Exercise 29:
Let the signal $s \in L^2(\mathbb{R})$ be Hölder continuous of order $\alpha \in [0, 1]$, i.e., there exists a $C_H > 0$ with
$$|s(x) - s(y)| \leq C_H |x - y|^{\alpha} \text{ for all } x, y \in \mathbb{R}.$$ Let $B: \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support that satisfies
$$\sum_{k \in \mathbb{Z}} B(\cdot - k) = 1.$$ We define the function $f$ by
$$f(\cdot) = \sum_{k \in \mathbb{Z}} s(hk)B(\cdot - k)$$ with $h > 0$. Show that
$$\|f(\cdot) - s(h \cdot)\|_{L^\infty} \leq C h^\alpha,$$ where $C$ is a positive constant.

Exercise 30:
Let $v \in \ell^1(\mathbb{Z})$ be real-valued. Moreover, let $h, \tilde{h}, g$ and $\tilde{g}$ be perfect reconstruction filters. Verify the following representation of $v$ with $j \in \mathbb{N}_0$:
$$v = (S^\uparrow)^{j+1} S^\uparrow \tilde{h} \ast R(S^\uparrow)^j h \ast v + (S^\uparrow)^{j+1} S^\downarrow \tilde{g} \ast R(S^\uparrow)^j g \ast v.$$ Herein, the operators $R, S^\downarrow$ and $S^\uparrow$ are given as in Exercise 28.

Exercise 31:
Let $\psi$ be a real-valued wavelet such that there exist positive constants $A$ and $B$ with
$$\frac{A}{2\pi} \leq \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \omega)|^2 \leq \frac{B}{2\pi} \text{ for all } \omega \in \mathbb{R}.$$ (a) Verify that
$$A \|f\|_{L^2(\mathbb{R})}^2 \leq \sum_{j \in \mathbb{Z}} 2^{-j} \|\sqrt{c_{\psi}} W_{\psi} f(2^j \cdot, \cdot)\|_{L^2(\mathbb{R})}^2 \leq B \|f\|_{L^2(\mathbb{R})}^2.$$ (b) Show: If $\varphi \in L^2(\mathbb{R})$ satisfies
$$\sum_{j \in \mathbb{Z}} \overline{\hat{\psi}(2^j \omega)} \hat{\varphi}(2^j \omega) = \frac{1}{2\pi} \text{ for all } \omega \in \mathbb{R} \setminus \{0\},$$ then (formally)
$$f = \sqrt{c_{\psi}} \sum_{j \in \mathbb{Z}} 2^{-j} W_{\psi} f(2^j \cdot, \cdot) \ast D^{2j} \varphi.$$ (c) Starting from (1), construct a $\varphi \in L^2(\mathbb{R})$ satisfying (2).

These exercises are discussed in the problem class on Thursday, January 16, 2014.