Resolution-controlled conductivity discretization in EIT

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Sensitivity of CEM
Optimal conductivity meshes
Adaptive conductivity meshes
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Optimal conductivity meshes

Adaptive conductivity meshes

Sensitivity of CEM
Complete Electrode Model (CEM)

- conductivity $\sigma$
- electrodes $E_1, \ldots, E_L$
- contact impedances $z_l > 0$

\[-\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega,\]
\[u + z_l \, i_\nu = U_l \quad \text{on } E_l,\]
\[i_\nu = 0 \quad \text{on gaps},\]
\[
\int_{E_l} i_\nu dS = I_l, \quad l = 1, \ldots, L,
\]

where $i_\nu = \nu \cdot \sigma \nabla u$ on $\partial \Omega$ is the boundary current.

(Weak formulation, existence & uniqueness: Somersalo, Cheney & Isaacson, 1992)

$R_\sigma : I \mapsto U$ \textbf{Current-to-Voltage or Neumann-to-Dirichlet map.}
How to descretize the conductivity?

- There is a rapid decay of the achievable resolution away from the boundary. (Dobson 1992, Palamodov 2002)
- This effect has been taken into account for numerical inversion by coarsening the conductivity discretization towards the interior. Different strategies have been suggested. For example,
  - use a distinguishability criterion for the continuum model (MacMillan, Manteuffel & McCormick 2004)
  - match the number of conductivity coefficients to the degrees of freedom available in the ND map (Cheney, Isaacson, Newell, Simske & Goble 1990, Borcea, Druskin & Vasquez 2008)

An explicit resolution-based quantification of the discretization size for CEM across the domain is still an open task.
Sensitivity and distinguishability

Relative sensitivity for distinguishing conductivities $\tilde{\sigma}, \sigma$:

$$\lambda_{\sigma, \tilde{\sigma}} := \frac{\|R_{\tilde{\sigma}} - R_{\sigma}\|_2}{\|R_{\sigma}\|_2}.$$ 

Remark: Isaacson 1986 introduced the notion of absolute sensitivity (detectability, distinguishability) which strongly depends on the background conductivity and on the contact impedances.

The conductivities $\sigma$ and $\tilde{\sigma}$ are distinguishable by the measurement setting if

$$\lambda_{\sigma, \tilde{\sigma}} \geq \frac{\|R_{\text{meas}, \sigma} - R_{\sigma}\|_2}{\|R_{\sigma}\|_2}$$

where the latter expression is the relative measurement noise.
Can we compute the sensitivity?

(I) Somersalo, Cheney and Isaacson 1992: Concentric conductivities and symmetric electrodes.

(II) Demidenko 2011: Homogeneous conductivity, arbitrary electrodes.

- Merge (I) and (II),
- Apply conformal mapping to determine $R_\sigma$ for

(III) Arbitrary circular perturbation, arbitrary electrodes. (Winkler and R. 2014)
Conformally mapped CEM

Investigate.
Conformally mapped CEM

Investigate.

map conformally

Solve.

\[ w \]

conclude

\[ w^{-1} \]
Optimal conductivity meshes
Resolution study for constant background conductivity and circular perturbations

\[ \sigma = 1, \quad \tilde{\sigma} = \tilde{\sigma}(x, r) = \sigma + \eta \chi_{B_r(x)}, \quad B_r(x) \subset B_1(0) = \Omega \]

Monotonicity: If \( r_1 \geq r_2 \) then

\[ \lambda_{\sigma, \tilde{\sigma}}(x,r_1) \geq \lambda_{\sigma, \tilde{\sigma}}(x,r_2). \]

For a given numerical value \( \varepsilon \in ]0, 1[ \) we determine the smallest radius \( r_{\text{min}} = r_{\text{min}}(x, \varepsilon) \) such that

\[ \varepsilon = \lambda_{\sigma, \tilde{\sigma}}(x,r_{\text{min}}). \]

The area of \( B_{r_{\text{min}}}(x) \) is a characteristic of the resolution in \( x \) w.r.t. the sensitivity/spectral noise level \( \varepsilon \).
Algorithm for generating optimal resolution meshes

Idea: Fill the disk with balls of roughly the same sensitivity and apply a Voronoi tessellation of $\Omega$ w.r.t. the centers.

Input: $\varepsilon, \eta$; finite set of test points $\mathcal{T} \subset \Omega$;

$\mathcal{C} := \emptyset$; $\mathcal{P} := \emptyset$;

repeat

Pick $p \in \mathcal{T}$; \hspace{1em} $\mathcal{T} := \mathcal{T} \setminus p$;

Find $r = r_{\min}(p, \varepsilon)$;

if $B_r(p) \subset \Omega \land B_r(p) \cap \mathcal{C} = \emptyset$ then

$\mathcal{P} := \mathcal{P} \cup p$; \hspace{1em} $\mathcal{C} := \mathcal{C} \cup B_r(p)$; \hspace{1em} $\mathcal{T} := \mathcal{T} \setminus B_r(p)$;

end if

until $\mathcal{T} = \emptyset$

Output: Voronoi tessellation of $\mathcal{P}$ truncated to $\Omega$;
Optimal resolution meshes

\[ \varepsilon = 0.02, \ 182 \text{ cells} \]

\[ \varepsilon = 0.01, \ 445 \text{ cells} \]

\[ \varepsilon = 0.02, \ 177 \text{ cells} \]
Numerical experiments on the unit disk

- Data generation by the analytic forward solver.
- Added component-wise uniform noise to the potential vector \( \| U - U^\delta \|_2 = 0.01 \).
- Inexact Newton solver REGINN as regularization scheme where
  - the linearized problems are solved by conjugate gradients,
  - all internal parameters of REGINN are the same for all experiments,
  - stopping criterion is Morozov’s discrepancy principle,
  - forward computations by FEM with meshes of 10 to 13,000 triangles refined near to the electrodes and being independent of the conductivity mesh.
True conductivity

$n_{it} = 8, e = 21.8\%$

$n_{it} = 10, e = 22.7\%$

$n_{it} = 21, e = 24.7\%

Optimal mesh (445 cells)

$n_{it} = 10, e = 16.2\%$

$n_{it} = 16, e = 17.4\%$

$n_{it} = 42, e = 18.8\%$

Triangle mesh (452 cells)

Uniform mesh (447 cells)
Adaptive conductivity meshes
The need for speed

- The computation of the optimal meshes shown before
  - required several thousand LU decompositions with up to 30,000 unknowns and
  - took days ($\varepsilon = 0.02$) up to weeks ($\varepsilon = 0.01$) in our MATLAB implementation on a 2.2 GHz workstation with 16 CPU cores and 128 GB RAM.

- Explicit expressions of conformal maps from arbitrary simply connected domains to the unit disk are not available in general.
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A fast and general heuristic is needed which relies on our theoretical results for the unit disk with circular inclusion.
The CEM resolution curve

\[ \sigma = 1, \quad 1 + \eta \chi_{B_{r_{\text{min}}}}(\bullet) \]

for \( \varepsilon = 5\% \).
The CEM resolution curve

\[ \sigma = 1, \quad \tilde{\sigma}(x, r_{\text{min}}) = 1 + \eta \chi_{B_{r_{\text{min}}}(\bullet)} \]
for \( \varepsilon = 5\% \).

area of \( B_{r_{\text{min}}}(\bullet) \)
vs. radial location of \( \bullet \).
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Area of \( B_{r_{\text{min}}}(\bullet) \) vs. radial location of \( \bullet \).
The CEM resolution curve

\[
\sigma = 1, \quad \tilde{\sigma}(x, r_{\text{min}}) = 1 + \eta \chi_{B_{r_{\text{min}}}}(\bullet)
\]
for \( \varepsilon = 5\% \).

area of \( B_{r_{\text{min}}} (\bullet) \)
vs. radial location of \( \bullet \).
The CEM resolution curve

\[ \sigma = 1, \quad \tilde{\sigma}(x, r_{\text{min}}) = 1 + \eta \chi B_{r_{\text{min}}}() \]

for \( \varepsilon = 5\% \).

area of \( B_{r_{\text{min}}}() \) vs. radial location of \( \bullet \).
The CEM resolution curve

\[ \sigma = 1, \quad \tilde{\sigma}(x, r_{\text{min}}) = 1 + \eta \chi_{B_{r_{\text{min}}}}(\bullet) \]
for \( \varepsilon = 5\% \).

area of \( B_{r_{\text{min}}}(\bullet) \)
vs. radial location of \( \bullet \).
The CEM resolution curve

\[ \sigma = 1, \quad \tilde{\sigma}(x, r_{\text{min}}) = 1 + \eta \chi_{B_{r_{\text{min}}}(\bullet)} \]

for \( \varepsilon = 5\% \).

Area of \( B_{r_{\text{min}}}(\bullet) \) vs. radial location of \( \bullet \).
The CEM resolution curve

\[ \sigma = 1, \quad \tilde{\sigma}(x, r_{\text{min}}) = 1 + \eta \chi_{B_{r_{\text{min}}}}(\bullet) \]
for \( \varepsilon = 5\% \).

The function on the right is called the resolution curve \( A_{\text{CEM}} \)

\[ A_{\text{CEM}} = A_{\text{CEM}}(|x|) = \pi r_{\text{min}}^2, \quad x \in \Omega. \]

It depends on \( \varepsilon, \eta \), and the electrode configuration.
Towards a heuristic on the unit disk

- \( \sigma = 1, \quad 1 + 10^6 \chi_{B_{r_{\text{min}}}}(x) \) where \( \lambda_{\sigma, \tilde{\sigma}(x, r_{\text{min}})} = 1.5\% \).
- Plots below show resolution curves \( A_{\text{CEM}}(|x|) = \pi r_{\text{min}}^2 \).
- The solid line belongs to the resolution curve \( A_{\text{cont}} \) of the continuum EIT model which can be approximated very fast.

![Graph showing resolution curves for different electrode coverages](image)

- 8 (▼), 16 (■) and 24 (●) electrodes.
- 16 electrodes covering 25% (▼), 50% (■) and 75% (●) of the boundary.
Heuristic on the unit disk

Idea: Lift $A_{\text{cont}}$ by an affine-linear term to match $A_{\text{CEM}}$ at both end points.

$$A_{\text{heu CEM}}(|x|) := A_{\text{cont}}(|x|) + \left(1 - \frac{|x|}{0.95}\right) (A_{\text{CEM}}(0) - A_{\text{cont}}(0))$$

$$+ \frac{|x|}{0.95} (A_{\text{CEM}}(0.95) - A_{\text{cont}}(0.95))$$

Then, work with

$$r_{\text{heu min}}(x) := \sqrt{\frac{A_{\text{heu CEM}}(|x|)}{\pi}}$$

in the algorithm for computing the meshes.

The resulting meshes are called adaptive meshes.
Optimal vs. adaptive mesh

Locations of the Voronoi cell centroids for the optimal mesh (×, 182 cells) and the adaptive mesh (●, 205 cells). Here, $\varepsilon = 2\%$.

Generating an adaptive mesh usually takes less than one second in MATLAB on an Intel i7 notebook.
True conductivity

Optimal mesh (445 cells)

Adaptive mesh (440 cells)

$\text{n_{it} = 8, e = 21.8\%}$

$\text{n_{it} = 9, e = 22.2\%}$

$\text{n_{it} = 10, e = 16.2\%}$

$\text{n_{it} = 12, e = 16.5\%}$
Heuristic for arbitrary domains

Let $\Omega$ an arbitrary, simply connected domain.

The relative distance of $z \in \Omega$ to the closest electrode is given by

$$d(z) = \frac{\min_{l=1,\ldots,L} \text{dist}(z, E_l)}{\max_{\zeta \in \Omega} \min_{l=1,\ldots,L} \text{dist}(\zeta, E_l)}.$$ 

Then, work with

$$r_{\min}^{\text{heu}}(x) := \sqrt{\frac{A_{\text{CEM}}^{\text{heu}} (1 - d(x))}{\pi}}$$

in the algorithm for computing the meshes.
Adaptive meshes for non-circular domains

16 electrodes, $\varepsilon = 2\%$
Numerical experiments: Simulated data 1

\[ \delta = 0.25\% \]

- Generic triangle mesh
  - \( n_{\text{it}} = 26, \ e = 15.5\% \)
  - Adaptive mesh
    - (5126 cells)
  - Triangle mesh
    - \( n_{\text{it}} = 102, \ e = 18.5\% \)
    - (5126 cells)
  - Uniform mesh
    - \( n_{\text{it}} = 84, \ e = 17.1\% \)
    - (5126 cells)
Numerical experiments: Simulated data 2

\[ \delta = 1\% \]

\[ n_{it} = 63, \; e = 17.8\% \]

Adaptive mesh (887 cells)

\[ n_{it} = 181, \; e = 20.8\% \]

Triangle mesh (888 cells)

\[ n_{it} = 414, \; e = 21.1\% \]

Uniform mesh (900 cells)
Numerical experiments: Simulated data 3

\[ \delta = 0.5\% \]

\[ n_{\text{it}} = 60, \; e = 19.6\% \]
adaptive mesh
(2152 cells)

\[ n_{\text{it}} = 198, \; e = 20.1\% \]
triangle mesh
(2152 cells)

\[ n_{\text{it}} = 124, \; e = 21.3\% \]
uniform mesh
(2152 cells)
Numerical experiments: Measured data

Saline water tank with metal inclusions
Estimated measurement tolerance: $\delta \approx 0.1–0.2\%$.
To account for model imperfections we set $\delta = 0.3\%$.

Data kindly provided by Aku Seppänen (University of Eastern Finland) and Stratos Staboulis (Aalto University).

$n_{\text{it}} = 26$
adaptive mesh
(1193 cells)

$n_{\text{it}} = 30$
triangle mesh
(1194 cells)

$n_{\text{it}} = 46$
uniform mesh
(1193 cells)
Summary

- We introduced an analytic method to quantify the sensitivity of CEM measurements to circular perturbations in conductivity on a disk.

- Based on this findings we discretized the conductivity space such that each cell in the mesh has the same impact on the measurements.

- Finally, we derived a heuristic approximation of sensitivity based discretizations for general domains and verified its performance with simulated and measured data.