Exercise 32:
Let \( \varphi \) be an orthogonal scaling function, that is, \( \langle \varphi, \varphi_{0,k} \rangle_{L^2(\mathbb{R})} = \delta_{0,k} \), and let \( h \) be its filter. We define a corresponding wavelet by
\[
\psi = \sum_{k \in \mathbb{Z}} g_k \varphi_{1,k} \quad \text{with} \quad g_k = (-1)^{1-k} h_{2l+1-k}
\]
for any choice of \( l \in \mathbb{Z} \). Show:

(a) \( \sum_{n \in \mathbb{Z}} h_n h_{n-2k} = \delta_{0,k} \),

(b) \( \langle \psi_{0,m}, \psi_{0,k} \rangle_{L^2(\mathbb{R})} = \delta_{m,k} \).

Exercise 33: (Shannon’s sampling theorem)
Let \( f \in L^2(\mathbb{R}) \) be \( b \)-bandlimited, that is, \( \hat{f}(\omega) = 0 \) for almost all \( \omega \in \mathbb{R} \setminus [-b,b] \), where \( b > 0 \). Show that
\[
f(\cdot) = \sum_{k \in \mathbb{Z}} f(\delta k) \operatorname{sinc}\left(\frac{\pi}{\delta}(\cdot - \delta k)\right)
\]
for any positive \( \delta \leq \pi/b \). Hence, a bandlimited signal is completely determined by its discrete samples.

Hint: Expand \( \hat{f} \) in a Fourier series in \( L^2(-\pi/\delta,\pi/\delta) \) and use \( \operatorname{sinc} = \sqrt{\pi/2} \chi_{[-1,1]} \).

Exercise 34:
Let \( h \) be a filter associated with a scaling function \( \varphi \). Prove: If \( H(\omega) \) has a zero of order \( p \) at \( \pi \), then \( \hat{\varphi}^{(l)}(2k\pi) = 0 \) for any \( k \in \mathbb{Z} \setminus \{0\} \) and any \( l < p \).

These exercises are discussed in the problem class on Thursday, January 23, 2014.