-\lambda}u$. Observe that $v > 0$ on $V \neq \emptyset$ and we have $v = 0$ on $\overline{V} \setminus V$ due to the initial and boundary conditions in (5.7). Equation (5.7) further yields
\[
(\partial_t - \Delta)v = -e_{-\lambda}u(1 - u) + \lambda e_{-\lambda}u = e_{-\lambda}u(\lambda - (1 - u)) \leq e_{-\lambda}u(\lambda - \sup(1 - u)) < 0 \text{ on } V.
\]
There is a maximum $v(t_1, x_1) = 0$ of $v$ on $\overline{V}$. Since $(t_1, x_1)$ must belong to $V$, Proposition 3.1.10 of [L] yields $(\partial_t - \Delta)v(t_1, x_1) \geq 0$, which is a contradiction. Hence, $V = \emptyset$ and so $u \geq 0$.

(Here we only need $u_0 \geq 0$ and use that $v \in \mathcal{C}^1((0, b], E) \cap \mathcal{C}([0, b], \mathcal{D}(A)).$) To prove that $u \leq 1$, suppose there are $t_2 \in [0, b(u_0))$ and $x_2 \in \overline{U}$ such that $u(t_2, x_2) > 1$. Choose $\beta \in (t_2, b(u_0))$ and set $W := \{(t, x) \in (0, \beta] \times U; u(t, x) > 1\} \neq \emptyset$ and $w := u - 1$. Then $w > 0$ on $W$ and $w = 0$ on $\overline{W} \setminus W$. Equation (5.7) now implies
\[
(\partial_t - \Delta)w = (\partial_t - \Delta)u = u(1 - u) < 0 \quad \text{on } W.
\]

\[\text{In what follows, we set of course } h|_{\partial U} := 0 \text{ for } h \in E.\]
On the other hand, \( w \) attains its maximum at a point \((t_3, x_3) \in \overline{W}\). Hence, \((\partial_t - \Delta)w(t_3, x_3) \geq 0\) as above. This contradiction forces that \( u \leq 1 \).

**Example 5.6** (nonlinear wave equation)

Let \( \emptyset \neq U \subseteq \mathbb{R}^3 \) be open and bounded with \( \mathcal{C}^2 \) boundary. For \( a \in \mathbb{R} \), \( u_0 \in Y := W^1_2(U) \) and \( u_1 \in L^2(U) \) we consider the problem

\[
\begin{cases}
\partial_t u(t) = \Delta u(t) - au(t)\|u(t)\|^2, & \text{on } U, \quad t > 0, \\
u(t) = 0, & \text{on } \partial U, \quad t > 0, \\
u(0) = u_0, \quad \partial_t u(0) = u_1.
\end{cases}
\]

We rewrite it as the evolution equation (5.1) on \( E = Y \times L^2(U) \) using \( \Delta_D u = \Delta u \), \( \mathcal{D}(\Delta_D) = W^2_2(U) \cap Y \), \( A = \left( \begin{array}{cc} 0 \\ \Delta_D \\ \end{array} \right) \) with \( \mathcal{D}(A) = \mathcal{D}(\Delta_D) \times Y \), and \( F(u, v) = (0, -au\|u\|^2) =: (0, F_0(u)) \) for \((u, v) \in E\). Note that \( W^2_2(U) \hookrightarrow L^6(U) \) by the (sharp) Sobolev embedding since \( 1 - \frac{3}{2} = -\frac{3}{6} \) (see, e.g., Corollary 3.22 of [ST]). Hence, \( F(u, v) \in E \). For \( u, w \in Y \) with \( \|u\|_{1,2}, \|w\|_{1,2} \leq r \) we obtain

\[
\|u\|^2 - w\|w\|^2 \leq |u - w| \cdot |u|^2 + |w| \cdot \|u\|^2 - \|w\|^2 \\
\leq |u - w| \cdot |u|^2 + |w| \cdot (\|u\| - |w|)|\|u\| + |w|| \\
\leq |u - w| \cdot |u|^2 + |u - w|(|u| \cdot |w| + \|w\|^2) \leq \frac{3}{2} (\|u\|^2 + \|w\|^2)|u - w|
\]

and

\[
\|u\|^2 - w\|w\|^2 \leq 3 \|u - w\|_6 \cdot \|u\|^2 + \|w\|^2 \leq 3 \left( \|u\|^2_6 + \|w\|^2 \right) \|u - w\|_6 \leq cr^2 \|u - w\|_{1,2}
\]

using Hölder’s inequality and the Sobolev embedding. Hence, \( F_0 : Y \rightarrow L^2(U) \) is locally Lipschitz so that Theorem 5.1 now gives a unique local mild solution of (5.8). (We point out that the map \( u \mapsto u\|u\|^2 \) does not act from \( L^2(U) \) to \( L^2(U) \).) One can also check that \( F \in \mathcal{C}^1(E) \) and thus this solution is a classical one if \( (u_0, u_1) \in \mathcal{D}(A) \). We compute \( F' \) in the real case. For \( u, w \in Y \) we then obtain above

\[
\|(u + w)^3 - u^3 - 3u^2w\|_2 = 3\|uw^2 + w^3\| \leq 3\|u\|_6 \cdot \|w^2\|_3 + \|w^3\|_2 = 3\|u\|_6 \cdot \|w^2\|_6 + \|w\|_6^3 \\
\leq c\|w\|_1,2 (\|u\|_{1,2} \cdot \|w\|_{1,2} + \|w\|^2_{1,2})
\]

Therefore \( F_0 : Y \rightarrow L^2(U) \) is differentiable with \( F'_0(u)w = 3u^2w \). As a result, \( F \) is differentiable in \( E \) with

\[
F'(u, v)(w, z) = (0, 3u^2w)
\]

for all \((u, v), (w, z) \in E\).

We show global existence for \( a > 0 \), where we suppose that \((u_0, u_1) \in \mathcal{D}(A)\) are real for simplicity. Let \((u, u') \) be the corresponding classical solution of (5.8) on \( J := [0, b(u_0, u_1)) \). Then \( \text{Im} \ u \) is also a classical solution with the initial value 0. Thus, \( \text{Im} \ u = 0 \), i.e., \( u \) is real. We define the “energy” of \( u \) by
As above one sees that the map \( \varphi(u) = \frac{1}{k} \int_\Omega u^k \, dx \) belongs to \( C^1(L^k(U), \mathbb{R}) \) for \( k \in \mathbb{N} \) with \( k \geq 2 \), where the derivative is given by

\[
\varphi'(u)v = \int_\Omega u^{k-1} v \, dx
\]

for \( u, v \in L^k(U) \). Since \( u \in \mathcal{C}^2(J, L^2(U)) \cap \mathcal{C}^1(J, Y) \cap \mathcal{C}(J, W^2_2(U)) \) (cf. Example 2.4), the chain rule, the embedding \( Y \hookrightarrow L^4(U) \) and integration by parts yield that \( E \in C^1(J) \) and

\[
E'(t) = \int_\Omega (\partial_t u(t) \partial_t u(t) + \nabla u(t) \cdot \nabla \partial_t u(t) + au(t)^3 \partial_t u(t)) \, dx
\]

\[
= \int_\Omega (\partial_{tt} u(t) - \Delta u(t) + au(t)^3) \partial_t u(t) \, dx = 0
\]

for all \( t \in J \). It follows that

\[
\frac{1}{2} \|(u(t), u'(t))\|^2_E \leq E(t) = E(0) = \frac{1}{2} \|(u_0, u_1)\|^2_E + \frac{a}{4} \| u_0 \|^4_1 = \text{const}.
\]

Theorem 5.1 now implies that \( b(u_0, u_1) = \infty \).

**Theorem 5.7** (principle of linearized stability)

*Let A generate the \( C_0 \)-semigroup \( T(\cdot) \) on \( X \) and let \( F : X \to X \) be locally Lipschitz and differentiable at \( x = 0 \) with \( F(0) = 0 \). If \( \omega_0(A + F'(0)) =: \omega_0 < 0 \), then for each \( \gamma \in (0, -\omega_0) \) there are constants \( c, \rho > 0 \) such that for all \( u_0 \in \overline{B}(0, \rho) \) we have \( b(u_0) = \infty \) and

\[
\|u(t)\| \leq ce^{-\gamma t} \|u_0\|, \quad t \geq 0,
\]

for the mild solution \( u \) of (5.1).*

**Remark**

Observe that \( A + F'(0) \) with domain \( \mathcal{D}(A) \) is a generator because of \( F'(0) \in \mathcal{B}(X) \), see Theorem 3.4. An analogous result holds if \( Au_0 + F(u_0) = 0 \) for some \( u_0 \in \mathcal{D}(A) \). We remark that then \( u(t) = u_0 \) for all \( t \geq 0 \) gives a stationary solution of (5.1). The condition \( \omega_0 < 0 \) can be checked by means of Gearhart’s stability theorem 4.4 provided that \( X \) is a Hilbert space or by the equality \( s(A + F'(0)) = \omega_0 \) if \( T(\cdot) \) is analytic, see Corollary 4.14 and Theorem 3.8. The above theorem says that the equilibrium \( u_0 = 0 \) attracts all solutions starting nearby even exponentially if the “linearization at 0” of (5.1) is exponentially stable.

**Proof.** Let \( \gamma \in (0, -\omega_0) \) and fix constants \( M \geq 1 \) and \( \delta \in (\gamma, -\omega_0) \) such that \( \| S(t) \| \leq M e^{-\delta t} \) for all \( t \geq 0 \), where \( S(\cdot) \) is generated by \( A + F'(0) \) with domain \( \mathcal{D}(A) \). Equations (5.2) and (3.4) yield
5.1. THE CASE OF A GENERATOR

\[ u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s)) \, ds \]

\[ = S(t)u_0 - \int_0^t S(t-s)F'(0)T(s)u_0 \, ds + \int_0^t S(t-s)F(u(s)) \, ds \]

\[ - \int_0^t \int_0^t S(t-\tau)F'(0)T(\tau-s)F(u(s)) \, d\tau \, ds \]

\[ = S(t)u_0 + \int_0^t S(t-\tau)(F(u(\tau)) - F'(0)T(\tau)u_0) \, d\tau \]

\[ - \int_0^t S(t-\tau)F'(0) \int_0^\tau T(\tau-s)F(u(s)) \, ds \, d\tau \]

\[ = S(t)u_0 + \int_0^t S(t-\tau)(F(u(\tau)) - F(0) - F'(0)u(\tau)) \, d\tau \]

for all \( t \in [0, b(u_0)) \), where we used Fubini’s theorem and \( F'(0) = 0 \). We set \( G(x) = F(x) - F(0) - F'(0)x \) for \( x \in X \). Fix \( \epsilon := \frac{\delta - \gamma}{M} \), i.e., \( \epsilon M - \delta = -\gamma \). By assumption there exists \( r > 0 \) such that \( \|G(x)\| \leq \epsilon \|x\| \) for all \( x \in X \) with \( \|x\| \leq r \). Set \( \rho := \frac{r}{2M} < r \). Let \( u_0 \in \overline{B}(0, \rho) \) and \( \tau := \sup\{t \in (0, b(u_0)); \sup_{0 \leq s \leq t} \|u(s)\| \leq r\} \in (0, b(u_0)) \). We can then estimate

\[ \| e^{\delta t} u(t) \| \leq \| e^{\delta t} S(t)u_0 \| + \int_0^t e^{\delta(t-s)} \| S(t-s) \| e^{\delta s} \| G(u(s)) \| \, ds \]

\[ \leq M \| u_0 \| + M \epsilon \int_0^t e^{\delta s} \| u(s) \| \, ds \]

for all \( 0 \leq t < \tau \) so that Gronwall’s inequality yields that

\[ \| e^{\delta t} u(t) \| \leq M \| u_0 \| e^{\epsilon M t} \]

and thus

\[ \| u(t) \| \leq M \| u_0 \| e^{-\gamma t} \leq M \rho e^{-\gamma t} \leq \frac{r}{2} \]

for all \( 0 \leq t < \tau \). If \( \tau < \infty \), then \( \tau < b(u_0) \) since for \( b(u_0) = \tau < \infty \) the blow up condition in Theorem 5.1 would be violated. Hence, \( \| u(\tau) \| \leq \frac{r}{2} \) by the above estimate which contradicts the definition of \( \tau \). Thus, \( \tau = b(u_0) = \infty \) and the theorem is established. \( \square \)
5.2. The analytic case

Proposition 5.8
Let $A$ generate an analytic semigroup $T(\cdot)$ on $X$ and $F : X \to X$ be locally Lipschitz. Then, for each $u_0 \in X$ the mild solution $u$ of (5.1) obtained in Theorem 5.1 is a classical one on $(0, b(u_0))$. If $u_0 \in \mathcal{D}(A)$, then it is a classical one on $[0, b(u_0))$.

Proof. Let $0 < \epsilon < b < b(u_0)$. Equation (5.2) and Theorem 2.2 (a) imply that $u \in C^\beta((\epsilon, b], X)$ for each $\beta \in (0, 1)$. Since $F$ is locally Lipschitz, $F \circ u$ is also Hölder continuous on $[\epsilon, b]$. As in the proof of Corollary 5.2 one sees that

$$u(t) = T(t - \epsilon)u(\epsilon) + \int_\epsilon^t T(t - s)F(u(s)) \, ds = T(t - \epsilon)u(\epsilon) + \int_0^{t - \epsilon} T(t - \epsilon - r)F(u(r + \epsilon)) \, dr$$

holds for all $t \in [\epsilon, b]$. Now Theorem 2.20 (b) shows that $u \in C^1((\epsilon, b], X) \cap C((\epsilon, b], [\mathcal{D}(A)])$ satisfies $u' = Au + F(u)$ on $[\epsilon, b]$. Since $0 < \epsilon < b < b(u_0)$ are arbitrary, the first assertion is shown. The second one also follows from Theorem 2.20, where we now may take $\epsilon = 0$. □

Let $A$ generate the analytic $C_0$-semigroup $T(\cdot)$. We want to use the regularity properties of $T(\cdot)$ to allow a more irregular nonlinearity $F$. For this purpose we assume that there is a Banach space $Y$ such that $[\mathcal{D}(A)] \hookrightarrow Y \hookrightarrow X$, $\mathcal{D}(A)$ is dense in $Y$ and there are constants $c_0 > 0$ and $\alpha \in [0, 1)$ such that

$$(5.9) \quad \|y\|_Y \leq c_0 \|y\|_A^\alpha \cdot \|y\|_X^{1 - \alpha}$$

for all $y \in \mathcal{D}(A)$. This “interpolative estimate” implies that

$$\|T(t)y\|_Y \leq c_0 \|((A + \omega I)T(t)y\| + \|T(t)y\|_A^\alpha \|T(t)y\|_X^{1 - \alpha}$$

$$\leq c \left( \left( \frac{1}{t} e^{\omega t} \|y\| + e^{\omega t} \|y\|_A^\alpha \right) (e^{\omega t} \|y\|)_1^{1 - \alpha} \leq c e^{\omega t} \max\{1, t^{-\alpha}\} \|y\|$$

for all $t > 0$ and $y \in \mathcal{D}(A)$ and some constants $c > 0$ (possibly varying from inequality to inequality) and $\omega \in \mathbb{R}$. Consequently, there exist $N, \gamma > 0$ such that

$$(5.10) \quad \|T(t)\|_{B(X,Y)} \leq N t^{-\alpha} e^{\gamma t}$$

for all $t > 0$. We further assume that

$$(5.11) \quad T(t)Y \subseteq Y \quad \text{and} \quad \|T(t)\|_{B(Y)} \leq c(t_0)$$

for all $0 \leq t \leq t_0$ and some constant $c(t_0)$. Clearly, $T(\cdot)|_Y$ is a semigroup on $Y$. It is strongly continuous on $Y$ since $\mathcal{D}(A)$ is dense in $Y$ and

$$\|T(t)y - y\|_Y \leq c_0 \|T(t)y - y\|_A^\alpha \cdot \|T(t)y - y\|_X^{1 - \alpha} \leq c \|y\|_A^\alpha \cdot \|T(t)y - y\|_X^{1 - \alpha} \to 0$$

as $t \to 0$ for all $y \in \mathcal{D}(A)$ and some constants $c$ (where we may assume that $t \in [0, 1]$).

Theorem 5.9
Let $A$ generate an analytic semigroup $T(\cdot)$ and $Y$ be a Banach space such that $[\mathcal{D}(A)] \hookrightarrow Y \hookrightarrow X$, $\mathcal{D}(A)$ is dense in $Y$ and (5.9) and (5.11) hold. Let $F : Y \to X$ be locally Lipschitz
continuous. Take any \( \rho > 0 \) and \( u_0 \in Y \) with \( \| u_0 \|_Y \leq \rho \). Then there are numbers \( b(u_0) \geq b_0(\rho) > 0 \) and a unique function \( u = u(\cdot ; u_0) \in C([0, b(u_0)), Y) \) satisfying

\[
(5.12) \quad u(t) = T(t)u_0 + \int_0^t T(t - s)F(u(s)) \, ds
\]

for all \( 0 \leq t < b(u_0) \). The map

\[
\overline{B}_Y(0, \rho) \to C([0, b_0(\rho)], Y); \quad u_0 \mapsto u(\cdot ; u_0)|_{[0, b_0(\rho)]}
\]

is Lipschitz continuous. If \( b(u_0) < \infty \), then \( \lim \sup_{t \to b(u_0)} \| u(t) \|_Y = \infty \).

**Remark**

For an appropriate choice of \( Y \), it can be proved that the solution in Theorem 5.9 is a classical solution on \((0, b(u_0))\), and on \([0, b(u_0))\) if \( u_0 \in \mathcal{D}(A) \), see Section 7.1 of \([L]\).

**Proof.** Fix \( \rho > 0 \) and take \( u_0 \in Y \) with \( \| u_0 \|_Y \leq \rho \). Fix \( r > c(1)\rho \) where \( c(1) \) is given by (5.11). Let \( L(r) \) be the Lipschitz constant of \( F \) on \( \overline{B}_Y(0, r) \). For \( 0 < b \leq 1 \), we define

\[
E(b) := \left\{ u \in C([0, b], Y); \| u \|_{\infty, Y} := \sup_{0 \leq t \leq b} \| u(t) \|_Y \leq r \right\}
\]

and

\[
\Phi(u)(t) = T(t)u_0 + \int_0^t T(t - s)F(u(s)) \, ds
\]

where \( u \in E(b) \) and \( 0 \leq t \leq b \). We observe that \( F \circ u \in C([0, b], X) \). Using (5.10) and (5.11), we estimate

\[
\| \Phi(u)(t) \|_Y \leq c(1)\| u_0 \|_Y + \int_0^t N e^{\gamma(t-s)}(t-s)^{-\alpha}(\| F(u(s)) - F(0) \|_X + \| F(0) \|_X) \, ds
\]

\[
\leq c(1)\rho + \frac{N e^\gamma t^{1-\alpha}}{1 - \alpha} \left( \max_{0 \leq s \leq b} \| u(s) \|_Y + \| F(0) \|_X \right)
\]

\[
\leq c(1)\rho + \frac{N e^\gamma}{1 - \alpha} (L(r)r + \| F(0) \|_X) b^{1-\alpha}
\]

as well as

\[
\| \Phi(u)(t) - \Phi(v)(t) \|_Y \leq \int_0^t N e^{\gamma(t-s)}(t-s)^{-\alpha}\| F(u(s)) - F(v(s)) \|_X \, ds
\]

\[
\leq \frac{N e^\gamma}{1 - \alpha} L(r)b^{1-\alpha}\| u - v \|_{\infty, Y}
\]

for all \( u, v \in E(b) \) and \( 0 \leq t \leq b \). Choosing a sufficiently small \( b_0 = b_0(\rho) \in (0, 1] \) we see that \( \Phi \) is a strictly contractive mapping on \( E(b_0) \) with the metric induce by \( \| \cdot \|_{\infty, Y} \). Therefore, there
exists a unique fixed point \( u(\cdot ; u_0) = \Phi(u(\cdot ; u_0)) \in E(b) \) (due to Banach’s fixed point theorem), i.e., (5.12) holds. The remaining assertions can be shown as in the proof of Theorem 5.1. □

**Example 5.10**

Let \( \emptyset \neq U \subseteq \mathbb{R}^d \) be open and bounded with \( C^2 \) boundary and \( E = L^p(U) \) for \( p > d \). Set \( Au = \Delta u \) with \( \mathcal{D}(A) = W^2_p(U) \cap W^1_p(U) \). Let \( f : \mathbb{C} \to \mathbb{C} \) be locally Lipschitz. Set \( Y = W^1_p(U) \).

Then \( \mathcal{D}(A) \hookrightarrow Y \hookrightarrow E \) and \( \mathcal{D}(A) \) is dense in \( Y \). Estimate (5.9) holds due to, e.g., Proposition 3.27 in [ST] and its proof. It can be seen that (5.11) holds since \( T(t) \) is bounded on \( E \) and \( \mathcal{D}(A) \approx W^2_p(U) \cap W^1_p(U) \) and \( Y \) is the “complex interpolation space” between \( E \) and \( \mathcal{D}(A) \), cf. Example IV.2.6.3 of [A] and Chapter 2 of [L-Int]. Sobolev’s embedding shows that \( Y \hookrightarrow C_0(U) \hookrightarrow E \) since \( p > d \). Example 5.5 says that the mapping

\[
F_\infty : C_0(U) \to C_0(U); \quad u \mapsto f \circ u
\]
is locally Lipschitz. Hence, \( F := J_2 F_\infty J_1 \) is locally Lipschitz from \( Y \) to \( E \) (and even \( C^1 \) if \( f \) is \( C^1 \)). So we can apply Theorem 5.9 to \( \Delta D \) and \( F \) on \( L^p(U) \) solving

\[
\begin{aligned}
\partial_t u(t) &= \Delta u(t) + f(u(t)), & \text{on } U, \quad t > 0, \\
u(t) &= 0, & \text{on } \partial U, \quad t > 0, \\
u(0) &= u_0 \in \mathcal{D}(A).
\end{aligned}
\]
Bibliography


