Exercise 5:
Verify that the Hermite polynomials $H_n : \mathbb{R} \to \mathbb{R}$ defined by
\[ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n \in \mathbb{N}_0, \]
satisfy the recursion formula
\[ H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \in \mathbb{N}. \]

Exercise 6:
Let $H_n$ be the Hermite polynomial of degree $n \in \mathbb{N}_0$. Prove that the function
\[ h_n(x) = H_n(x) e^{-x^2/2} \]
is an eigenfunction of the Fourier transform with eigenvalue $(-i)^n$, that is,
\[ \hat{h}_n = (-i)^n h_n. \]

Hint: Use the recursion formula and an inductive argument.

Exercise 7:
Show that for $d \in \mathbb{N}$ the identity
\[ \int_{\mathbb{R}^d} e^{-|x|^2/2} \, dx = (2\pi)^{d/2} \]
holds.

Hint: Start calculating $\left( \int_{\mathbb{R}} e^{-t^2/2} \, dt \right)^2$.

Exercise 8:
Let $h \in L^2(\mathbb{R}^2)$ and $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ with $\|\varphi\|_{L^2} = 1$. Show the following statement:
There exists an $f \in L^2(\mathbb{R}^2)$ such that $h = F_\varphi f$ if and only if
\[ h(\xi_0, b_0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} h(\xi, b) K(\xi_0, \xi, b_0, b) \, d\xi \, db, \]
where
\[ K(\xi_0, \xi, b_0, b) = \langle E^{-\xi} T^b \varphi, E^{-\xi_0} T^{b_0} \varphi \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \varphi(t - b) \varphi(t - b_0) e^{-i(\xi_0 - \xi)t} \, dt. \]