Seismic tomography is locally ill-posed

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Acoustic wave equation

\[ u(t, x) \in \mathbb{R} \text{ acoustic potential in } x \in \Omega \subset \mathbb{R}^d \text{ at time } t \geq 0: \]

\[ c \partial_t^2 u - \nabla_x \cdot \left( r \nabla_x u \right) = f(x, t), \quad u|_{\partial \Omega} = 0, \]

with initial data \( u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1 \) and coefficients

\[ c := \frac{1}{\rho \nu^2} \quad \text{and} \quad r := \frac{1}{\rho} \]

where \( \rho = \rho(x) \) mass density, \( \nu = \nu(x) \) speed of sound.

Remark: The Dirichlet boundary restriction is quite meaningful in the framework of seismic wave propagation. According to finite wave speed and finite observation time, homogeneous boundary conditions can be assumed if \( \Omega \) is chosen sufficiently large.
Acoustic wave equation: weak formulation

assumptions/notations: $c, r \in L_{\infty}^+ (\Omega), V = H_0^1(\Omega), H = L^2(\Omega)$,

$u_0 \in V, u_1 \in H, f \in L^2((0,T), H) = L^2((0,T) \times \Omega)$

$a_r : V \times V \to \mathbb{R}, \quad a_r(\psi, \varphi) = \int_{\Omega} r \nabla_x \psi \cdot \nabla_x \varphi \, dx.$

$X := C^0([0,T], V) \cap C^1([0,T], H), \quad \|u\|_X^2 := \max_{0 \leq t \leq T} \|u(t)\|_V^2 + \max_{0 \leq t \leq T} \|\dot{u}(t)\|_H^2$

Find $u \in X$ with $u(0) = u_0$ and $\dot{u}(0) = u_1$ such that

$$\int_0^T \left( a_r(u(t), v(t)) - \langle c\dot{u}(t), \dot{v}(t) \rangle_H \right) dt = \int_0^T \langle f(t), v(t) \rangle_H dt$$

for all $v \in C_0^\infty([0,T], V)$. 
Properties of the weak solution

- The weak wave equation has a unique solution, which depends continuously on the data and satisfies (Lions & Magenes 1972, Stolk 2000)

\[ \|u(t)\|_V^2 + \|\dot{u}(t)\|_H^2 \lesssim \|u_0\|_V^2 + \|u_1\|_H^2 + \int_0^T \|f(\tau)\|_H^2 \, d\tau. \]

- For almost all \( s \in ]0, T[ \),

\[ a_T(u(s), w) + \langle c\ddot{u}(s), w \rangle_{V' \times V} = \langle f(s), w \rangle_H \quad \text{for all } w \in V. \]

- \( c\ddot{u} \in L^2([0, T], V') \) and \( \ddot{u} \in L^2([0, T], V') \) provided \( c \in W^{1,\infty}(\Omega) \).

- The weaker assumption \( f \in L^2([0, T], V') \) is not sufficient to guarantee \( u \in L^2([0, T], V) \).
The inverse problem and its ill-posedness
Seismic reflection inverse problem

Seismic tomography forward operator

\[ F : D(F) \subset L^\infty(\Omega)^2 \to X, \quad (c, r) \mapsto u, \]

where \( D(F) = \{(c, r) \in L^\infty(\Omega)^2 : c(x) \geq k_-, r(x) \geq k_-, \text{ a.e.}\} \)

- Let \( M \subset \Omega \) be the (smooth) measurement submanifold.
- Let \( \Psi : C^0([0, T], V) \to L^2([0, T] \times M) \) be the measurement operator. For instance, \( \Psi : u \mapsto u\big|_M \) (trace map).

Given \( w \in L^2([0, T] \times M) \) find \( (c, r) \in D(F) \) such that

\[ \Psi F(c, r) = w. \]

Solving above problem is called full waveform inversion in seismic imaging.
Local ill-posedness in Banach spaces

\[ T : D(T) \subset X \rightarrow Y, \quad X, Y \text{ infinite dim. Banach spaces} \]

**Def.:** The equation \( T(x) = y \) is called **locally ill-posed** in \( x^+ \in D(T) \) satisfying \( T(x^+) = y \) if in any neighborhood of \( x^+ \) a sequence \( \{x_k\}_{k \in \mathbb{N}} \subset D(T) \) can be found such that

\[
\lim_{k \to \infty} \| T(x_k) - T(x^+) \|_Y = 0, \quad \text{however} \quad \| x_k - x^+ \|_X \not\to 0 \text{ for } k \to \infty.
\]

(Hofmann 1997)
A criterion for local ill-posedness

Lemma  The problem \( T(x) = y \) is locally ill-posed in \( x^+ \in D(T) \) if

- \( T \) is compact, weak-\( \star \)-to-weak continuous, and
- there is \( \{e_k\}_{k \in \mathbb{N}} \subset D(T), \|e_k\|_X = 1 \), which converges weakly-\( \star \) to 0 such that \( \{x^+ + re_k\} \subset D(F) \) for any \( r \in ]0, 1] \).

Proof: Define \( x_k := x^+ + \rho e_k \in B_r(x^+) \cap D(T) \) for any \( 0 < \rho < r \). We have \( \|x_k - x^+\|_X = \rho \) but \( x_k \not\to x^+ \).

\( T \) weak-\( \star \)-to-weak continuous and compact: \( \|T(x_k) - T(x^+)\|_Y \to 0 \). \( \bigvee \)
Weak-$\star$-to-weak continuity (part 1)

$$ F : D(F) \subset L^\infty(\Omega)^2 \to L^2([0, T] \times \Omega) \quad (c, r) \mapsto u, $$

$$ w_k \xrightarrow{\ast} w \text{ in } L^\infty(\Omega) \iff \int_\Omega w_k v \, dx \xrightarrow{k \to \infty} \int_\Omega w v \, dx \quad \forall v \in L^1(\Omega) $$

**Proposition** $F$ is weak-$\star$-to-weak continuous.

**Proof:**

- $(c_m, r_m) \xrightarrow{\ast} (c, r) \in D(F); \ u_m = F(c_m, r_m), u = F(c, r) \in X.$
- $\{u_m\}$ and $\{u_m\}$ are bounded in $L^2([0, T], V)$ and $L^2([0, T], H)$, resp.
- weakly convergent subsequences $\{u_m\}_{l \in \mathbb{N}}$ and $\{u_m\}_{l \in \mathbb{N}}$ with limits $\eta$ and $\xi$, resp.
- Observe $\dot{\eta} = \xi$.

We will show now that $\eta$ solves the wave equation.
Weak-⋆-to-weak continuity (part 2)

Let \( v \in C_0^\infty([0, T], V) \) and consider

\[
\int_0^T \left( a_{rm_l} (u_{ml}(t), v(t)) - \langle c_{ml} u_{ml}(t), \dot{v}(t) \rangle_H \right) dt = \int_0^T \langle f(t), v(t) \rangle_H dt.
\]

We are going to show that the left hand side converges to

\[
\int_0^T \left( a_r (\eta(t), v(t)) - \langle c \dot{\eta}(t), \dot{v}(t) \rangle_H \right) dt.
\]

Indeed,

\[
\int_0^T \left( a_{rm_l} (u_{ml}(t), v(t)) - a_r (\eta(t), v(t)) \right) dt = \\
\int_0^T a_{rm_l} - r (u_{ml}(t), v(t)) dt + \int_0^T a_r (u_{ml}(t) - \eta(t), v(t)) dt, \quad \rightarrow 0 \quad \text{as} \quad u_{ml} \rightarrow \eta.
\]
Weak-⋆-to-weak continuity (part 3)

Further,
\[
\left| \int_{0}^{T} a_{r_{ml}} - r \left( u_{ml}(t), v(t) \right) \, dt \right| \leq \|(r_{ml} - r) \nabla_{x} v\|_{L^{2}([0,T],H^{d})} \|u_{ml}\|_{L^{2}([0,T],V)}
\]
and \(\|(r_{ml} - r) \nabla_{x} v\|^{2} \lesssim |\nabla_{x} v|^{2}\) a.e. in \(\Omega \times [0,T]\).

By the dominated convergence theorem,
\[
\int_{0}^{T} \left( a_{r_{ml}} (u_{ml}(t), v(t)) - a_{r} (\eta(t), v(t)) \right) \, dt \xrightarrow{l \to \infty} 0.
\]

Analogously,
\[
\int_{0}^{T} \left( \langle c_{ml} \dot{u}_{ml}(t), \dot{v}(t) \rangle_{H} - \langle c\dot{\eta}(t), \dot{v}(t) \rangle_{H} \right) \, dt \xrightarrow{l \to \infty} 0.
\]

Hence, \(\eta\) satisfies the wave equation in weak form.
Moreover,

$$\eta(0) = u_0 \text{ and } \dot{\eta}(0) = u_1.$$ 

Thus, $\eta = u$ and the whole sequence $\{u_m\}$ converges weakly to $u$ because all convergent subsequences of $\{u_m\}$ have the limit $u$. ✓
Compactness (part 1)

\[ F : \mathcal{D}(F) \subset L^\infty(\Omega)^2 \to L^2([0,T] \times \Omega) \quad (c, r) \mapsto u \]

**Proposition**  
\( F \) is compact, that is, \( F \) maps bounded sets to relatively compact ones.

**Proof:** Let \( Q \subset \mathcal{D}(F) \) be bounded. We show that \( F(Q) \) is relatively compact in \( C([0,T],H) \) by the (general) theorem of Arcela-Ascoli.

By the energy estimate, for \( t \in [0,T] \),

\[ \{ u(t) : u \in F(Q) \} \subset \{ v \in V : \| \nabla_x v \|_{L^2(\Omega)^d} \leq \hat{c} \} \]

and the latter set is relatively compact in \( H = L^2(\Omega) \).
Compactness (part 2)

Furthermore, $F(Q)$ is equicontinuous because

$$
\|u(t_2) - u(t_1)\|_H = \sup_{\|\psi\|_H = 1} \langle u(t_2) - u(t_1), \psi \rangle_H = \sup_{\|\psi\|_H = 1} \int_{t_1}^{t_2} \frac{d}{ds} \langle u(s), \psi \rangle_H \, ds
$$

$$
= \sup_{\|\psi\|_H = 1} \int_{t_1}^{t_2} \langle \dot{u}(s), \psi \rangle_H \, ds \leq |t_2 - t_1| \|\dot{u}\|_{C([0,T],H)}
$$

and $\|\dot{u}\|_{C([0,T],H)}$ is uniformly bounded for $(c, r) \in Q$.

The continuous embedding $C([0,T], H) \hookrightarrow L^2([0,T], H)$ finishes the proof. ✓
Main result

Given $w \in L^2([0, T] \times M)$ find $(c, r) \in D(F)$ such that

$$\Psi F(c, r) = w.$$

**Theorem** The above inverse problem of seismic imaging is locally ill-posed in any point $(c_0, r_0) \in D(F)$.

**Proof:** The assertion follows readily from the abstract criterion as soon as we have found a sequence

$$\{e_n\} \subset L^\infty(\Omega), \ e_n \geq 0 \text{ a.e.}, \ \|e_n\|_\infty = 1, \ e_n \overset{*}{\rightharpoonup} 0.$$

Let $\xi \in \Omega$ and $\rho_n \searrow 0$. Define $e_n := \chi_{B_{\rho_n}(\xi)}$.

Obviously, $e_n \geq 0$ and $\|e_n\|_\infty = 1$ and

$$\int_\omega e_n v \, dx \overset{n \to \infty}{\longrightarrow} 0 \quad \forall v \in L^1(\Omega)$$

by the dominated convergence theorem. √
Final remarks
Discussion

- Ill-posedness of seismic tomography is an empirical fact known in the geophysical community for quite some time.

- Our result shows that ill-posedness is an intrinsic feature of the mathematical model which does not originate from too few measurements.

- Thus, regularization is not only advisable but inevitable.

- We are cautiously optimistic that our theory carries over to other boundary settings if the corresponding parameter-to-solution map can be defined in a functional analytic framework and if an energy estimate holds.