The Poisson Process SS 2016
Exercise sheet 1

The exercises will be discussed in the tutorial on Thursday, April 28, 2016.

Exercise 1.1:
Let \( \eta_1, \eta_2, \ldots \) be a sequence of point processes and define 
\[ \eta := \eta_1 + \eta_2 + \ldots, \]
that is 
\[ \eta(\omega, B) := \eta_1(\omega, B) + \eta_2(\omega, B) + \ldots, \quad \omega \in \Omega, B \in \mathcal{X}. \]

(a) Prove that \( \eta \) is a s-finite measure.

(b) Show that \( \eta \) is a point process.

Exercise 1.2:
Let \( X_1, X_2 \) be uniformly distributed on \( B(0,1) := \{ x \in \mathbb{R}^2 : \|x\| < 1 \} \), \( X_3 \) be uniformly distributed on the set \( \{ (0,0), (0,1), (1,0), (1,1) \} \) and \( X_4 \) be uniformly distributed on the line segment \( [-1,1] \times \{0\} \).

Let \( \eta \) be the point process defined by 
\[ \eta := \sum_{i=1}^{4} \delta_{X_i}. \]

(a) Determine the intensity measure \( \lambda \) of \( \eta \).

(b) Compute \( \mathbb{E} \int_{[-1,1]^2} x^2 + y^2 \eta(d(x,y)) \).

Exercise 1.3:
Let \( Q \) be a probability measure on \( Q \) and let \( X_1, X_2, \ldots \) be identically distributed random variables on \( X \) with distribution \( Q \). Further, let \( \kappa \) be a \( \mathbb{N}_0 \)-valued random variable, independent of \( (X_i)_{i \in \mathbb{N}} \).

Determine the intensity measure \( \lambda \) of 
\[ \eta := \sum_{i=1}^{\kappa} \delta_{X_i}. \]

Exercise 1.4:
Suppose that \( X = [0,1] \).

(a) Find a \( \sigma \)-field \( \mathcal{X} \) and a measure \( \mu \) on \( (X, \mathcal{X}) \) such that \( \mu(X) = 1 \) and \( \mu(B) \in \{0,1\} \) for all \( B \in \mathcal{X} \) which is not of the form \( \mu = \delta_x \) for some \( x \in X \).

\textit{Hint:} Take the system of all countably infinite subsets of \( X \) as a generator of \( \mathcal{X} \).

(b) Let \( \mathcal{X} \) be the Borel \( \sigma \)-field and \( \mu \) be a measure on \( (X, \mathcal{X}) \) with the properties \( \mu(X) = 1 \) and \( \mu(B) \in \{0,1\}, B \in \mathcal{X} \). Show that there is a \( x \in X \) such that \( \mu = \delta_x \).

Exercise 1.5:
Show that Proposition 2.10 remains valid if \( B_1, \ldots, B_m \in \mathcal{X} \) in (ii) are assumed to be pairwise disjoint.