Problem 1 (Numerical Simulation of SDEs: Euler-Maruyama method) 3 Points

In Part I of the lecture, the price of the underlying asset was modeled by the Geometric Brownian Motion (GBM)

\[ S(t) = S_0 \exp(\alpha t + \sigma W(t)), \quad \alpha = \mu - \sigma^2/2, \]

with \( \mu, \sigma > 0 \), initial condition \( S_0 \geq 0 \) and \( W(t), \ t \in [0, T] \) denoting the Wiener process. Furthermore, the process \( S(t) \) is the solution of the SDE

\[ dS(t) = \mu S(t) dt + \sigma S(t) dW(t), \quad t \in [0, T], \quad S(0) = S_0. \]

A numerical simulation of this SDE can be obtained by employing the Euler-Maruyama method. Choose \( N \in \mathbb{N} \) and define step-size \( \tau = T/N \). For \( n = 1, \ldots, N \) let \( t_n = n\tau \) and compute the approximation \( S_n \approx S(t_n) \) as

\[ S_n = S_{n-1} + \mu S_{n-1} \tau + \sigma S_{n-1} \Delta W_{n-1} \]

with \( \Delta W_{n-1} = W(t_n) - W(t_{n-1}) \).

(a) Write a MATLAB program that implements the Euler-Maruyama method for the GBM. As a first step you should numerically simulate the Wiener process. The corresponding pseudocode is:

choose \( N \) and define step-size \( \tau = T/N \)
set \( W_0 = 0 \)
for \( n = 0, \ldots, N - 1 \)
generate random number \( Z_n \sim \mathcal{N}(0,1) \)
\[ W_{n+1} = W_n + \sqrt{\tau} Z_n \]
endfor.

Try to vectorize your implementation instead of using a for loop. Hint: take a look at the MATLAB functions cumsum and randn.

Next, implement the Euler-Maruyama method and test your program for \( T = 1, \ S_0 = 1, \ \mu = 0.3, \ \sigma = 0.4 \) and different step-sizes \( \tau = \{2^{-8}, 2^{-7}, 2^{-6}\} \). For all tests, use the same discretized Wiener path over \([0, 1]\) with timestep \( 2^{-8} \).

Plot the Euler-Maruyama approximations versus time and compare them with the true solution of the GBM \( S(t) \).

(b) Generate \( m = 10000 \) different Wiener paths over the interval \([0, 10]\) using \( \tau = 10/2^9 \approx 0.0195 \). For each path, apply the Euler-Maruyama method with \( \mu = 0.3, \ \sigma = 0.4 \) and \( S_0 = 1 \). Use different step-sizes \( \tau_i = 2^{-i}, \ i = 0, \ldots, 5 \), and examine the strong order of convergence at \( T = 10 \) for each of the chosen step-sizes. Plot the error versus the step-size in logarithmic scale using the MATLAB function loglog. Confirm the order of strong convergence by plotting a line of slope 1/2 in the same figure.

(c) By sampling over \( m = 500000 \) Wiener paths, examine the weak order of convergence for the Euler-Maruyama method at \( T = 10 \) with \( \mu = 0.3, \ \sigma = 0.4 \) and \( S_0 = 1 \). Use different step-sizes \( \tau_i = 2^{-i}, \ i = 0, \ldots, 5 \) and different paths for each step-size. Similarly to part (b), plot the error versus the step-size and confirm the order of weak convergence by plotting a line with slope 1 in the same figure. For the geometric Brownian motion we have that \( \mathbb{E}(S(t)) = S_0 \exp(\mu t) \).

Note: This is a programming assignment, so there is no one correct implementation of the program besides a working code. However, when implementing your program you should keep in mind the following:

- **Design** your code such that it is reusable and modular.
- **Document** your code. Each function should contain a header describing the Input, Optional and Output parameters as well as a short description of the function. Also add comments where appropriate.

The MATLAB programs can be solved in pairs or alone, and with the assistance of the tutor. The students should present their programs in the exercise class taking place on 13th May, 2013 at the latest.